PROBLEMS; "RIEMANNIAN GEOMETRY"

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This is a collection of problems for the course "Riemannian Geometry", 1MA196, fall 2017, at Uppsala University.
(http://www.math.uu.se/∼astrombe/riemanngeometri2017/rg2017.html)
I remark that the purpose of many of the problems below is mainly to fill in or explain some (pedantic) facts or details which I felt were appropriate to mention in my lectures, and which I couldn't find in Jost's book. In a later version, I will probably move the content of these problems into some kind of appendices in the lecture notes.

Contents

1. Problems

Problem 1. **[Manifolds are path-connected]** Prove that if M is a topological manifold (in the sense defined in the course, in particular M is connected) then M is *path-connected,* i.e. for any two points $p, q \in M$ there is a curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$.

Solution: p. 45.

Problem 2. **[A criterion for paracompactness.]**

(a). Let M be any topological space which is locally Euclidean. Prove that M is second countable iff M has a countable atlas.

[Pedantically, in Lecture #1 we only defined the notation of an "atlas" when M is connected and Hausdorff; however the same defnition applies to any locally Euclidean topological space.]

(b). Let M be a connected Hausdorff space which is locally Euclidean. Prove that M is paracompact iff M has a countable atlas.

Solution: p. 46.

Problem 3. [Invariance of dimension.] Brouwer's Theorem on invariance of dimension states: *If nonempty open sets $U \subset \mathbb{R}^{d_1}$ and $V \subset \mathbb{R}^{d_1}$ are homeomorphic, then $d_1 = d_2$.* (Cf., e.g., Hatcher, [7, Thm. 2.26].) Using this result, prove the following: If M is a connected Hausdorff space for which every point has an open neighborhood U which is homeomorphic to an open subset Ω of $\mathbb{R}^d$ for some $d \in \mathbb{Z}_{\geq 1}$ (which apriori may depend on U), then in fact all the dimensions d appearing must be one and the same.

(Thus, in Def. 1 in Lecture #1, we would not obtain any new objects if we modify the definition so that the dimension is allowed to depend on U.)

Solution: p. 47.

Problem 4. [Every C^∞ atlas determines a unique C^∞ structure.] Prove the following statement from Lecture #1 (here made slightly more precise): "Any C^∞ atlas on a topological manifold M is contained in a unique C^∞ structure on M, namely the family of all charts which are compatible with every chart in the given atlas."

Solution: p. 47.

Problem 5. [Basic property of C^∞ structures.]

(a). Let $B_r(0)$ be the open ball in $\mathbb{R}^d$ of radius $r > 0$, centered at the origin. Prove that there exists an uncountable family $\mathcal{H}$ of homeomorphisms of $B_1(0)$ onto itself, with each $h \in \mathcal{H}$ satisfying $h(x) = x$ for all $x \notin B_{1/2}(0)$, such that for any two $h_1 \neq h_2 \in \mathcal{H}$, the function $h_1 \circ h_2^{-1}$ is not C^∞.

[Hint. One can e.g. take each h to be of the form $h(x) = f(\|x\|)\|x\|^{-1}x$ (for $x \neq 0$) where f is a piecewise linear function on $(0, 1)$.]

(b). Let M be a topological manifold. Prove that if M has one C^∞ structure then there exists an uncountable family $\mathcal{F}$ of *distinct* C^∞ structures on M such that for any two structures in $\mathcal{F}$, the corresponding C^∞ manifolds are diffeomorphic.

[Hint. One approach is as follows. Let $\mathcal{H}$ be as in part (a) and fix a chart (U, x) on M with $x(U) = B_1(0)$ (prove that such a chart exists). Now for each $h \in \mathcal{H}$ we can define a homeomorphism $\varphi_h : M \to M$ by letting φ_h be "given by h inside U and the identity map everywhere else". Now we get a new C^∞ structure by "composing the given C^∞ structure with φ_h". (These things have to be made precise.)]

Remark: The problem shows why it is much more interesting to ask for the number of *diffeomorphism classes* of C^∞ structures on a given topological manifold M. (Cf. the end of Lecture #1.)

Solution: p. 49.

Problem 6. [Open submanifolds] Let M be a C^∞ manifold and let U be an open subset of M.

(a). Prove that U inherits from M a natural structure of a (not necessarily connected) C^∞ manifold. This C^∞ manifold is called an *open submanifold* of M.

(b). Prove that the inclusion map $i : U \to M$ is C^∞.

(c). Let N be another C^∞ manifold and f a map from M to N. Prove that if f is C^∞, then so is the map $f_{|U} : U \to N$ for every open subset $U \subset M$ (with its inherited C^∞ manifold structure). Prove also the following converse: If $\{U_\alpha\}$ is a family of open sets covering M and $f_{|U_\alpha}$ is C^∞ for every α, then f itself is C^∞.

Solution: p. 54.

Problem 7. [Existence of C^∞ functions with desired properties.] Let M be a C^∞ manifold.

(a). Let f be a function from M to $\mathbb{R}$ and let U be an open subset of M such that $f_{|U} \in C^\infty(U)$ and $\text{supp}(f) \subset U$. (Recall that $\text{supp}(f)$ is the closure in M of the set $\{p \in M \ : \ f(p) \neq 0\}$.) Prove that $f \in C^\infty(M)$.

(b). Let U be an open subset of M and let $f : U \to \mathbb{R}$ be a C^∞ function with compact support. Prove that the function

$$\widetilde{f} : M \to \mathbb{R}, \qquad \widetilde{f}(p) = \begin{cases} f(p) & \text{if } p \in U \\ 0 & \text{if } p \notin U \end{cases}$$

is C^∞.

(c). Prove that for every $p \in M$ and every open subset $U \subset M$ with $p \in U$, there exists a C^∞ function $f : M \to [0,1]$ which has compact support contained in U and which satisfies $f(p) = 1$.

(d). (A strengthening of (c).) Prove that if K is compact and U is open with $K \subset U \subset M$, then there exists a C^∞ function $f : M \to [0,1]$ which has compact support contained in U, and which satisfies $f_{|K} \equiv 1$.

[Hint: When $M = \mathbb{R}^d$ the claim is a well-known fact of analysis; cf., e.g., [10, Thm. 1.4.1]. Thus it remains to reduce to this Euclidean setting...]

(e). (A simple consequence of (d) and (a).) Prove that if K is compact and U is open with $K \subset U \subset M$, and if $f : U \to \mathbb{R}$ is a C^∞ is a function, then there exists a C^∞ function $f_1 : M \to \mathbb{R}$ which satisfies $f_{1|K} \equiv f_{|K}$.

Solution: p. 55.

Problem 8. [Basic facts about product manifolds]

(a) Prove that if M and N are C^∞ manifolds then the Cartesian product $M \times N$ also naturally carries the structure of a C^∞ manifold. (Cf. [12, p. 4 (Ex. 4)].)

(b) Prove that the projection maps $\mathrm{pr}_1 : M \times N \to M$ and $\mathrm{pr}_2 : M \times N \to N$ are C^∞.

(c) Prove that if $f : M \to N_1$ and $g : M \to N_2$ are C^∞ maps of manifolds then also the map $(f, g) : M \to N_1 \times N_2$, defined by

$$(f, g)(p) := (f(p), g(p)),$$

is C^∞.

(d) Prove that if $f : M_1 \to N_1$ and $g : M_2 \to N_2$ are C^∞ maps of manifolds then also the map

$$M_1 \times M_2 \to N_1 \times N_2, \qquad (p, q) \mapsto (f(p), g(q)),$$

is C^∞.

Solution: p. 57.

Problem 9. [Definition of quotient manifold.] Let M be a topological manifold, and let Homeo(M) be the group of all homeomorphisms of M onto itself (the group operation is composition). Let Γ be a subgroup of Homeo(M). We assume that Γ acts *freely* on M, meaning that for any $\gamma \in \Gamma$ and $p \in M$, if $\gamma(p) = p$ then $\gamma = \mathrm{Id}$. We also assume that Γ acts *properly discontinuously* on M, meaning that for any compact set $K \subset M$, the set $\{\gamma \in \Gamma : \gamma(K) \cap K \neq \emptyset\}$ is finite. Let us define the relation $\sim$ on M by $[p \sim q \text{ iff } \exists \gamma \in \Gamma \text{ s.t. } \gamma(p) = q]$.

(a) Prove that $\sim$ is an equivalence relation.

(b) Let us write $[p]$ for the $\sim$ equivalence class of a point $p \in M$; let $\Gamma\backslash M := \{[p] : p \in M\}$ be the set of all equivalence classes, and let $\pi : M \to \Gamma\backslash M$, $\pi(p) := [p]$, be the corresponding projection map. Define a topology on $\Gamma\backslash M$ by declaring $U \subset \Gamma\backslash M$ to be open iff $\pi^{-1}(U)$ is open in M. Prove that this indeed is a topology; it is called the *quotient topology.* Prove also that $\Gamma\backslash M$ with this topology is a topological manifold of the same dimension as M.

(c) Now on top of the previous assumptions we assume that M is a C^∞ manifold, and that every $\gamma \in \Gamma$ is a *diffeomorphism* of M. (In other words, $\Gamma \subset \mathrm{Diff}(M)$.) Prove that $\Gamma\backslash M$ inherits from M a natural C^∞ structure, and that π is a C^∞ map.

Solution: p. 60.

Problem 10. [Constructing a C^∞ manifold without requiring from start that it is a topological space.]

(a) Prove that if $\{(U_\alpha, x_\alpha)\}$ is an atlas on a (topological) manifold M, and V is any subset of M, then V is open iff $V \cap U_\alpha$ is open in U_α for every α.

(b) Let us define a "(d-dimensional) C^∞ fold" to be a *set* M together with a family $\{(U_\alpha, x_\alpha)\}_{\alpha \in A}$ where for each $\alpha \in A$, U_α is a subset of M and x_α is a bijection from U_α onto an open subset of $\mathbb{R}^d$, such that $M = \cup_{\alpha \in A} U_\alpha$ and for any $\alpha, \beta \in A$, $x_\alpha(U_\alpha \cap U_\beta)$ is an open subset of $x_\alpha(U_\alpha)$, and the map $x_\beta \circ x_\alpha^{-1}$ on $x_\alpha(U_\alpha \cap U_\beta)$ is C^∞.

Given a "C^∞ fold" M, let us call a subset $V \subset M$ "open" if $x_\alpha(V \cap U_\alpha)$ is open in $\mathbb{R}^d$ for every $\alpha \in A$. Prove that this defines a topology on M. Prove also – by giving an example – that this topology is *not* always Hausdorff.

(c) Prove that a sufficient criterion for the topology defined in part (b) to be Hausdorff is that for any two points $p, q \in M$ there is $\alpha \in A$ such that $p, q \in U_\alpha$. (You may also like to prove the following partial converse: If M is a C^∞ manifold then for any two points $p, q \in M$ there is a C^∞ chart (U, x) on M such that $p, q \in U$.)

(d) Let M be a "C^∞ fold" and assume that the topology defined above is Hausdorff, and also connected and paracompact. Prove that then M is a C^∞ manifold, with $\{(U_\alpha, x_\alpha)\}$ being a C^∞ atlas.

Solution: p. 64.

Problem 11. [Partition of unity: Some variants.]

(a). Prove the following variation of [12, Lemma 1.1.1]: Let M be a C^∞ manifold and let $\mathcal{U} = (U_\alpha)_{\alpha\in A}$ be an open cover of M. Then there exist C^∞ functions $\varphi_\alpha : M \to [0,1]$ ($\alpha \in A$) such that $\operatorname{supp}\varphi_\alpha \subset U_\alpha$ for every $\alpha \in A$, and $\sum_{\alpha\in A} \varphi_\alpha(x) = 1$ for all $x \in M$.

(Remark: Note that in the above statement it is *not* always possible to make each φ_α have compact support; consider e.g. the case $\mathcal{U} = \{M\}$; then we are forced to choose the single φ-function to be $\varphi \equiv 1$.)

(Hint: The above statement can e.g. be deduced as a *consequence* of [12, Lemma 1.1.1].)

(b). Prove that both in [12, Lemma 1.1.1], and in the statement of (a) above, we can further require that all functions φ_α are such that also $\sqrt{\varphi_\alpha}$ is C^∞.

Solution: p. 67.

Problem 12. [Extending a function from a curve to a manifold.]

Let M be a C^∞ manifold, let $c : [a,b] \to M$ be a C^∞ curve, let $s \in (a,b)$, and assume $\dot{c}(s) \neq 0$.

(a) Prove that there is $\varepsilon > 0$ and a (C^∞) chart (U,x) for M such that $a < s - \varepsilon < s + \varepsilon < b$ and

$$c(t) \in U \text{ and } x(c(t)) = (t - s, 0, \ldots, 0), \qquad \forall t \in (s - \varepsilon, s + \varepsilon).$$

(b) Prove that given any C^∞ function $f : [a,b] \to \mathbb{R}$, there is $\varepsilon > 0$ and a C^∞ function $g : M \to \mathbb{R}$ such that $a < s - \varepsilon < s + \varepsilon < b$ and $g(c(t)) = f(t)$ for all $t \in (s - \varepsilon, s + \varepsilon)$.

Solution: p. 69.

Problem 13. [Details in the definition of tangent space.]

In the following all references are to Lecture #2:

(a). In Definition 3, verify that $\sim$ is an equivalence relation.

(b). On p. 4 (below Definition 3): Prove that for any fixed chart (U,x) with $p \in U$, the map $u \mapsto [(U,x,u)]$ is indeed a bijection from $\mathbb{R}^d$ onto T_pM.

(c). On p. 5: Verify the claim that if M is a ((connected)) open subset of a finite dimensional vector space V over $\mathbb{R}$, then there is a natural identification "$T_pM = V$", for every $p \in M$.

(d). On p. 7: Verify that df_p is well-defined.

(e). On p. 7: Verify the chain rule $d(g\circ f)_p = dg_{f(p)} \circ df_p$, when $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are C^∞ maps between C^∞ manifolds.

(f). On p. 8–9: Verify the three facts stated here!

Solution: p. 70.

Problem 14. [Tangent vector of a curve.] Let M be a C^∞ manifold of dimension d, let $c : I \to M$ be a C^∞ curve, and let (U, x) be a chart on M. For $t \in I$ with $c(t) \in U$, we define $c^1(t), \ldots, c^d(t) \in \mathbb{R}$ by

$$x(c(t)) = (c^1(t), \ldots, c^d(t)).$$

Then prove that

$$\dot{c}(t) = \dot{c}^j(t) \frac{\partial}{\partial x^j}.$$

Also explain how this formula shows that the two definitions of "tangent vector of a curve" in Lecture #2 (p. 2 and 8) are consistent with each other.

Solution: p. 76.

Problem 15. [Alternative definition of tangent space.]

(a). Let M be a C^∞ manifold and let $p \in M$. By definition, a *derivation at* p is an $\mathbb{R}$-linear map $D : C^\infty(M) \to \mathbb{R}$ that satisfies the Leibniz identity

$$D(fg) = D(f) \cdot g(p) + f(p) \cdot D(g), \qquad \forall f, g \in C^\infty(M).$$

Prove that there is a natural bijection between the set of all derivations at p and the tangent space $T_p(M)$.

(b). A *vector field* X on M is by definition a C^∞ map $X : M \to TM$ satisfying $\pi \circ X = 1_M$. (Thus using notation from Lecture #7, a vector field on M is the same as a section in $\Gamma(TM)$.) Also by definition, a *derivation of* $C^\infty(M)$ is an $\mathbb{R}$-linear map $D : C^\infty(M) \to C^\infty(M)$ which satisfies $D(fg) = D(f)g + fD(g)$ for all $f, g \in C^\infty(M)$. Prove that there is a natural bijection between the set of vector fields on M and the set of derivations of $C^\infty(M)$.

Solution: p. 78.

Problem 16. [The definition of the tangent bundle TM.] Prove that the construction in Lecture #2, p. 10, leads to a well-defined C^∞ manifold TM, and that the projection map $\pi : TM \to M$ is C^∞.

[Hint: Use Problem 10.]

Solution: p. 79.

Problem 17. [Some facts about df.]

Let M, N be C^∞ manifolds and let $f : M \to N$ be a C^∞. Let $\pi : TM \to M$ and $\pi' : TN \to N$ be the standard projection maps.

(a). Prove that $df : TM \to TN$ is a C^∞ map and $\pi' \circ df = f \circ \pi$. (Facts from Lecture #2.)

(b). Prove that for any C^∞ map $\varphi : N \to \mathbb{R}$ and any $X \in TM$,

$$df(X)(\varphi) = X(\varphi \circ f).$$

(c). Prove that if $f : M_1 \to M_2$ and $g : M_2 \to M_3$ are C^∞ maps between C^∞ manifolds, then $d(g \circ f) = dg \circ df$ (equality between maps $TM_1 \to TM_3$).

Solution: p. 81.

Problem 18. [Riemannian structure on a submanifold of a Riemannian manifold.] Let $f : M \to N$ be a C^∞ immersion of C^∞ manifolds, and assume that N is equipped with a Riemannian metric.

(a). Prove that then also M gets naturally equipped with a Riemannian metric, by setting, for any $p \in M$ and $v, w \in T_pM$:

$$\langle v, w \rangle := \langle df_p(v), df_p(w) \rangle.$$

(In particular this means that any immersed submanifold of a Riemannian manifold gets naturally equipped with a Riemannian metric.)

(b). Prove also that for any piecewise C^∞ curve $\gamma : [a, b] \to M$ we have $L(\gamma) = L(f \circ \gamma)$ and $E(\gamma) = E(f \circ \gamma)$.

(c). Prove that $d(p, q) \geq d(f(p), f(q))$ for all $p, q \in M$, and give an example where strict inequality holds.

Solution: p. 81.

Problem 19. [Existence of a C^∞ curve between any two points.]

Let M be a C^∞ manifold.

(a). Prove that for any two points $p, q \in M$ there exists a piecewise C^∞ curve $\gamma : [0, 1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$.

(b). Show that "piecewise C^∞" can be sharpened to "C^∞" in the previous statement.

Solution: p. 83.

Problem 20. [Basic properties of the hyperbolic space H^n.]

Go through the discussion in [12, Sec. 5.4], and verify all claims up until the computation of the curvature using Jacobi fields! In particular:

(a). Verify that if $p \in H^n$ then T_pH^n is orthogonal to p wrt the form $\langle \cdot, \cdot \rangle$, and the restriction of I to T_pH^n is positive definite, so that we obtain a Riemannian metric on H^n.

(b). Prove that $O(1, n)$ [1] is a group, and that $O(1, n)$ has a normal subgroup of index 2, which we call $O^+(1, n)$, such that each $T \in O^+(1, n)$ acts on H^n by isometries.

(c). Prove that for any $p \in H^n$ and $v \in T_pH^n$, $v \neq 0$, there is a transformation $R \in O^+(1, n)$ whose set of fixed points in $\mathbb{R}^{n+1}$ equals the 2-dimensional plane spanned by p and v. (Hint: The map can be constructed as the "$\langle \cdot, \cdot \rangle$-reflection" in said plane.)

(d). Conclude by proving the formula which Jost states for a geodesic with an arbitrary starting condition.

Solution: p. 85.

[1] I think the group which Jost calls "$O(n, 1)$" is more appropriately called "$O(1, n)$", in view of the definition of $\langle \cdot, \cdot, \rangle$.

Problem 21. [The maximal domain for exp**, and the geodesic flow.]** The goal of this problem is to prove Theorem 2 in Lecture #4. Note that the proof basically just consists in squeezing as much information as possible out of the local ODE existence and uniqueness result (Theorem 1).

(a). For each $p \in M$ and $v \in T_pM$ there is a uniquely determined open interval $I_v \subset \mathbb{R}$ containing 0 such that (i) there exists a geodesic $c_v : I_v \to M$ with $c_v(0) = p$, $\dot{c}_v(0) = v$ and (ii) given *any* open interval $J \subset \mathbb{R}$ containing 0 and any geodesic $\gamma : J \to M$ with $\gamma(0) = p$, $\dot{\gamma}(0) = v$, then $J \subset I_v$ and $\gamma \equiv c_{v|J}$.

We call the above curve c_v the *(unique) maximal geodesic* starting at $v \in T_pM$.

(b). Set $W = \{(t, v) \in \mathbb{R} \times TM : t \in I_v\}$ and define the map $\theta : W \to TM$ by $\theta(t, v) := \dot{c}_v(t)$. Prove that for all $v \in TM$ and $s \in I_v$ we have $\theta(0, v) = v$, $I_{\theta(s,v)} = I_v - s$, [2] and $\theta(\theta(s, v), t) = \theta(t + s, v)$ $(\forall t \in I_{\theta(s,v)})$.

(c). There exist an open subset $\mathcal{D} \subset TM$ and a C^∞ map $\exp : \mathcal{D} \to M$ such that for each $p \in M$ and $v \in T_pM$, $I_v := \{t \in \mathbb{R} : tv \in \mathcal{D}\}$ is an an open interval containing 0, and the curve $t \mapsto \exp(tv)$, $I_v \to M$, is the unique maximal geodesic starting at v. (Note that it is obvious that $\mathcal{D}$ and exp are uniquely determined by the required properties.)

(d). Note that by (c), the set W in part (b) equals

$$W = \{(t, v) \in \mathbb{R} \times TM : tv \in \mathcal{D}\},$$

and that this is an open subset of $\mathbb{R} \times TM$. For $t \in \mathbb{R}$, set

$$W_t := \{v \in TM : (t, v) \in W\}$$

Prove that for each $t \in \mathbb{R}$, W_t is an open subset of TM, and the map $\theta(t, \cdot)$ is a C^∞ diffeomorphism of W_t onto W_{-t} with inverse $\theta(-t, \cdot)$.

(The map $\theta : W \to TM$ is called the *geodesic flow* on TM.)

Solution: p. 90.

[2] Here we use the natural notation $I_v - s := \{x - s : x \in I_v\}$.

Problem 22. [Varying the center of normal coordinates.]

(a). Prove Theorem 3' in Lecture #4.

[Hint: One approach is as follows. First prove that the differential of the map $(\pi, \exp) : \mathcal{D} \to M \times M$ at 0_p is non-singular; hence by the Inverse Function Theorem there is a neighborhood of 0_p in which $(\pi, \exp)$ is a diffeomorphism.]

(b). Let $r > 0$ and let U be an open subset of a Riemannian manifold M, and assume that for every $p \in U$, $B_r(0_p) \subset \mathcal{D}$ and $\exp_{p|B_r(0_p)}$ is a diffeomorphism onto an open subset of M; let us agree to write simply $\exp_p^{-1}$ for the inverse map. Set

$$V := \{(p, \exp_p(v)) \, : \, p \in U, \, v \in B_r(0_p)\} \subset M \times M.$$

Prove that V is an open subset of $M \times M$, and that the map $V \to TM$, $(p, q) \mapsto \exp_p^{-1}(q)$ is C^∞. (More generally one may let r be a continuous function of p.)

Solution: p. 94.

Problem 23. [The Riemannian metric wrt polar coordinates.]

(The point of this problem is to go through the details in the proof of Jost's [12, Thm. 1.4.5].)

Let M be a Riemannian manifold, $p \in M$, and take $r > 0$ so that $\exp_p$ restricted to $B_r(0) \subset T_p(M)$ is a diffeomorphism onto an open subset $U \subset M$. Let (U, x) be the corresponding normal coordinates. Let also (V, φ) be a chart on S^{d-1}, and define the ("polar coordinates") chart $(\mathbb{R}^+ V, y)$ on $\mathbb{R}^d$ by

$$\mathbb{R}^+ V := \{rv \, : \, r \in \mathbb{R}^+, \, v \in V\}$$

(an open cone) and

$$(y^1, \ldots, y^d) = \Big(\|x\|, \varphi\Big(\frac{x}{\|x\|} \Big) \Big).$$

Set $U' = x^{-1}(\mathbb{R}^+ V \cap B_r(0))$; then $(U', y \circ x)$ is a chart on M. Prove that in the coordinates defined by this chart, the Riemannian metric satisfies

$$(h_{ij}(y)) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_{22}(y) & \cdots & h_{2d}(y) \\ \vdots & \vdots & & \vdots \\ 0 & h_{d2}(y) & \cdots & h_{dd}(y) \end{pmatrix}, \qquad \forall y \in (0, r) \times \varphi(V).$$

Solution: p. 96.

Problem 24. [Any (pw C^∞) curve realizing $d(p, q)$ is a geodesic.]

Prove Theorem 2 in Lecture #5: Let M be a Riemannian manifold and let $\gamma : [a, b] \to M$ be a pw C^∞ curve which is parametrized by arc length. Assume that $L(\gamma) = d(\gamma(a), \gamma(b))$. Then γ is a geodesic.

Solution: p. 98.

Problem 25. [Completeness.]

Let $M = \mathbb{R}^d$ with its standard C^∞ manifold structure. Give an example of a complete Riemannian metric on M, and also an example of one non-complete Riemannian metric on M.

(Thus, the parenthesis in Jost's [12, Thm. 1.7.1(i)] is misleading; completeness is *not* a property of the topology, but depends on the choice of metric.)

Solution: p. 99.

Problem 26. [A closed embedded submanifold is complete.]

(a). Let N be a complete Riemannian manifold and let M be an embedded submanifold of N which is *closed.* Prove that M is complete.

(b). Prove that if we replace "embedded submanifold" by "immersed submanifold" in (a), then the conclusion is no longer valid, in general!

Solution: p. 100.

Problem 27. [Spheres and distances.]

The following properties play a role in the proof of the Hopf-Rinow Theorem. Let (X, d) be an arbitrary metric space. Recall that for $p \in X$ and $r > 0$ we write $B_r(p)$ for the open ball $B_r(p) := \{q \in X \,:\, d(p,q) < r\}$.

(a). Prove that d is a continuous function ($X \times X \to \mathbb{R}_{\geq 0}$).

(b). Prove that for any $p \in X$, $r > 0$,

$$\partial B_r(p) \subset \{q \in X \,:\, d(p,q) = r\},$$

and both these sets are closed. Furthermore if (X, d) is a Riemannian manifold[3] then equality holds: $\partial B_r(p) = \{q \in X \,:\, d(p,q) = r\}$.

(c). Continue to assume that (X, d) is a Riemannian manifold. Let $p, q \in X$, $r > 0$, and assume $d(p,q) > r$. Assume that p_0 is a point on $\partial B_r(p)$ where $d(\cdot, q)_{\partial B_r(p)}$ is minimal. Prove that $d(p,q) = d(p,p_0) + d(p_0,q)$.

Solution: p. 101.

Problem 28. [Consequences of $B_r(0_p) \subset \mathcal{D}_p$.]

Let M be a Riemannian manifold, let $p \in M$ and $R > 0$, and assume $B_R(0_p) \subset \mathcal{D}_p$. Prove that then for every point $q \in B_R(p)$, the distance $d(p,q)$ is realized by a geodesic, and hence $B_R(p) = \exp_p(B_R(0_p))$.

Solution: p. 102.

Problem 29. [Existence of geodesics in homotopy classes.]

Prove that Theorem 1 in Lecture #5 remains true for any *complete* (instead of compact) Riemannian manifold.

Solution: p. 103.

[3] By this we mean: X is a Riemannian manifold and d is the metric on X which comes from the Riemannian structure.

Problem 30. [Injectivity radius on a surface of revolution.]

(The following problem is a slight variation of [12, Ch. 1, Problem 11].)

Consider the surface of revolution

$$S := \{(x, e^x \cos\alpha, e^x \sin\alpha) \,:\, x, \alpha \in \mathbb{R}\}.$$

(a). Prove that S is a closed differentiable submanifold of $\mathbb{R}^3$ (cf. the notes to Lecture #2).

(b). Equip S with the Riemannian metric induced by the standard Riemannian metric on $\mathbb{R}^3$ (cf. Problem 18; note that S is complete by Problem 26). Fix $x_0 \in \mathbb{R}$ and let $p_0 = (x_0, e^{x_0}, 0) \in S$. Prove that the injectivity radius of p_0 satisfies $i(p_0) \leq \pi e^{x_0}$.

Solution:
p. 104.

Problem 31. [The fundamental group of the n-punctured plane.]

Let $p_1, \ldots, p_n$ be n distinct points in $\mathbb{R}^2$. Compute $\pi_1(\mathbb{R}^2 \setminus \{p_1, \ldots, p_n\})$.

Solution: p. 107.

Problem 32. [Covering space; lifting of structure.]

A *covering space* of a topological space X is a topological space $\widetilde{X}$ together with a continuous map $\pi : \widetilde{X} \to X$ satisfying the following condition: Each point $x \in X$ has an open neighborhood U in X such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{X}$, each of which is mapped homeomorphically onto U by π.

(a). Let M be a topological manifold of dimension d and let $\pi : \widetilde{M} \to M$ be a covering space of M which is connected and second countable. Prove that then also $\widetilde{M}$ is a topological manifold of dimension d. (In fact the assumption that $\widetilde{M}$ is second countable is redundant; see the remark at the end of the solution.)

(b). Let M be a C^∞ manifold of dimension d and let $\pi : \widetilde{M} \to M$ be a covering space of M which is connected and second countable. Prove that then $\widetilde{M}$ has a unique structure as a C^∞ manifold such that π is C^∞ and each point $p \in M$ has an open neighborhood U in M such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{M}$, each of which is mapped *diffeomorphically* onto U by π.

(c). Let M be a *Riemannian* manifold of dimension d and let $\pi : \widetilde{M} \to M$ be a covering space of M which is connected and second countable. Prove that then $\widetilde{M}$ has a unique structure as a Riemannian manifold such that π is C^∞ and each point $p \in M$ has an open neighborhood U in M such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{M}$, each of which is mapped (C^∞) *isometrically* onto U by π.

(d). Prove that for any topological manifold M and any subgroup $\Gamma <$ Homeo(M) acting freely and properly discontinuously on M, if $\Gamma\backslash M$ and $\pi : M \to \Gamma\backslash M$ are as in Problem 9, then $\pi : M \to \Gamma\backslash M$ is a covering space of $\Gamma\backslash M$.

Solution: p. 108.

Problem 33. [Trivial vector bundle; basis of sections.]

Let (E, π, M) be a vector bundle of rank n and let U be an open subset of M. Prove that the following statements are equivalent:

(a) $E_{|U}$ is trivial;

(b) there is some φ such that (U, φ) is a bundle chart for E;

(c) there is a *basis of sections* in $\Gamma E_{|U}$, i.e. sections $s_1, \ldots, s_n \in \Gamma E_{|U}$ such that $s_1(p), \ldots, s_n(p)$ is a basis of E_p for every $p \in U$.

Solution: p. 113.

Problem 34. [Trivial vector bundle; one more (very!) basic fact.]

Let (E, π, M) be a vector bundle of rank n, let U be an open subset of M, and let $s_1, \ldots, s_n \in \Gamma E_{|U}$ be a basis of sections in $\Gamma E_{|U}$ (cf. Problem 33(c)). Prove that for every section $s \in \Gamma E_{|U}$ there exists a unique n-tuple of functions $\alpha^1, \ldots, \alpha^n \in C^\infty(U)$ such that $s = \alpha^j s_j$.

Solution: p. 115.

Problem 35. [About sections: restrictions and surjectivity to fibers.]

Let (E, π, M) be a vector bundle over a C^∞ manifold M.

(a) Prove that for every open set $U \subset M$, every section $s \in \Gamma(E_{|U})$, and every point $p \in U$, there exists a section $s' \in \Gamma(E)$ such that $s'_{|V} = s_{|V}$ for some open set $V \subset U$ containing p.

(b) Prove that for every point $p \in M$ there exist an open set $V \subset M$ with $p \in V$ and sections $b_1, \ldots, b_n \in \Gamma(E)$ such that $b_{1|V}, \ldots, b_{n|V}$ form a basis of sections of $E_{|V}$.

(c) Prove that for every $p \in M$ and every $v \in E_p$, there is some $s \in \Gamma E$ such that $s(p) = v$.

Solution: p. 115.

Problem 36. [Defining a vector bundle without requiring from start that it is a manifold.] Let M be a C^∞ manifold, let E be a set and let $\pi : E \to M$ be a surjective map. Assume that for every $p \in M$, $E_p := \pi^{-1}(p)$ carries the structure of an n-dimensional real vector space. Also let $\{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ be a family such that for each $\alpha \in A$, U_α is an open subset of M and φ_α is a bijection of $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{R}^n$ such that for every $p \in U_\alpha$, the map $(\varphi_\alpha)_p := (\varphi_\alpha)_{|E_p}$ is a linear isomorphism of E_p onto $\{p\} \times \mathbb{R}^n$. Assume that $M = \cup_{\alpha \in A} U_\alpha$, and that for any $\alpha, \beta \in A$, the map $\varphi_\beta \circ \varphi_\alpha^{-1}$ from $(U_\alpha \cap U_\beta) \times \mathbb{R}^n$ to itself is C^∞. Prove that then E has a unique C^∞ manifold structure such that (E, π, M) is a vector bundle of rank n, and $(U_\alpha, \varphi_\alpha)$ is a bundle chart for every $\alpha \in A$.

Solution: p. 116.

Problem 37. [Classifying all vector bundles over S^1.]

(a). Prove that the Möbius bundle over S^1 (cf. Lecture #7, p. 2) is not trivial.

(b). Classify all vector bundles over S^1 up to isomorphism.

Solution: p. 119.

Problem 38. [Finite cover of trivializing sets.]

Let M be a C^∞ manifold of dimension d and let E be a vector bundle over M. Prove that then there exists an open cover $U_1, \ldots, U_{d+1}$ of M such that $E_{|U_j}$ is trivial for each $j = 1, \ldots, d+1$.

[Remark: We will need to make use of this result a few times later. Then what will matter for us is the fact that $U_1, \ldots, U_{d+1}$ is a *finite* open cover; the exact number of open sets used will not be of importance.

[Hint: You may make use of the following theorem from dimension theory: Let M be a topological manifold of dimension d. Then every open cover $\mathcal{U}$ of M has a refinement $\mathcal{W}$ such that for any $d+2$ tuple of *distinct* open sets $W_1, \ldots, W_{d+2} \in \mathcal{W}$, one has $W_1 \cap \cdots \cap W_{d+2} = \emptyset$.
Cf. [9, Thm. V.8 and p. 25 (Ex. III.4)].]

Solution: p. 124.

Problem 39. [Definitions of $E_1 \otimes E_2$, Hom(E_1, E_2), E^*.]

Let (E_1, π_1, M) and (E_2, π_2, M) be vector bundles over a C^∞ manifold M.

(a) Verify that $E_1 \otimes E_2$, as defined in Lecture #7, is indeed a vector bundle over M.

(b) Similarly define the vector bundle $\text{Hom}(E_1, E_2)$.

(c) Similarly define the vector bundle E_1^*.

Hint for parts (a)-(c): See Problem 36!

Solution: p. 124.

Problem 40. [Γ(Hom(E_1, E_2)) = bundle homomorphisms $E_1 \to E_2$.]

Let (E_1, π_1, M) and (E_2, π_2, M) be vector bundles over a C^∞ manifold M. Prove that there is a natural bijection between $\Gamma(\text{Hom}(E_1, E_2))$ and the set of bundle homomorphisms $E_1 \to E_2$.

[Remarks: (1) From now on we will often *identify* these two sets, i.e. a bundle homomorphism $f : E_1 \to E_2$ is automatically viewed as an element in $\Gamma(\text{Hom}(E_1, E_2))$, and vice versa. (2) See also Problem 43 below for another important property of $\Gamma(\text{Hom}(E_1, E_2))$.]

Solution: p. 128.

Problem 41. [Definition of subbundle.] Let (E, π, M) be a vector bundle of rank n. Recall that in Lecture #7 we defined a *subbundle* of E to be a subset $E' \subset E$ such that for every $p \in M$ there exists a bundle chart (U, φ) for E such that $p \in U$ and

$$\varphi(E' \cap \pi^{-1}(U)) = U \times \mathbb{R}^m \tag{1}$$

for some $m \leq n$, where we view $\mathbb{R}^m \subset \mathbb{R}^n$ through

$$\mathbb{R}^m = \{(x^1, \ldots, x^n) \in \mathbb{R}^n \,:\, x^{m+1} = \cdots = x^n = 0\}.$$

In this situation, prove that

(a) m is independent of p and (U, φ);

(b) $(E', \pi_{|E'}, M)$ is a vector bundle of rank m, and for every bundle chart (U, φ) satisfying (1), $(U, \varphi_{|E' \cap \pi^{-1}(U)})$ is a bundle chart for E'.

[Hint: cf. Problem 36.]

(c) E' is a differentiable submanifold of E.

Solution: p. 130.

Problem 42. [Basic facts about the pulled back bundle f^*E.]

Let $f : M \to N$ be a C^∞ map and let (E, π, N) be a vector bundle.

(a). Prove that the pulled back bundle, f^*E, defined in Lecture #7 as a subset of $M \times E$ with extra structure, really is a vector bundle over M.

[Hint: cf. Problem 36.]

(b). Prove that f^*E is a differentiable submanifold of $M \times E$.

Solution: p. 133.

Problem 43. [Properties of the functor Γ.]

Let (E_1, π_1, M) and (E_2, π_2, M) be vector bundles over a C^∞ manifold M. Prove that there exist natural identifications (isomorphisms of $C^\infty(M)$-modules) as follows:

(a). $\Gamma(E_1 \oplus E_2) = \Gamma(E_1) \oplus \Gamma(E_2)$.

(b). $\Gamma(E_1^*) = (\Gamma E_1)^*$.

(c). $\Gamma(\mathrm{Hom}(E_1, E_2)) = \mathrm{Hom}(\Gamma E_1, \Gamma E_2)$.

(d). $\Gamma(E_1 \otimes E_2) = \Gamma(E_1) \otimes \Gamma(E_2)$.

[Remarks: As we stressed in the lecture, any space of sections ΓE is a *$C^\infty M$-module,* and when applying dual, "Hom" or "$\otimes$" to spaces of sections, it should always be viewed as *operations on $C^\infty M$-modules!* Thus $(\Gamma E_1)^*$ is the $C^\infty M$-module of $C^\infty M$-linear maps from ΓE_1 to $C^\infty M$, "$\mathrm{Hom}(\Gamma E_1, \Gamma E_2)$" is the $C^\infty M$-modules of $C^\infty M$-linear maps from ΓE_1 to ΓE_2, and "$\Gamma(E_1) \otimes \Gamma(E_2)$" is the $C^\infty M$-module which in a more precise notation would be denoted $\Gamma(E_1) \otimes_{C^\infty(M)} \Gamma(E_2)$.]

Solution: p. 135.

Problem 44. [Sections along a function; $\Gamma_f E$.]

Let $f : M \to N$ be a C^∞ map and let (E, π, N) be a vector bundle.

(a). A *section of* E *along* f (or "a lift of f to E") is a C^∞ map $\sigma : M \to E$ such that $\pi \circ \sigma = f$. The set of sections of E along f is denoted $\Gamma_f E$. Prove that $\Gamma_f E$ has a structure as a $C^\infty M$-module, and that there is a natural isomorphism of $C^\infty M$-modules $\Gamma f^* E \cong \Gamma_f E$.

[Remark: From now on we will often use the above isomorphism to *identify* $\Gamma f^* E$ and $\Gamma_f E$. As will be seen, to view a section $s \in \Gamma f^* E$ as an element in $\Gamma_f E$ simply means considering $\text{pr}_2 \circ s : M \to E$, i.e. "forgetting the first component of s, which anyway contains redundant information about the base point". On the other hand, one can *not* in any reasonable way define $f^* E$ directly as a subset of E, unless f is injective; indeed, for any two points $p \neq q$ in M with $f(p) = f(q)$ we want $(f^* E)_p$ and $(f^* E)_q$ to be two *disjoint* copies of $E_{f(p)}$.]

(b). Note that for any $s \in \Gamma E$ we have $s \circ f \in \Gamma_f E = \Gamma f^* E$; we call $s \circ f$ the (f-)*pullback* of s. Prove that if V is an open set in N and U is an open set in M with $f(U) \subset V$, and if $s_1, \ldots, s_n$ is a basis of sections in $\Gamma E_{|V}$, then $s_1 \circ f, \ldots, s_n \circ f$ is a basis of sections in $\Gamma(f^* E)_{|U}$.

(c). Prove any section of $f^* E$ can be expressed as a function-linear combination of f-pullbacks of sections of E. (In other words: Any $\sigma \in \Gamma f^* E$ can be expressed as a finite sum $\sigma = \sum_{j=1}^m \alpha_j \cdot (s_j \circ f)$ where $\alpha_1, \ldots, \alpha_m \in C^\infty(M)$ and $s_1, \ldots, s_m \in \Gamma E$.) [Hint: Problems 11 and 38 may be useful.]

Solution: p. 141.

Problem 45. [Interpreting $\Gamma(\mathbf{Hom}(E_1, f^* E_2))$.]

Let $f : M \to N$ be a C^∞ map and let (E_1, π_1, M) and (E_2, π_2, N) be vector bundles. We say that a map $h : E_1 \to E_2$ is a *bundle homomorphism along* f if h is C^∞, $\pi_2 \circ h = f \circ \pi_1$, and for each $x \in M$ the fiber map $h_x := h_{|E_{1,x}} : E_{1,x} \to E_{2,f(x)}$ is linear.

(a). Prove that there is a natural bijection between $\Gamma(\text{Hom}(E_1, f^* E_2))$ and the set of bundle homomorphisms $E_1 \to E_2$ along f.

(b). Explain how the result in (a) can be seen to generalize both Problem 40 and Problem 44(a).

Solution: p. 144.

Problem 46. [Extending a section from a curve to the whole space.]

Let (E, π, M) be a vector bundle, let $c : (a, b) \to M$ be a C^∞ curve, let $s \in \Gamma_c E$ (cf. Problem 44(a)), let $t_0 \in (a, b)$, and assume $\dot{c}(t_0) \neq 0$. Prove that there exist $\varepsilon > 0$ and a section $s_1 \in \Gamma E$ such that $a < t_0 - \varepsilon < t_0 + \varepsilon < b$ and $s_1(c(t)) = s(t)$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$.
(Hint: cf. Problems 12 and 35.)

Solution: p. 146.

Problem 47. [Lie product of vector fields.]

Let M be a C^∞ manifold.

(a). For any vector fields X, Y on M, prove that there exists a unique vector field Z on M satisfying $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$. By definition, this vector field Z is denoted "$[X, Y]$", and called the *Lie product* of X and Y.

[Hint: Use Problem 15(b).]

(b). Prove that our definition in part a is equivalent with Jost, [12, Def. 2.2.4].

(c). Prove the Jacobi identity:

$$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0, \qquad \forall X, Y, Z \in \Gamma(TM).$$

(d). Prove that for any $X, Y \in \Gamma(TM)$ and $f \in C^\infty(M)$,

$$[X, fY] = (Xf) \cdot Y + f \cdot [X, Y]$$

and

$$[fX, Y] = -(Yf) \cdot X + f \cdot [X, Y].$$

Solution: p. 147.

Problem 48. [Basic properties of the exterior derivative.]

Let M be a C^∞ manifold.

(a) Following Jost, [12, Def. 2.1.15], we define $d : \Omega^r(M) \to \Omega^{r+1}(M)$ by the requirement that for any $\omega \in \Omega^r(M)$ and any C^∞ chart (U, x) on M, if $\omega_{|U} = \sum_I \omega_I \, dx^I$ (with $\omega_I \in C^\infty(U)$) then $(d\omega)_{|U} = \sum_I d\omega_I \wedge dx^I$. [4] Prove that this indeed gives a well-defined, $\mathbb{R}$-linear map $d : \Omega^r(M) \to \Omega^{r+1}(M)$. (In other words, explain in detail what happens in [12, Cor. 2.1.2].)

(b) Prove that if $f : M \to N$ is a C^∞ map then $d(f^*(\omega)) = f^*(d\omega)$ for all $\omega \in \Omega^r(N)$. (In other words, provide more details for [12, Lemma 2.1.3].)

(c) Prove that for any $\omega \in \Omega^r(M)$ and $X_0, \ldots, X_r \in \Gamma(TM)$,

$$[d\omega](X_0, \ldots, X_r) = \sum_{j=0}^{r} (-1)^j X_j\big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_r)\big) + \sum_{0 \le j < k \le r} (-1)^{j+k} \omega([X_j, X_k], X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r).$$

[Explanation of notation: "$X_0, \ldots, \hat{X}_j, \ldots, X_r$" denotes "$X_0, X_1, X_2, \ldots, X_r$ but with the term X_j removed". Similarly "$[X_j, X_k], X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r$" denotes "$[X_j, X_k], X_0, X_1, X_2, \ldots, X_r$ but with both X_j and X_k removed". Also, the sum in the second line runs through all pairs $\langle j, k \rangle \in \mathbb{Z}^2$ satisfying $0 \le j < k \le r$.]

Solution: p. 148.

[4] Note that $d\omega_I = \frac{\partial \omega_I}{\partial x^j} \, dx^j$; hence our definition indeed agrees with [12, Def. 2.1.15].

Problem 49. [Wedge product of vector valued forms]

(a). Let E_1 and E_2 be vector bundles over a C^∞ manifold M. We define the *wedge product* $\wedge : \Omega^r(E_1) \times \Omega^s(E_2) \to \Omega^{r+s}(E_1 \otimes E_2)$, for any $r, s \geq 0$, to be the unique $C^\infty(M)$-bilinear map satisfying

$$(\mu_1 \otimes \omega_1) \wedge (\mu_2 \otimes \omega_2) = (\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2),$$
$$\forall \mu_1 \in \Gamma(E_1),\ \omega_1 \in \Omega^r(M),\ \mu_2 \in \Gamma(E_2),\ \omega_2 \in \Omega^s(M).$$

Prove that this indeed makes $\wedge$ a well-defined $C^\infty(M)$-bilinear map . Note also that in the special case $E_1 = E_2 = M \times \mathbb{R}$, this gives back the standard wedge product $\Omega^r(M) \times \Omega^s(M) \to \Omega^{r+s}(M)$.

(b). [Associativity and "commutativity".] Let E_1, E_2, E_3 be vector bundles over M and let $r, s, t \geq 0$. Prove that

$$s_1 \wedge (s_2 \wedge s_3) = (s_1 \wedge s_2) \wedge s_3, \qquad \forall s_1 \in \Omega^r(E_1),\ s_2 \in \Omega^s(E_2),\ s_3 \in \Omega^t(E_3).$$

(Here both expressions lie in $\Omega^{r+s+t}(\widetilde{E})$, where $\widetilde{E} = E_1 \otimes E_2 \otimes E_3 = E_1 \otimes (E_2 \otimes E_3) = (E_1 \otimes E_2) \otimes E_3$.) Prove also that

$$s_1 \wedge s_2 = (-1)^{rs} \cdot J(s_2 \wedge s_1), \qquad \forall s_1 \in \Omega^r(E_1),\ s_2 \in \Omega^s(E_2),$$

where J is the isomorphism of $C^\infty(M)$-modules

$$J : \Omega^{r+s}(E_2 \otimes E_1) \xrightarrow{\sim} \Omega^{r+s}(E_1 \otimes E_2)$$

which maps $J(\mu_2 \otimes \mu_1 \otimes \omega) = \mu_1 \otimes \mu_2 \otimes \omega$ for all $\mu_1 \in \Gamma E_1$, $\mu_2 \in \Gamma E_2$, $\omega \in \Omega^{r+s}(M)$.

(c). ["vector-wedge-product"; extending commutativity.] Let $E_1, E_2, \widetilde{E}$ be vector bundles over M and assume given a "multiplication rule" from E_1, E_2 to $\widetilde{E}$, i.e. a $C^\infty(M)$-linear map $m : \Gamma(E_1 \otimes E_2) \to \Gamma(\widetilde{E})$. By extending with the identity map on $\Omega^r(M)$, this defines for each $r \geq 0$ a $C^\infty(M)$-linear map $\Omega^r(E_1 \otimes E_2) \to \Omega^r(\widetilde{E})$, which we *also* call m. Let m' be the multiplication rule $m' : \Gamma(E_2 \otimes E_1) \to \Gamma(\widetilde{E})$ defined by $m'(s_2 \otimes s_1) = m(s_1 \otimes s_2)$ for all $s_1 \in \Gamma E_1$, $s_2 \in \Gamma E_2$, and call m' also the corresponding map $\Omega^r(E_2 \otimes E_1) \to \Omega^r(\widetilde{E})$. Prove that

$$m(s_1 \wedge s_2) = (-1)^{rs} m'(s_2 \wedge s_1), \qquad \forall s_1 \in \Omega^r(E_1),\ s_2 \in \Omega^s(E_2). \tag{2}$$

[Comments: In many cases we will write simply "$m(s_1, s_2)$" *or* "$s_1 \wedge s_2$" to denote the combined vector-wedge-product $m(s_1 \wedge s_2)$! For example this appears in [12, (4.1.26)]; "$A \wedge A$", wherein $E_1 = E_2 = \widetilde{E} = \operatorname{End} E$ and m is – of course – composition. Other examples appear in the computation of DF a bit further down on [12, p. 139]; e.g. "$[A, F]$"; here again $E_1 = E_2 = \widetilde{E} = \operatorname{End} E$ but m is Lie bracket. Another example, in a slightly generalized setting, is in [12, p. 154]; "$\widetilde{P}(F, \ldots, F)$". A main example where the relation (2) applies is when $E := E_1 = E_2 = \widetilde{E}$ is a *commutative* (weak) algebra bundle over M (with m being the multiplication rule). In this case $m' = m$, and so (2) shows how the commutativity of E extends to $\Omega(E)$. On the other hand, a natural example with $E_1 \neq E_2$ is when $E_1 = E$ (an arbitrary vector bundle over M), $E_2 = E^*$ and $\widetilde{E} = M \times \mathbb{R}$, with the multiplication rule m (as well as m') being the standard contraction from $\Gamma(E \otimes E^*)$ (or $\Gamma(E^* \otimes E)$) to $C^\infty(M)$.]

(d). [extension of associativity.] Let $E_1, E_2, E_3, E_{12}, E_{23}, E_{123}$ be vector bundles over M and assume given multiplication rules

$$\Gamma(E_1 \otimes E_2) \to \Gamma(E_{12}); \qquad \Gamma(E_{12} \otimes E_3) \to \Gamma(E_{123});$$
$$\Gamma(E_2 \otimes E_3) \to \Gamma(E_{23}); \qquad \Gamma(E_1 \otimes E_{23}) \to \Gamma(E_{123}).$$

For each of these, we denote the image of $s \otimes s'$ simply by "$s \cdot s'$". Assume that these multiplication rules satisfy the associativity relation

$$(s_1 \cdot s_2) \cdot s_3 = s_1 \cdot (s_2 \cdot s_3), \qquad \forall s_1 \in \Gamma E_1, s_2 \in \Gamma E_2, s_3 \in \Gamma E_3.$$

In line with the above comments, let us write $s_1 \wedge s_2 \in \Omega^{r+s}(E_{12})$ for the combined vector-wedge-product of any $s_1 \in \Omega^r(E_1)$ and $s_2 \in \Omega^s(E_2)$; and similarly for the other three product rules. Then prove that

$$(s_1 \wedge s_2) \wedge s_3 = s_1 \wedge (s_2 \wedge s_3), \qquad \forall s_1 \in \Omega^r(E_1),\ s_2 \in \Omega^s(E_2),\ s_3 \in \Omega^t(E_3).$$

[Comments: A main example of the above situation is of course when $E := E_1 = E_2 = E_3 = E_{12} = E_{23} = E_{123}$ is an *associative* (weak) algebra bundle over M. A general example where E_1, E_2, E_3 may be distinct vector bundles is when $E_j := \mathrm{Hom}(F_{j+1}, F_j)$ for $j = 1, 2, 3$, where F_1, F_2, F_3, F_4 are four arbitrary vector bundles over M, and all multiplication rules are *composition* (thus $E_{12} = \mathrm{Hom}(F_3, F_1)$, etc., and the associativity relation holds).]

Solution: p. 155.

Problem 50. [Wedge-product of matrix valued forms made explicit.]

Let E_1, E_2, E_3 be vector bundles over M; then we have a standard multiplication rule "$\circ$" (composition of homomorphisms)

$$\Gamma(\mathrm{Hom}(E_2, E_3)) \times \Gamma(\mathrm{Hom}(E_1, E_2)) \to \Gamma(\mathrm{Hom}(E_1, E_3)).$$

Let us write "$\circ$" also for the corresponding vector-wedge-product

$$\Omega^r(\mathrm{Hom}(E_2, E_3)) \times \Omega^s(\mathrm{Hom}(E_1, E_2)) \to \Omega^{r+s}(\mathrm{Hom}(E_1, E_3))$$

(cf. Problem 49(c)). Let U be an open subset of M such that there exist bases of sections

$$\alpha_1, \ldots, \alpha_{n_1} \in \Gamma E_{1|U} \quad \text{and} \quad \beta_1, \ldots, \beta_{n_2} \in \Gamma E_{2|U} \quad \text{and} \quad \gamma_1, \ldots, \gamma_{n_3} \in \Gamma E_{3|U}$$

(here $n_\ell = \mathrm{rank}\, E_\ell$). Let

$$\alpha^{1*}, \ldots, \alpha^{n_1*} \in \Gamma E^*_{1|U} \quad \text{and} \quad \beta^{1*}, \ldots, \beta^{n_2*} \in \Gamma E^*_{2|U} \quad \text{and} \quad \gamma^{1*}, \ldots, \gamma^{n_3*} \in \Gamma E^*_{3|U}$$

be the dual bases.

Then for each $\mu \in \Omega^r(\mathrm{Hom}(E_2, E_3))$ there exist unique r-forms $\mu^k_j \in \Omega^r(U)$ such that $\mu_{|U} = \beta^{j*} \otimes \gamma_k \otimes \mu^k_j$, and similarly for each $\eta \in \Omega^s(\mathrm{Hom}(E_1, E_2))$ there exist unique s-forms $\eta^k_j \in \Omega^s(U)$ such that $\eta_{|U} = \alpha^{j*} \otimes \beta_k \otimes \eta^k_j$. Prove that in terms of this representation,

$$(\mu \circ \eta)_{|U} = \alpha^{i*} \otimes \gamma_k \otimes (\mu^k_\ell \wedge \eta^\ell_i).$$

[Comment: Note that "$\mu_{|U} = \beta^{j*} \otimes \gamma_k \otimes \mu^k_j$" means that if we use the given bases to identify $E_{2|U}$ with $U \times \mathbb{R}^{n_2}$ and $E_{3|U}$ with $U \times \mathbb{R}^{n_3}$, then $\mu_{|U}$ is represented by the *matrix*

$$(\mu^k_j) = \begin{pmatrix} \mu^1_1 & \cdots & \mu^1_{n_2} \\ \vdots & & \vdots \\ \mu^{n_3}_1 & \cdots & \mu^{n_3}_{n_2} \end{pmatrix}$$

(wherein each entry is an r-form). Similarly "$\eta_{|U} = \alpha^{j*} \otimes \beta_k \otimes \eta^k_j$" means that $\eta_{|U}$ is represented by the matrix (η^k_j) and "$(\mu \circ \eta)_{|U} = \alpha^{i*} \otimes \gamma_k \otimes (\mu^k_\ell \wedge \eta^\ell_i)$" means that $(\mu \circ \eta)_{|U}$ is represented by the matrix $(\mu^k_\ell \wedge \eta^\ell_i)_{k,i}$. Hence when $r = s = 0$, the formula gives back the usual formula for matrix product; $(\mu \circ \eta)_{|U} = (\mu^k_\ell \cdot \eta^\ell_i)_{k,i}$, as it should.]

Solution: p. 158.

Problem 51. [Wedge product; alternative definition]

(a). Let (E, π, M) be a vector bundle. Prove that there is a natural identification of $\Omega^r(E)$ with the space of alternating $C^\infty(M)$-multilinear maps

$$\Gamma(TM)^{(r)} = \underbrace{\Gamma(TM) \times \cdots \times \Gamma(TM)}_{r \text{ times}} \longrightarrow \Gamma E.$$

(b). Let E_1 and E_2 be vector bundles over M. Prove that using the identification in part (a), the wedge product $s_1 \wedge s_2$ (cf. Problem 49(a)) of any $s_1 \in \Omega^r(E_1)$ and $s_2 \in \Omega^s(E_2)$ is given by

$$(s_1 \wedge s_2)(X_1, \ldots, X_{r+s}) \\ = \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma)\, s_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \otimes s_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}), \\ \forall X_1, \ldots, X_{r+s} \in \Gamma(TM),$$

where $\mathfrak{S}_{r+s}$ is the group of all permutations of $\{1, \ldots, r+s\}$. Prove also a similar formula for the product "$s_1 \cdot s_2 \in \Omega^{r+s}(\widetilde{E})$", in the case when there is given a multiplication rule from E_1, E_2 to $\widetilde{E}$ (cf. Problem 49(c)).

Solution: p. 159.

Problem 52. [Restricting a connection to open sets.]

Complete the proof of Lemma 1 in Lecture #9; that is, prove the following:

Let (E, π, M) be a vector bundle. For (a) and (b), let D be a connection on E and let $U \subset M$ be open.

(a) $\forall s_1, s_2 \in \Gamma E \;:\; s_{1|U} = s_{2|U} \Rightarrow (Ds_1)_{|U} = (Ds_2)_{|U}$.

(b) There is a unique connection "$D_{|U}$" on $E_{|U}$ satisfying $(Ds)_{|U} = D_{|U}(s_{|U})$ for all $s \in \Gamma(E)$.

(c) Let $(U_\alpha)_{\alpha\in A}$ be an open covering of M, and for each $\alpha \in A$ let D_α be a connection on $E_{|U_\alpha}$. Assume that for any two $\alpha, \beta \in A$, if $V := U_\alpha \cap U_\beta \neq \emptyset$ then $(D_\alpha)_{|V} = (D_\beta)_{|V}$. Then there exists a unique connection D on E satisfying $D_{|U_\alpha} = D_\alpha$ for every $\alpha \in A$.

Solution: p. 162.

Problem 53. .
[$D_v s$ depends only on the values of s along a curve with $\dot c(0) = v$.]

Let (E, π, M) be a vector bundle and let D be a connection on E. Let $v \in TM$ and let $s_1, s_2 \in \Gamma E$. Assume that there exists a C^∞ curve $c : (-\varepsilon, \varepsilon) \to M$ such that $\dot c(0) = v$ and $s_1(c(t)) = s_2(c(t))$ for all $t \in (-\varepsilon, \varepsilon)$. Prove that then $D_v s_1 = D_v s_2$.

Solution: p. 164.

Problem 54. [The connection "d" (for given local coordinates).]

Let (U, φ) be a bundle chart of a vector bundle (E, π, M), let $s_1, \ldots, s_n \in \Gamma(E_{|U})$ be the corresponding basis of sections and define the map $d : \Gamma(E_{|U}) \to \Gamma((E \otimes T^*M)_{|U})$ by $d(a^k s_k) = s_k \otimes da^k$ for any $a^1, \ldots, a^n \in C^\infty U$. Prove that this is a connection on $E_{|U}$. (Cf. p. 6 in Lecture #9.)

Solution: p. 165.

Problem 55. [Restriction of a connection to a subbundle.]

Let D be a connection on a vector bundle (E, π, M) and let E' be a vector subbundle of E. Then also $E' \otimes T^*M$ is a vector subbundle of $E \otimes T^*M$. Assume that $Ds \in \Gamma(E' \otimes T^*M)$ for all $s \in \Gamma E'$. Prove that then the restriction of D to $\Gamma E'$ is a connection on E'. Also give an example to show that the given condition is not always satisfied.

Solution: p. 165.

Problem 56. [Alternative definition of the dual of a connection.]

Let D be a connection on a vector bundle (E, π, M), and let D^* be the dual connection on E^*. (Cf. Lecture #10.) Recall that given any C^∞-curve $\gamma : (-\varepsilon, \varepsilon) \to M$ we have a linear isomorphism

$$\mathbb{P}_{\gamma(0) \underset{\gamma}{\to} \gamma(h)} : E_{\gamma(0)} \to E_{\gamma(h)}$$

for each $h \in (-\varepsilon, \varepsilon)$; let us write $\mathbb{P}^*_{\gamma,h}$ for the *dual* of that map; this is a linear isomorphism $E^*_{\gamma(h)} \to E^*_{\gamma(0)}$. Prove that for any $\mu \in \Gamma E^*$,

$$D^*_{\dot\gamma(0)}(\mu) = \lim_{h \to 0} \frac{\mathbb{P}^*_{\gamma,h}(\mu(\gamma(h))) - \mu(\gamma(0))}{h} \quad \text{in } E_{\gamma(0)}.$$

Problem 57. [Defining the pullback of a connection.]

(a). Let $f : M \to N$ be a C^∞ map and let D be a connection on a vector bundle (E, π, N). Prove that there exists a unique connection f^*D on f^*E such that for any $s \in \Gamma E$,

$$(f^*D)(s \circ f) = D_{df(\cdot)}(s) \in \Gamma(\mathrm{Hom}(TM, f^*E)) = \Gamma(f^*E \otimes T^*M).$$

[Explanation: "$D_{df(\cdot)}(s)$" stands for the map

$$TM \to E, \qquad [v \mapsto D_{df(v)}(s)],$$

which is a bundle homomorphism along f, and hence can be viewed as an element of $\Gamma(\mathrm{Hom}(TM, f^*E))$ by Problem 45.]

(b). (Comparing with Jost's definition of f^*D, [12, p. 205].) Prove that f^*D in part (a) is the unique connection on f^*E such that the following holds: For any $s \in \Gamma f^*E = \Gamma_f E$ and any C^∞ curve $c : (-\varepsilon, \varepsilon) \to M$, if $s_1 \in \Gamma E$ satisfies $s_1(f(c(t))) = s(c(t))$ for all $t \in (-\varepsilon, \varepsilon)$, then $(f^*D)_{\dot c(0)}(s) = D_{df(\dot c(0))}(s_1)$.

(c). Let $c : (-\varepsilon, \varepsilon) \to M$ be any C^∞ curve such that $f \circ c$ is a *constant* point $q \in N$. Prove that for any $s \in \Gamma f^*E$,

$$(f^*D)_{\dot c(0)}(s) = \Big(\frac{d}{dt}(s \circ c)(t)\Big)_{|t=0} \in E_q,$$

where $\frac{d}{dt}(s \circ c)(t) \in T_{s(c(t))}(E_q) = E_q$ stands for the tangent vector of the curve $s \circ c$ in E_q.

(Comments: In the situation in (c), if s is not constant along c, the formula in (b) cannot be used *directly* to compute $(f^*D)_{\dot c(0)}(s)$, since there cannot exist any $s_1 \in \Gamma E$ satisfying $s_1(f(c(t))) = s(c(t))$, $\forall t \in (-\varepsilon, \varepsilon)$. Note also that the tangent vector of the curve $s \circ c$, $\frac{d}{dt}(s \circ c)(t)$, is *always* a well-defined vector in $T_{s(c(t))}(E)$; however in the situation in (c) we can view $s \circ c$ as a curve in the fiber E_q; hence $\frac{d}{dt}(s \circ c)(t) \in T_{s(c(t))}(E_q)$, and this last tangent space can naturally be identified with E_q by Problem 13(c).)

Solution:
p. 165.

Problem 58. [**The tensor product of two connections.**]

Prove Proposition 2 in Lecture #10, i.e. the following: Let E_1, E_2 be vector bundles over M with connections D_1, D_2, respectively. Then there is a unique connection D on $E_1 \otimes E_2$ such that

$$D(\mu \otimes \nu) = (D_1\mu) \otimes \nu + \mu \otimes (D_2\nu), \qquad \forall \mu \in \Gamma E_1,\ \nu \in \Gamma E_2.$$

Solution: p. 171.

Problem 59. [**A Leibniz rule for general connections.**]

Let E_1, E_2, E_3 be vector bundles over M, each equipped with a connection "D". Let us write "D" also for the corresponding connections on E_1^* and $E_1 \otimes E_2$ and $\text{Hom}(E_1, E_2) = E_1^* \otimes E_2$, etc.

(a). Given any

$$\alpha \in \Gamma(E_1 \otimes E_2) \quad \text{and} \quad \beta \in \Gamma(E_1^* \otimes E_3),$$

let us write "(α, β)" for the section in $\Gamma(E_2 \otimes E_3)$ obtained by contracting the E_1-part of α against the E_1^*-part of β. Prove that then

$$D(\alpha, \beta) = (D\alpha, \beta) + (\alpha, D\beta) \qquad \text{in } \Omega^1(E_2 \otimes E_3).$$

(Here "$(D\alpha, \beta)$" is again defined by contracting the E_1-part of $D\alpha$ against the E_1^*-part of β, and similarly for "$(\alpha, D\beta)$"; note that these can be viewed as vector-wedge-products à la Problem 49(c), from $\Omega^r(E_1 \otimes E_2) \times \Omega^s(E_1^* \otimes E_3)$ to $\Omega^{r+s}(E_2 \otimes E_3)$, coming from the given product $(\cdot, \cdot)$ from $\Gamma(E_1 \otimes E_2) \times \Gamma(E_1^* \otimes E_3)$ to $\Gamma(E_2 \otimes E_3)$.)

(b). Prove that for any $\alpha \in \Gamma(\text{Hom}(E_2, E_3))$ and $\beta \in \Gamma(\text{Hom}(E_1, E_2))$,

$$D(\alpha \circ \beta) = (D\alpha) \circ \beta + \alpha \circ (D\beta) \qquad \text{in } \Omega^1(\text{Hom}(E_1, E_3)).$$

(c). Prove that for any $\alpha \in \Gamma(\text{Hom}(E_1, E_2))$ and $\beta \in \Gamma E_1$,

$$D(\alpha(\beta)) = (D\alpha)(\beta) + \alpha(D\beta) \qquad \text{in } \Omega^1(E_2).$$

Solution: p. 174.

Problem 60. [The exterior covariant derivative.]

Let D be a connection on a vector bundle (E, π, M).

(a). Prove Proposition 4 in Lecture #10, i.e. the following: Then for any $p \geq 0$ there exists a unique $\mathbb{R}$-linear map $D : \Omega^p(E) \to \Omega^{p+1}(E)$ satisfying

$$D(\mu \otimes \omega) = (D\mu) \wedge \omega + \mu \otimes d\omega, \qquad \forall \mu \in \Gamma E, \ \omega \in \Omega^p(M).$$

(b). Let (U, φ) be a fixed bundle chart for E; let d be the corresponding naive connection on $E_{|U}$ and set $A = D - d \in \Omega^1(\operatorname{End} E_{|U})$, as usual. Prove that for any $\mu \in \Omega^p(E_{|U})$,

$$D\mu = d\mu + A \wedge \mu \qquad \text{in } \Omega^{p+1}(E_{|U}),$$

where d is the naive exterior covariant derivative $\Omega^p(E_{|U}) \to \Omega^{p+1}(E_{|U})$ coming from the given bundle chart, and $A \wedge \mu$ is the image of A and μ under the combined vector-wedge-product (cf. Problem 49(c)) $\Omega^1(\operatorname{End} E_{|U}) \times \Omega^p(E_{|U}) \to \Omega^{p+1}(E_{|U})$ coming from the standard contraction ("evaluation") $\Gamma(\operatorname{End} E_{|U}) \times \Gamma E_{|U} \to \Gamma E_{|U}$.

(c). Let $E_1, E_2, \widetilde{E}$ be vector bundles over M, each equipped with a connection "D". Assume given a multiplication rule from E_1, E_2 to $\widetilde{E}$, i.e. a $C^\infty(M)$-linear map $\Gamma(E_1 \otimes E_2) \to \Gamma(\widetilde{E})$. We write $s_1 \cdot s_2 \in \Gamma\widetilde{E}$ for the product of $s_1 \in \Gamma E_1$, $s_2 \in \Gamma E_2$, and we write "$\wedge$" for the corresponding vector-wedge-product as in Problem 49(c). Assume that the connections *respect* the multiplication rule, in the sense that

$$D(s_1 \cdot s_2) = (Ds_1) \wedge s_2 + s_1 \wedge (Ds_2), \qquad \forall s_1 \in \Gamma E_1, \ s_2 \in \Gamma E_2.$$

Prove that then for any $r, s \geq 0$,

$$D(\mu_1 \wedge \mu_2) = (D\mu_1) \wedge \mu_2 + (-1)^r \mu_1 \wedge D\mu_2, \qquad \forall \mu_1 \in \Omega^r(E_1), \ \mu_2 \in \Omega^s(E_2),$$

where "$\wedge$" is the vector-wedge-product as in Problem 49(c).

(d). Addendum to (c): Let m be the multiplication rule in (c), i.e. a $C^\infty(M)$-linear map $\Gamma(E_1 \otimes E_2) \to \Gamma(\widetilde{E})$. By Problem 43(c), the multiplication rule can be identified with a section $m \in \Gamma(\operatorname{Hom}(E_1 \otimes E_2, \widetilde{E}))$. Prove that the given connections on $E_1, E_2, \widetilde{E}$ respect the multiplication rule iff

$$Dm = 0.$$

(Here D is the connection on $\operatorname{Hom}(E_1 \otimes E_2, \widetilde{E})$ induced by the given connections on $E_1, E_2, \widetilde{E}$.)

Solution: p. 175.

Problem 61. [Explicit formula for exterior covariant derivative (using Lie product of vector fields).]

Let D be a connection on a vector bundle (E, π, M); let $r \geq 0$, and write "D" also for the corresponding exterior covariant derivative $\Omega^r(E) \to \Omega^{r+1}(E)$. Prove that for any $s \in \Omega^r(E)$ and $X_0, \ldots, X_r \in \Gamma(TM)$,

$$[Ds](X_0, \ldots, X_r) = \sum_{j=0}^{r} (-1)^j D_{X_j}\big(s(X_0, \ldots, \hat{X}_j, \ldots, X_r)\big) + \sum_{0 \leq j < k \leq r} (-1)^{j+k} s([X_j, X_k], X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r).$$

[Here "$[Ds](X_0, \ldots, X_r)$" stands for the contraction of the form part of $Ds \in \Omega^{r+1}(E)$ against $X_0, \ldots, X_r$; and similarly for all "$s(\cdots)$" in the right hand side. For the rest of the notation, cf. Problem 48(c).]

Solution: p. 180.

Problem 62. [One more explicit formula for exterior covariant derivative.]

Let D be a connection on a vector bundle (E, π, M); let $r \geq 0$, and write "D" also for the corresponding exterior covariant derivative $\Omega^r(E) \to \Omega^{r+1}(E)$. Recall from the solution of Problem 51 that $E \otimes \bigwedge^r M$ is in a natural way a subbundle of $E \otimes T_r^0(M)$; accordingly for any section $s \in \Omega^r(E)$ let us write "$\widetilde{s}$" for s viewed as a section in $\Gamma(E \otimes T_r^0(M))$. Furthermore let ∇ be an arbitrary torsion free connection on TM, and let us write "$[{}^{\nabla}_{D}]$" for the connection on $E \otimes T_r^0(M)$ induced by D and ∇. Then prove that for any $s \in \Omega^r(E)$ and $X_0, \ldots, X_r \in \Gamma(TM)$,

$$[Ds](X_0, \ldots, X_r) = \sum_{j=0}^{r} (-1)^j \Big([{}^{\nabla}_{D}]_{X_j} \widetilde{s} \Big)(X_0, \ldots, \hat{X}_j, \ldots, X_r).$$

Solution: p. 181.

Problem 63. [Basic facts about AdE (for E with a bundle metric).]

Let (E, π, M) be a vector bundle equipped with a bundle metric. Recall that $\mathrm{Ad}E$ (as a subset of $\mathrm{End}\, E$) was defined in Lecture #11, p. 11.

(a). Prove that $\mathrm{Ad}E$ is a vector subbundle of $\mathrm{End}\, E$.

(b). Prove that if D is any metric connection on E, and if we write D also for the corresponding connection on $\mathrm{End}\, E$, then $Ds \in \Omega^1(\mathrm{Ad}E)$ for all $s \in \Gamma(\mathrm{Ad}E) \subset \Gamma(\mathrm{End}\, E)$. (Hence by Problem 55, the connection D on $\mathrm{End}\, E$ descends to give a connection on $\mathrm{Ad}E$.)

Solution: p. 183.

Problem 64. [Some facts about $\bigwedge^r(V)$ for V a vector space]

(See Sec. 7.2 for the definition and some basic properties of $\bigwedge^r(V)$.)

Let V be a finite dimensional vector space over $\mathbb{R}$ and let $r \geq 1$.

(a). For any $v_1, \ldots, v_r, w_1, \ldots, w_r \in V$, the following statement about vectors in $\bigwedge^r(V)$:

$$\Big[v_1 \wedge \cdots \wedge v_r = c \cdot w_1 \wedge \cdots \wedge w_r \text{ for some } c \in \mathbb{R}, \text{ and } \ v_1 \wedge \cdots \wedge v_r \neq 0\Big]$$

holds if and only if $v_1, \ldots, v_r$ are linearly independent and $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ span the same r-dimensional linear subspace of V.

(b). Prove that if V is equipped with a scalar product $\langle \cdot, \cdot \rangle$ then there is a corresponding scalar product $\langle \cdot, \cdot \rangle$ on $\bigwedge^r(V)$ which has the following two properties:

(i) If $e_1, \ldots, e_n$ is any ON-basis for V then (e_I) is an ON-basis for $\bigwedge^r(V)$, where I runs through all r-tuples $I = (i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$ with $i_1 < \cdots < i_r$, and $e_I := e_{i_1} \wedge \cdots \wedge e_{i_r}$.

(ii) For any $v_1, \ldots, v_r, w_1, \ldots, w_r \in V$,

$$\langle v_1 \wedge \cdots \wedge v_r, w_1 \wedge \cdots \wedge w_r \rangle = \det(\langle v_i, \beta_j \rangle)_{i,j}.$$

Prove also that this scalar product on $\bigwedge^r(V)$ is uniquely determined by the requirement that *either* (i) or (ii) hold.

(c). With notation as in (b), for any $v_1, \ldots, v_r \in V$, the "length"

$$\|v_1 \wedge \cdots \wedge v_r\| := \sqrt{\langle v_1 \wedge \cdots \wedge v_r, v_1 \wedge \cdots \wedge v_r \rangle}$$

equals the volume of the r-dimensional parallelotope spanned by $v_1, \ldots, v_r$ (wrt. the natural r-dimensional volume measure induced by the the scalar product $\langle \cdot, \cdot \rangle$ on V).

Solution:
p. 186.

Problem 65. [Equivalent criteria for a manifold being orientable.]

(a). Let M be a C^∞ manifold of dimension d. Prove that the following three statements are equivalent:

(i) M possesses an oriented C^∞ atlas, i.e. an atlas such that all chart transition maps have everywhere positive Jacobian determinant.

(ii) There exists an atlas of bundle charts for the vector bundle (TM, π, M) which makes it an *oriented* vector bundle ($\Leftrightarrow$ makes it have structure group $\mathrm{GL}_d^+(\mathbb{R})$; cf. Lecture #12, Def. 4).

(iii) There exists a nowhere vanishing d-form $\omega \in \Omega^d(M)$.

[Comment: M is said to be *orientable* if one and hence all of the conditions (i)–(iii) hold. Note that (i) is the definition given in Jost, [12, Def. 1.1.3].]

(b). Prove that TM is *always* an orientable manifold, regardless of whether M is orientable or not.

Solution: p. 188.

Problem 66. [Total covariant derivative of a tensor field.]

(a). Let M be a C^∞ manifold of dimension d and let ∇ be a connection on TM. Write also ∇ for the corresponding connection on $T_s^r M$, for any $r, s \geq 0$. Let A be a tensor field in $\Gamma(T_1^1 M)$, and let A_i^j be the coefficients of A wrt a given C^∞ chart (U, x) on M. (Thus: $A_i^j \in C^\infty(U)$ for all $i, j \in \{1, \ldots, d\}$ and $A_{|U} = A_i^j \cdot dx^i \otimes \frac{\partial}{\partial x^j}$.) Also for each $k \in \{1, \ldots, d\}$ let $A_{i;k}^j$ be the coefficients of $\nabla_{\frac{\partial}{\partial x^k}} A$. Prove that

$$A_{i;k}^j = \frac{\partial}{\partial x^k} A_i^j - \Gamma_{ki}^\ell \cdot A_\ell^j + \Gamma_{k\ell}^j \cdot A_i^\ell \qquad \text{in } U.$$

(b). Generalize the above to the case of a tensor field $A \in \Gamma(T_s^r M)$, for any $r, s \geq 0$.

Solution: p. 191.

Problem 67. [Some explicit computations in vector bundles over S^d.] Consider the sphere

$$S^d = \{x = (x^1, \ldots, x^{d+1}) \in \mathbb{R}^{d+1} \ : \ (x^1)^2 + \cdots + (x^{d+1})^2 = 1\}$$

with its standard C^∞ manifold structure (cf. [12, p. 3, Ex. 1]), and let (U, y) be the chart on S^d given by

$$U = S^d \setminus \{(0, \ldots, 0, -1)\}; \qquad y(x) = \Big(\frac{x^1}{1 + x^{d+1}}, \ldots, \frac{x^d}{1 + x^{d+1}}\Big).$$

(a). For $d = 2$, prove that the vector field $y^1 \dfrac{\partial}{\partial y^1}$ on U can not be extended to a (C^∞) vector field on S^2.

(b). For $d = 3$, prove that the vector field

$$(y^1 y^3 - y^2)\frac{\partial}{\partial y^1} + (y^2 y^3 + y^1)\frac{\partial}{\partial y^2} + \frac{1 - (y^1)^2 - (y^2)^2 + (y^3)^2}{2}\,\frac{\partial}{\partial y^3}$$

on U *can* be extended to a (C^∞) vector field on S^3.

(c). Prove that for $d = 2$, the section

$$\frac{1}{(1 + (y^1)^2 + (y^2)^2)^4}\big(dy^1 \otimes dy^1 + dy^2 \otimes dy^2\big)$$

of $T_2^0(U)$ has a unique extension to a (C^∞) section of $T_2^0(S^2)$. Prove also that the above section defines a Riemannian metric on U, but its extension to $T_2^0(S^2)$ does *not* define a Riemannian metric on S^2.

(d). Let $d = 2$ and set $V = S^2 \setminus \{(0, 0, 1)\}$ (recall $U = S^2 \setminus \{(0, 0, -1)\}$). Fix an integer m, and define the function $\mu : U \cap V \to \mathrm{GL}_2(\mathbb{R})$ by

$$\mu(y) = \begin{pmatrix} \cos(m\,\alpha(y)) & -\sin(m\,\alpha(y)) \\ \sin(m\,\alpha(y)) & \cos(m\,\alpha(y)) \end{pmatrix}, \qquad \text{where } \ \alpha(y) := \arg(y^1 + iy^2).$$

(Thus $\alpha(y)$ is the argument of the complex number $y^1 + iy^2$; note that this number is non-zero for all points in $U \cap V$.) Prove that there exists a vector bundle E of rank 2 over S^2 which has bundle charts (U, ϕ) and (V, ψ) with transition function μ, that is, so that $\psi_p = \mu(p) \cdot \phi_p : E_p \to \mathbb{R}^2$ for every $p \in U \cap V$. (Hint: Problem 36 may be useful.)

Solution: p. 192.

Problem 68. [More about the pullback of a connection.]

Let $f : M \to N$ be a C^∞ map; let (E, π, N) be a vector bundle of rank n, and let D be a connection on E.

(a). Let (U, x) be a chart for N and let $s_1, \ldots, s_n$ be a basis of sections in $\Gamma(E_{|U})$. Also let (V, y) be a chart for M with $V \subset f^{-1}(U)$, and recall that then $s_1 \circ f, \ldots, s_n \circ f$ is a basis of sections in $\Gamma((f^*E)_{|V})$; cf. Problem 44(b). Set $d = \dim N$ and $d' = \dim M$. Let $\Gamma_{ij}^k \in C^\infty(U)$ be the Christoffel symbols of D with respect to the bases $s_1, \ldots, s_n$ and $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d}$, and let $\widetilde{\Gamma}_{ij}^k \in C^\infty(V)$ be the Christoffel symbols of f^*D with respect to the bases $s_1 \circ f, \ldots, s_n \circ f$ and $\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{d'}}$. Give a formula for $\widetilde{\Gamma}_{ij}^k$ in terms of Γ_{ij}^k !

(b). Let D be a connection on E. For clarity in this problem let us write $d^D : \Omega^r(E) \to \Omega^{r+1}(E)$ (instead of just "D") for the exterior covariant derivative corresponding to D; then also write $d^{f^*D} : \Omega^r(f^*E) \to \Omega^{r+1}(f^*E)$ for the exterior covariant derivative corresponding to the connection f^*D on f^*E (cf. Problem 57). Prove that for every $r \geq 0$ there is a unique $\mathbb{R}$-linear map $f^* : \Omega^r(E) \to \Omega^r(f^*E)$ satisfying

$$f^*(\mu \otimes \omega) = (\mu \circ f) \otimes f^*(\omega) \qquad \text{for all } \mu \in \Gamma E \text{ and } \omega \in \Omega^r(N).$$

Next prove that for any $s \in \Omega^r(E)$,

$$(d^{f^*D})(f^*(s)) = f^*(d^D s).$$

[Comment: In particular for $r = 0$ we have $f^*(\mu) = \mu \circ f$ for all $\mu \in \Gamma E$, and $d^D = D : \Omega^0(E) \to \Omega^1(E)$ and $d^{f^*D} = f^*D : \Omega^0(f^*E) \to \Omega^1(f^*E)$. In this case the above formula says:

$$(f^*D)(f^*(s)) = f^*(D(s)),$$

which can be viewed as a (nicer!) reformulation of the formula in Problem 57(a)!]

Solution: p. 199.

Problem 69. [Basic about sectional curvature.]

In Lecture #15, Def. 1, prove that $K(X \wedge Y)$ indeed only depends on the 2-dimensional plane spanned by X, Y in T_pM.

Problem 70. [Scaling a Riemannian metric.]

Let M be a C^∞ manifold equipped with a Riemannian metric $\langle \cdot, \cdot \rangle$, and let $c > 0$ be a constant. Let $[\cdot, \cdot]$ be the Riemannian metric on M defined by $[\cdot, \cdot] := c \langle \cdot, \cdot \rangle$ (that is, $[v, s] = c \langle v, w \rangle$ for any $p \in M$, $v, w \in T_pM$). Prove that the two Riemannian manifolds $(M, \langle \cdot, \cdot \rangle)$ and $(M, [\cdot, \cdot])$ have the same Levi-Civita connection and curvature tensor, but that the sectional curvatures K on $(M, \langle \cdot, \cdot \rangle)$ and $\widetilde{K}$ on $(M, [\cdot, \cdot])$ are related by

$$\widetilde{K}(X \wedge Y) = c^{-1} K(X \wedge Y)$$

for any $p \in M$ and any linearly independent $X, Y \in T_pM$.

Solution: p. 203.

Problem 71. [Ricci curvature as average of sectional curvatures.]

(a). Let M be a Riemannian manifold of dimension d. Prove that there is a constant $C_d > 0$ which only depends on d such that for any $p \in M$ and $X \in T_pM$ with $\|X\| = 1$, the Ricci curvature in direction X, $\mathrm{Ric}(X, X)$, equals C_d times the uniform average of the sectional curvatures of all planes in T_pM containing X. Also determine the constant C_d explicitly.

(b). Similarly, prove that there is a constant $C'_d > 0$ such that the scalar curvature at any point $p \in M$ equals C'_d times the uniform average of the Ricci curvatures of all unit vectors in T_pM.

Solution: p. 204.

Problem 72. [Explicit formula for the curvature tensor in terms of sectional curvature.] Let V be a vector space over $\mathbb{R}$ and let

$$\mathcal{R} : V \times V \times V \times V \to \mathbb{R}$$

be a multilinear form having the same symmetries as the curvature tensor field Rm (cf. Lemma 1 in Lecture #14); that is, for all $X, Y, Z, W \in V$:

$$\mathcal{R}(X, Y, Z, W) = -\mathcal{R}(Y, X, Z, W) = -\mathcal{R}(X, Y, W, Z) = \mathcal{R}(Z, W, X, Y)$$

and

$$\mathcal{R}(X, Y, Z, W) + \mathcal{R}(Y, Z, X, W) + \mathcal{R}(Z, X, Y, W) = 0.$$

Set

$$K(X, Y) := \mathcal{R}(X, Y, Y, X).$$

Find an explicit formula expressing $\mathcal{R}(X, Y, Z, W)$ in terms of the function K. Note that this gives a proof of a corrected version of [12, Lemma 4.3.3].

Solution: p. 206.

[Hint: One way to obtain this is by appropriately working through the steps in proof of the uniqueness Lemma 1 in lecture #15.]

Problem 73. [Analogue of Schur's Theorem for Ricci curvature.]

Prove the second part of Theorem 1 in Lecture #15 (=[12, Thm. 4.3.2]); "if $\dim M \geq 3$ and the Ricci curvature is constant at each point then M is Einstein".

Solution: p. 208.

Problem 74. [The pullback of a metric connection is metric.]

Let $F : H \to M$ be a C^∞ map of manifolds, let (E, π, M) be a vector bundle equipped with a bundle metric $\langle \cdot, \cdot \rangle$, and let D be a metric connection on E preserving the bundle metric. Prove that $\langle \cdot, \cdot \rangle$ in a natural way gives rise to a bundle metric on $F^*(E)$ (which we may also denote $\langle \cdot, \cdot \rangle$), and that the pullbacked connection $F^*(D)$ is metric with respect to this bundle metric.

(Comment: This fact is used in the proof of Lemma 1 in Lecture #16, and also in Jost, [12, p. 206, lines -5 to -4].)

Problem 75. [Pullback and torsion.]

Let $F : H \to M$ be a C^∞ map of manifolds and let ∇ be a connection on TM.

(a). Prove that the map

$$\begin{aligned} S \ &: \Gamma(TH) \times \Gamma(TH) \to \Gamma(F^*(TM)); \\ &S(X, Y) = (F^*\nabla)_X (dF \circ Y) - (F^*\nabla)_Y (dF \circ X) - dF \circ [X, Y], \end{aligned}$$

is well-defined and $C^\infty(H)$-bilinear. Conclude that S can be identified with a section in $\Gamma(T^*H \otimes T^*H \otimes F^*(TM))$.

(b). Prove that if ∇ is torsion free then $S = 0$.

(c). Use the above to give a detailed justification of the identity

$$\nabla_{\frac{\partial}{\partial s}} \dot{c} = \nabla_{\frac{\partial}{\partial t}} c'$$

appearing in the proof of Lemma 1 in Lecture #16 (and also in Jost, [12, p. 206 (line -4 to -3)]).

Problem 76. [Pullback of curvature.]

(a). Let $f : M \to N$ be a C^∞ map, and let D be a connection on a vector bundle (E, π, N), with curvature tensor $R \in \Omega^2(\operatorname{End} E)$. Also let $\widetilde{R} \in \Omega^2(\operatorname{End}(f^*E))$ be the curvature tensor of the connection f^*D on f^*E. Prove that for any $p \in M$ and $X, Y \in T_p(M)$,

$$\widetilde{R}(X, Y) = R(df(X), df(Y)) \qquad \text{in } \operatorname{End}(f^*E)_p = \operatorname{End}(E_{f(p)}).$$

(b). Use the above to give a detailed justification of the identity

$$\nabla_{\frac{\partial}{\partial s}} \nabla_{\frac{\partial}{\partial t}} \frac{\partial c}{\partial s}(t, s) = \nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} \frac{\partial c}{\partial s}(t, s) + R\Big(\frac{\partial c}{\partial t}, \frac{\partial c}{\partial s}\Big) \frac{\partial c}{\partial s}$$

appearing in the proof of Theorem 1 in Lecture #16 (and also in Jost, [12, p. 208 (lines 4,7,8)]).

Solution:
p. 209.

Problem 77. [Interpretation of curvature in terms of parallel transport around a 'square']

Let D be a connection on a vector bundle (E, π, M) and let $F = F_D$ be its curvature. Given $p \in M$, $X, Y \in T_pM$ and $v \in E_p$, prove the following formula for $F(X,Y)(v)$: Let $\eta > 0$ and let f be a C^∞ function from

$$(-\eta, \eta)^2 = \{(x, y) \in \mathbb{R}^2 \ : \ -\eta < x, y < \eta\}$$

to M satisfying $f(0,0) = p$, $df_{(0,0)}(\frac{\partial}{\partial x}) = X$ and $df_{(0,0)}(\frac{\partial}{\partial y}) = Y$. For $0 < \varepsilon < \eta$, let $\mathbb{P}_\varepsilon : E_p \to E_p$ denote parallel transport around the ("square") curve

$$c(t) = \begin{cases} f(t, 0) & \text{if } \ 0 \le t \le \varepsilon \\ f(\varepsilon, t - \varepsilon) & \text{if } \ \varepsilon \le t \le 2\varepsilon \\ f(3\varepsilon - t, \varepsilon) & \text{if } \ 2\varepsilon \le t \le 3\varepsilon \\ f(0, 4\varepsilon - t) & \text{if } \ 3\varepsilon \le t \le 4\varepsilon. \end{cases}$$

Then

$$F(X,Y)(v) = -\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon^2}\big(\mathbb{P}_\varepsilon(v) - v\big).$$

Problem 78. [Constant curvature metrics in normal coordinates.]

Let M be a Riemannian manifold with constant sectional curvature ρ. Let $p \in M$ and let (U, x) be normal coordinates with center p, and let $(g_{ij}(x))$ represent the Riemannian metric with respect to (U, x). Prove that for any $x \in x(U) \setminus \{0\}$:

$$g_{ij}(x) = \begin{cases} \dfrac{x_i x_j}{\|x\|^2} + \dfrac{\sin^2(\rho^{1/2}\|x\|)}{\rho\,\|x\|^2}\Big(\delta_{ij} - \dfrac{x_i x_j}{\|x\|^2}\Big) & \text{if } \ \rho > 0 \\ \delta_{ij} & \text{if } \ \rho = 0 \\ \dfrac{x_i x_j}{\|x\|^2} + \dfrac{\sinh^2(|\rho|^{1/2}\|x\|)}{|\rho|\,\|x\|^2}\Big(\delta_{ij} - \dfrac{x_i x_j}{\|x\|^2}\Big) & \text{if } \ \rho < 0. \end{cases}$$

(Verify also that the above expression extends to a C^∞ function on all of $x(U)$, as it should.)

Solution: p. 211.

Problem 79. [Some relations for (g_{ij}) in normal coordinates.]

Let M be a Riemannian manifold, let $p \subset M$, and let (U, x) be a chart on M which gives normal coordinates centered at p. Let the Riemannian metric be represented by $(g_{ij}(x))$ with respect to (U, x). Prove that for every i,

$$g_{ii,ii}(0) = 0$$

and for any $i \neq j$,

$$g_{ii,jj}(0) = g_{jj,ii}(0) = -2g_{ij,ij}(0).$$

Here $g_{ij,k\ell}(x) := \frac{\partial}{\partial x^k \partial x^\ell} g_{ij}(x)$.

[Some hints/suggestions: For symmetry reasons we may assume $i, j \in \{1, 2\}$ and then it suffices to study $g_{ij}(x)$ for $x = (x_1, x_2, 0, \ldots, 0)$. One can show that Jost's [12, Thm. 1.4.5] ($\Leftrightarrow$ Problem 23) implies that at any point $x = (x_1, x_2, 0, \ldots, 0)$ the vector $x_1 \frac{\partial}{\partial x_1} + x_2 \frac{\partial}{\partial x_2}$ has length $\sqrt{x_1^2 + x_2^2}$, and is orthogonal to the vector $-x_2 \frac{\partial}{\partial x_1} + x_1 \frac{\partial}{\partial x_2}$. Now investigate carefully what these facts imply for the functions $g_{ij}(x_1, x_2, 0, \ldots, 0)$ for $i, j \in \{1, 2\}$.]

Solution: p. 214.

Problem 80. [A formula for sectional curvature.]

Let M be a Riemannian manifold, let $p \subset M$, and let Π be a plane in T_pM (viz., a 2-dimensional linear subspace of T_pM). Let $D_r \subset T_pM$ be the open disc of radius r in the plane Π, centered at 0. For r sufficiently small, we know (by Theorem 3 in Lecture #4) that $\exp_p(D_r)$ is an embedded 2-dimensional submanifold of M; call its area A_r. Prove that

$$K(\Pi) = \lim_{r \to 0^+} 12 \, \frac{\pi r^2 - A_r}{\pi r^4}.$$

(The Riemannian metric on $\exp_p(D_r)$ is the one induced from M; cf. Problem 18. Also the "area" of $\exp_p(D_r)$ is the same as its "volume"; cf. p. 1 in Lecture #12.) [Hint: The results from Problem 79 may be useful.]

Solution: p. 216.

Problem 81. [On a surface of revolution: geodesics, parallel transport and sectional curvature.]

Let f be a C^∞ function from $\mathbb{R}$ to $\mathbb{R}_{>0}$, and consider a surface of revolution

$$S := \{(x, f(x)\cos\alpha, f(x)\sin\alpha) \,:\, x, \alpha \in \mathbb{R}\}.$$

We take it as known[5] that S is a closed differentiable submanifold of $\mathbb{R}^3$, and that for any real interval $J = (a, b)$ with $b < a + 2\pi$ the inverse of the map $(x, \alpha) \mapsto (x, f(x)\cos\alpha, f(x)\sin\alpha)$ from $\mathbb{R} \times J$ to S is a chart on S. Equip S with the Riemannian metric induced from the standard Riemannian metric on $\mathbb{R}^3$.

(a). Make explicit the ode describing an arbitrary geodesic on S, $\nabla_{\dot\gamma}\dot\gamma = 0$ (cf. p. 9 in Lecture #13), in the (x, α) coordinates. Your answer should be of the form

$$\begin{cases} \ddot{x} + \boxed{*}\,\dot{x}\dot{x} + \boxed{*}\,\dot{x}\dot{\alpha} + \boxed{*}\,\dot{\alpha}\dot{\alpha} = 0 \\ \ddot{\alpha} + \boxed{*}\,\dot{x}\dot{x} + \boxed{*}\,\dot{x}\dot{\alpha} + \boxed{*}\,\dot{\alpha}\dot{\alpha} = 0, \end{cases}$$

with each " $\boxed{*}$ " being an explicit expression in x, α, f. Prove also from this equation that $f(x)^2 \cdot \dot{\alpha}$ remains constant along any geodesic. Finally, describe all geodesics which have $x \equiv$ constant or $\alpha \equiv$ constant.

(b). Given $x \in \mathbb{R}$, consider the closed curve $c(t) = (x, f(x)\cos t, f(x)\sin t)$, $t \in [0, 2\pi]$, in S. Describe explicitly the parallel transport of an arbitrary tangent vector $v \in T_{c(0)}S$ along c.

(c). Compute the sectional curvature of S at an arbitrary point

$$(x, f(x)\cos\alpha, f(x)\sin\alpha).$$

(In particular, where is this sectional curvature positive/negative? Also, as a consistency check, verify that you get back the known answer for the case $f(x) = \sqrt{r^2 - x^2}$, $|x| < r$.)

Solution: p. 218.

[5](cf. Problem 30(a))

Problem 82. [A formula involving ∇^2 of a 1-form.]

Let M be a Riemannian manifold and let ∇ be the Levi-Civita connection on TM. By the standard definitions of dual and tensor product connections (cf. Propositions 1,2 in Lecture #10) ∇ gives rise to a connection on any tensor bundle

$$T_s^r(M) = \underbrace{TM \otimes \cdots \otimes TM}_{r \text{ times}} \otimes \underbrace{T^*M \otimes \cdots \otimes T^*M}_{s \text{ times}},$$

which we *also* call ∇. This ∇ is a map from $\Gamma T_s^r(M)$ to

$$\Omega^1(T_s^r(M)) = \Gamma(T_s^r(M) \otimes T^*M) = \Gamma(T_{s+1}^r(M)). \tag{3}$$

Prove that for any $\eta \in \Gamma(T_1^0(M))$, the tensor field

$$\nabla^2\eta := \nabla(\nabla\eta) \quad \text{in } \Gamma(T_3^0(M))$$

satisfies

$$(\nabla^2\eta)(X,Y,Z) - (\nabla^2\eta)(X,Z,Y) = \eta\Big(R(Y,Z)X\Big),$$

for all vector fields $X, Y, Z \in \Gamma(TM)$.

(Remark: We stress that the "new" T^*M-factor is put *last* in (3); thus for any $F \in \Gamma(T_s^r(M))$ and any $\omega^1, \ldots, \omega^r \in \Gamma(T^*M)$, $Y_1, \ldots, Y_s \in \Gamma(TM)$, $X \in \Gamma(TM)$,

$$(\nabla F)\big(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s, X\big) = (\nabla_X F)\big(\omega^1, \ldots, \omega^r, Y_1, \ldots, Y_s\big).$$

Note also that the connections $\nabla : \Gamma(T_s^r(M)) \to \Gamma(T_{s+1}^r(M))$ considered here should not be confused with the exterior covariant derivative defined in Proposition 4 in Lecture #10.)

(Hint: The formula can be proved either by expressing everything in local coordinates using Christoffel symbols, *or* by working through the definitions expressing all "∇" appearing in terms of the original Levi-Civita connection $\nabla : \Gamma(TM) \to \Omega^1(TM)$.)

Solution: p. 223.

Problem 83. [Basic fact on existence of variations of a curve.]

Let M be a C^∞ manifold, let $c : [0,1] \to M$ be a C^∞ curve, and let Y be a vector field along c.

(a). Prove that there exists a variation of c with $c' = Y$, and that if $Y(0) = 0 = Y(1)$ then this variation can be taken to be proper.

(b). Prove that if $\gamma_0, \gamma_1 : (-\varepsilon', \varepsilon') \to M$ are C^∞ curves with $\gamma_0(0) = c(0)$, $\dot\gamma_0(0) = Y(0)$, $\gamma_1(0) = c(1)$, $\dot\gamma_1(0) = Y(1)$, then there exists a variation of c with $c' = Y$ such that $c(0,s) = \gamma_0(s)$ and $c(1,s) = \gamma_1(s)$ for all small s.

Solution: p. 225.

Problem 84. [On proper variations through geodesics.]

Prove that if $c(t,s)$ is a proper variation of a geodesic c through geodesics (viz., c_s is a geodesic for every s), then $E(s)=E(c_s)$ and $L(s)=L(c_s)$ are *constant* functions of s.

(Comment: This means that Jost's sentence in [12, p. 216 (lines 13–14)] is somewhat misleading; namely the length is *always* constant on the whole family, for a proper variation through geodesics.)

Solution: p. 226.

Problem 85. [Around Cor. 3 in Lecture #17 ≈ Jost's Cor. 5.2.4.]

(a). Let $M=S^d$ with its standard Riemannian metric, and let $p\in M$. Give an example of a piecewise smooth curve $\gamma:[0,1]\to T_pM$ such that $L(\exp_p\circ\gamma)=\|\gamma(1)\|$ but γ is *not* a reparametrization of the curve $t\mapsto t\cdot\gamma(1)$ ($t\in[0,1]$).

(Comment: This shows that the last statement in Jost's [12, Cor. 5.2.4], i.e. the criterion for when equality holds, is incorrect.)

(b). Use "Gauss Lemma" (= Cor. 2 in Lecture #17 = Jost's [12, Cor. 5.2.3]) to derive the following strengthening of a result from Problem 23: Let M be a Riemannian manifold, let $p\in M$, and let $\mathcal{D}_p=T_pM\cap\mathcal{D}$ be the maximal domain of $\exp_p$ (cf. Problem 21). Let (W,y) be a C^∞ chart on T_pM with $W\subset\mathcal{D}_p$, which we assume is "polar coordinates" in the sense that

$$y^1(w)=\|w\|,\qquad \forall w\in W,$$

and

$$(y^2(cw),\ldots,y^d(cw))=(y^2(w),\ldots,y^d(w))\ \text{ whenever } c>0,\ w\in W,\ cw\in W.$$

Prove that at every point $\tilde{y}\in y(W)$, the matrix representing the symmetric bilinear form

$$(v,w)\mapsto\big\langle d(\exp_p\circ y^{-1})_{\tilde{y}}(v),d(\exp_p\circ y^{-1})_{\tilde{y}}(w)\big\rangle,\qquad v,w\in\mathbb{R}^d,$$

is of the form

$$(h_{ij}(\tilde{y}))=\begin{pmatrix}1 & 0 & \cdots & 0\\ 0 & h_{22}(\tilde{y}) & \cdots & h_{2d}(\tilde{y})\\ \vdots & \vdots & & \vdots\\ 0 & h_{d2}(\tilde{y}) & \cdots & h_{dd}(\tilde{y})\end{pmatrix}.$$

(Comment: As explained in Lecture #17, the above fact can be used to prove Cor. 3 in Lecture #17, which is [12, Cor. 5.2.4] with a modified criterion for equality.)

(c). Prove the following alternative criterion for equality in #17, Cor. 3: "If equality holds, and there does not exist a point conjugate to $c(0)$ along c, then γ must be a reparametrization of the curve $t\mapsto tv$ ($t\in[0,1]$)."

Solution: p. 226.

Problem 86. [Remark 2 in Lecture #18]

Let $c : [a,b] \to M$ be a geodesic and let $t_0 \neq t_1 \in [a,b]$. Prove that $c(t_0)$ and $c(t_1)$ are conjugate along c iff the differential

$$\big(d\exp_{c(t_0)}\big)_{(t_1-t_0)\cdot\dot c(t_0)} : \; T_{c(t_0)}M \to T_{c(t_1)}M$$

is singular.

Solution: p. 229.

Problem 87. [On the metric space C_M of C^∞ curves on M.]

Let M be a Riemannian manifold. Introduce the space C_M with its metric d as on p. 3 in Lecture #18.

(a). Prove that d is well-defined, and is indeed a metric on C_M.

(b). Prove that the metric space (C_M, d) is not complete.

(c). Prove that neither E nor L are continuous on (C_M, d); in fact for every $c \in C_M$ and $\delta > 0$, both E and L are *unbounded* on the open ball $B_\delta(c)$.

(d). As a small consolation, prove that both E and L are lower semicontinuous on (C_M, d).

Problem 88. [Approximating a non-C^∞ vector field along a curve.]

Let $c : [a,b] \to M$ be a geodesic and let Y be a "pw C^∞ vector field along c", i.e. Y is a continuous function $Y : [a,b] \to TM$ such that $Y(t) \in T_{c(t)}(M)$ for all $t \in [a,b]$, and such that there exist a finite number of 'break-points' $a = t_0 < t_1 < \cdots < t_m = b$ such that the restricted function $Y_{|[t_{j-1},t_j]}$ is C^∞ for each $j = 1, 2, \ldots, m$. Prove that then for every $\varepsilon > 0$ there exists some C^∞ vector field Z along c such that

$$Z(t) = Y(t) \qquad \forall t \in [a,b] \setminus \bigcup_{j=1}^{m-1} (t_j - \varepsilon, t_j + \varepsilon)$$

and

$$\big| I(Z,Z) - I(Y,Y) \big| < \varepsilon.$$

(Of course here "$I(Y,Y)$" is well-defined, for example it can be defined as $\sum_{j=1}^m I(Y_{[t_{j-1},t_j]}, Y_{[t_{j-1},t_j]})$.)

Solution: p. 230.

Problem 89. [Equivalence of definitions of injectivity radius.]

Let M be a Riemannian manifold and let $p \in M$. Let $r > 0$ be such that $\exp_p$ is defined and injective on the open ball $B_r(0)$ in $T_p(M)$. Prove that then $\exp_{p|B_r(0)}$ is a diffeomorphism of $B_r(0)$ onto an open subset of M.

(Comment: This proves that the injectivity radius of p can be defined either as the supremum of all $r > 0$ for which $\exp_p$ is defined and injective on $B_r(0) \subset T_p(M)$, as in Jost [12, Def. 1.4.6], *or* as the supremum of all $r > 0$ for which $\exp_{p|B_r(0)}$ is a diffeomorphism.)

Solution:
p. 231.

Problem 90. [Vanishing derivatives up to order k.]

Let M be a C^∞ manifold and let $f \in C^\infty(M)$, $p \in M$ and $k \in \mathbb{Z}_{\geq 0}$. We say that f has *vanishing derivatives up to order k at p* if for some chart (U, x) for M with $p \in U$, any $1 \leq r \leq k$ and any $j_1, \ldots, j_r \in \{1, \ldots, d\}$ $(d = \dim M)$,

$$\frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_r}} f = 0 \qquad \text{at } p.$$

Prove that when this holds, it follows that *every* chart (U, x) with $p \in U$ has the same property.

Solution: p. 232.

Problem 91. [Geodesics and conjugate points on a perturbed sphere.]

Let S^d be unit sphere equipped with its standard Riemannian metric, which we denote by $\langle \cdot, \cdot \rangle$. For any function $f \in C^\infty(M)$ which is everywhere positive, we write S^d_f for S^d equipped with the Riemannian metric

$$[v, w] := f(p) \cdot \langle v, w \rangle, \qquad \forall p \in S^d, \ v, w \in T_p S^d.$$

Fix a geodesic $c : [0, \pi] \to S^d$ parametrized by arc length (thus the endpoints $c(0)$ and $c(\pi)$ are antipodal points). For $k \in \mathbb{Z}_{\geq 0}$, let $\mathcal{F}_k$ be the family of all positive functions $f \in C^\infty(M)$ such that for every point p along c we have $f(p) = 1$ and f has vanishing derivatives up to order k at p (cf. Problem 90).

(a). Prove that c is a geodesic in S^d_f for every $f \in \mathcal{F}_1$.

(b). Prove that for every $f \in \mathcal{F}_2$, the following holds in S^d_f: c is a geodesic, $c(0)$ and $c(\pi)$ are conjugate along c, and there is no point before $c(\pi)$ conjugate to $c(0)$ along c.

(c). Let $U \subset S^d$ be an open set which has nonempty intersection with the geodesic c, and let f be any function in $\mathcal{F}_1$ which satisfies $f \geq 1$ on all S^d and $f(p) > 1$ for all $p \in U \setminus c([0, \pi])$. Prove that then c is a strict local minimum for L in S^d_f among pw C^∞ curves with fixed endpoints.

(d). Take U as in part (c), and let f be any function in $\mathcal{F}_1$ which satisfies $f \leq 1$ on all S^d and $f(p) < 1$ for all $p \in U \setminus c([0, \pi])$. Prove that then c is *not* a local minimum for L in S^d_f among pw C^∞ curves with fixed endpoints.

[Comment: It is a standard fact from analysis that there *exist* functions f as in (c) and (d), also in $\mathcal{F}_k$ with k arbitrarily large. It then follows from (b), (c), (d) that in the situation described in the remark immediately below Theorem 1 in Lecture #18 – i.e. when the endpoints of c are conjugate but there is no previous point along c conjugate to the starting point – one *cannot* make any general statement about c being or not being a (strict or non-strict) local minimum for L!]

Solution: p. 233.

Problem 92. [A comparison result for lengths of curves.]

Let M_0 and M be d-dimensional complete Riemannian manifolds such that M_0 has constant sectional curvature μ and the sectional curvature of M is everywhere $\leq \mu$. Fix points $p \in M$ and $p_0 \in M_0$, and identify both T_pM and $T_{p_0}M$ with $\mathbb{R}^d$ in a way carrying the respective Riemannian scalar products to the standard scalar product in $\mathbb{R}^d$. Take $r > 0$ so small that $\exp_{p_0}$ restricted to the open ball $B_r(0) \subset \mathbb{R}^d$ is a diffeomorphism onto an open subset of M_0. Prove that for any pw C^∞ curve $c : [a, b] \to B_r(0)$,

$$L(\exp_p \circ c) \geq L(\exp_{p_0} \circ c).$$

(Here $\exp_p \circ c$ is a curve on M while $\exp_{p_0} \circ c$ is a curve on M_0.)

[Hint: Try to prove a stronger statement comparing the norms of $d(\exp_p)_x(v)$ and $d(\exp_{p_0})_x(v)$ for any $x \in B_r(0)$ and $v \in \mathbb{R}^d$. Here use can be made of Corollaries 1 and 2 in Lecture #17 and Theorem 1 in Lecture #19 (the Rauch Comparison Theorem).]

Solution: p. 237.

Problem 93. [Focal points (special case).]

Let $\gamma : [-\eta, \eta] \to M$ and $c : [a, b] \to M$ be geodesics on the Riemannian manifold M, satisfying $c(a) = \gamma(0)$, $\dot{c}(a) \neq 0$, and $\langle \dot{c}(a), \dot{\gamma}(0) \rangle = 0$. For $\tau \in (a, b]$, $c(\tau)$ is called a *focal point* of γ along c if there exists a nontrivial Jacobi field X along c such that $X(\tau) = 0$, and

$$X(a) \in \mathrm{Span}(\dot{\gamma}(0)) \quad \text{and} \quad \dot{X}(a) \perp \dot{\gamma}(0) \qquad \text{in } T_{c(a)}(M).$$

Prove that if there is some $\tau \in (a, b)$ such that $c(\tau)$ is a focal point of γ along c, then there exists a variation $c : [a, b] \times (-\varepsilon, \varepsilon) \to M$ of the curve c such that $c(a, s) \in \gamma([-\eta, \eta])$ and $c(b, s) = c(b)$ for all $s \in (-\varepsilon, \varepsilon)$ and $L(s) < L(0)$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$ (with $L(s) := L(c(\cdot, s))$, as usual).

[Hint: If γ is a constant point then the result follows from Theorem 1 in Lecture #18; thus try to extend the proof of that theorem to the present situation. See also Problem 83(b).]

[Comment: More generally one can define the notion of “focal point” for any *submanifold* of M (in the place of γ above); cf., e.g., [2, p. 23TM]. The general definition looks different from our definition above, however in the special case which we consider, i.e. that of a submanifold which is a geodesic, the two formulations can be shown to be equivalent. Note also that the definition given in Jost, [12, Exc. 5.2], is completely incorrect.]

Solution: p. 239.

Problem 94. [Local isometry ⇒ covering map.]

Let $\widetilde{M}$ and M be Riemannian manifolds with $\widetilde{M}$ complete, and let

$$\pi : \widetilde{M} \to M$$

be a local isometry. Prove that then M is complete and π is a covering map.

Solution: p. 241.

Problem 95. [The Killing-Hopf Theorem.]

Prove the Killing-Hopf Theorem: Let M be an n-dimensional complete, simply connected Riemannian manifold with constant sectional curvature. Then M is isometric to $\mathbb{R}^n$ (with its standard Riemannian metric) or a sphere of radius $r > 0$ in $\mathbb{R}^{n+1}$ (with its standard Riemannian metric) or the hyperbolic space $H^n(\rho)$ introduced in [12, Sec. 5.4] (cf. Problem 20).

Solution: p. 241.

2. Solution suggestions

Problem 1: For any $p \in M$, let U_p be the set of points $q \in M$ for which there exists a curve from p to q. Using the fact that M is locally Euclidean one verifies that

$$\forall p \in M \; : \; \big[U_p \text{ is open}\big]. \tag{4}$$

Next let us note:

$$\forall p, q \in M \; : \; \big[U_p \cap U_q \neq \emptyset \Rightarrow q \in U_p\big]. \tag{5}$$

[Proof: Assume $U_p \cap U_q \neq \emptyset$; then there is a point $q' \in U_p \cap U_q$. Now $q' \in U_p$ means that there is a curve γ_1 in M from p to q', and $q' \in U_q$ means that there is a curve γ_2 in M from q to q'. Then the "product path" of γ_1 and the "inverse path" of γ_2 [6] is a curve in M from p to q. Hence $q \in U_p$.]

Now for any $p \in M$, if $q \in \complement U_p$ (complement wrt M) then also $U_q \subset \complement U_p$, by (5), and U_q is open (by (4)), and $q \in U_q$ (immediate from the definition of U_q). Hence every point in $\complement U_p$ has an open neighborhood which is contained in $\complement U_p$. Hence $\complement U_p$ is open (viz., U_p is closed).

Hence for every $p \in M$, both U_p and $\complement U_p$ are open. Furthermore M equals the disjoint union of these two sets. Hence since M is connected, either U_p or $\complement U_p$ must be empty. But $p \in U_p$; hence $\complement U_p = \emptyset$, i.e. $U_p = M$. By the definition of U_p, this means that for every $q \in M$ there exists a curve from p to q. $\square$

[6] we will discuss these notions in Lecture #6, and the product path in question will be denoted "$\gamma_1 \cdot \overline{\gamma}_2$"; however it should hopefully be clear already at this point how the curve in question is constructed; just draw a picture!

Problem 2:

(a). First assume that M has a countable atlas $\mathcal{A}$. For each chart $(U,x) \in \mathcal{A}$, since $x(U)$ (an open subset of $\mathbb{R}^d$) is second countable, we can choose a base $\mathcal{U} = \mathcal{U}_{(U,x)}$ for the topology of $x(U)$. Then set

$$\mathcal{U}' = \mathcal{U}'_{(U,x)} := \{x^{-1}(V) \,:\, V \in \mathcal{U}_{(U,x)}\}.$$

This is a countable family of open subsets of U. Next let $\mathcal{U}''$ be the union of all families $\mathcal{U}'_{(U,x)}$ as (U,x) runs through $\mathcal{A}$. This is a countable family of open subsets of M. We claim that $\mathcal{U}''$ is a base for the topology of M. In order to prove this, let Ω be an arbitrary open set in M, and let $p \in \Omega$. Take a chart $(U,x) \in \mathcal{A}$ with $p \in U$. Then $\Omega \cap U$ is an open set in U containing p, and so $x(\Omega \cap U)$ is an open subset of $x(U)$ and $x(p) \in x(\Omega \cap U)$. Hence, since $\mathcal{U}_{(U,x)}$ is a base for $x(U)$, there is $V \in \mathcal{U}_{(U,x)}$ such that

$$x(p) \in V \subset x(\Omega \cap U).$$

Then $x^{-1}(V) \in \mathcal{U}' \subset \mathcal{U}''$ and

$$p \in x^{-1}(V) \subset \Omega \cap U \subset \Omega.$$

This proves that $\mathcal{U}''$ is a base for the topology of M. Done!

We now prove the opposite implication. Thus assue that M is second countable; let $\mathcal{U}$ be a countable base for the topology of M. Also let $\mathcal{A}$ be the family of *all* charts on M; this is an atlas for M. Set

$$\mathcal{U}' := \{U \in \mathcal{U} \,:\, \text{there is some } x : U \to \mathbb{R}^d \text{ s.t. } (U,x) \in \mathcal{A}\}.$$

We claim that $\mathcal{U}'$ covers M, i.e. $\cup_{U \in \mathcal{U}'} U = M$. To prove this, take an arbitrary point $p \in M$. Then there is some chart $(U,x) \in \mathcal{A}$ with $p \in U$, and since $\mathcal{U}$ is a base for M there is $V \in \mathcal{U}$ such that $p \in V \subset U$. Now $(V, x_{|V})$ is also a chart for M (since the restriction of a homeomorphism to an open subset is itself a homeomorphism onto its image), i.e. $(V, x_{|V}) \in \mathcal{A}$, and thus $V \in \mathcal{U}'$. Hence $\mathcal{U}'$ indeed covers M. It follows that if for each $U \in \mathcal{U}'$ we choose *one* map $x_U : U \to \mathbb{R}^d$ such that $(U, x_U) \in \mathcal{A}$, then

$$\{(U, x_U) \,:\, U \in \mathcal{U}'\}$$

is an atlas for M. This atlas is countable since $\mathcal{U}'$ is countable (since $\mathcal{U}' \subset \mathcal{U}$). Done! □

(b). By the notes to Lecture #1, this is clear from part (a). □

Problem 3: Let M be a connected topological space for which every point has an open neighborhood U which is homeomorphic to an open subset Ω of $\mathbb{R}^d$ for some $d \in \mathbb{Z}_{\geq 1}$ (which apriori may depend on U). Note that we actually don't need to assume that M is Hausdorff for the following argument to work.

For each $d \in \mathbb{Z}_{\geq 1}$, let $\mathcal{F}_d$ be the family of *all* open sets $U \subset M$ which are homeomorphic to an open subset of $\mathbb{R}^d$. Then the assumption on M implies that

$$M = \bigcup_{d=1}^{\infty} \Big(\bigcup_{U \in \mathcal{F}_d} U \Big). \tag{6}$$

Using Brouwer's Theorem on invariance of dimension, we now have:

$$\forall d \neq d' \in \mathbb{Z}_{\geq 1} : \quad \forall U \in \mathcal{F}_d,\ V \in \mathcal{F}_{d'} : \quad U \cap V = \emptyset. \tag{7}$$

[Detailed proof: Take such U, V and set $W := U \cap V$. Note that W is an open subset of both U and V. Now $U \in \mathcal{F}_d$ implies that U is homomorphic to an open subset of $\mathbb{R}^d$; this homeomorphism then restricts to a homeomorphism of W to a (smaller) open subset of $\mathbb{R}^d$. Similarly $V \in \mathcal{F}_{d'}$ implies that W is *also* homeomorphic to an open subset of $\mathbb{R}^{d'}$. Hence by Brouwer's Theorem on invariance of dimension, using $d = d'$, we must have $W = \emptyset$, qed.]

The property (7) implies that the unions $\cup_{U \in \mathcal{F}_d} U$ are pairwise disjoint for $d = 1, 2, \ldots$. Also each such union is an open set, since it is a union of open sets. Hence (6) expresses M as a union of *disjoint* open sets. But M is connected; therefore $\cup_{U \in \mathcal{F}_d} U$ must be *empty* for all except (at most) one d, say d_0. This means that $\mathcal{F}_d = \{\emptyset\}$ for all $d \neq d_0$, and this implies the desired result. □

Problem 4: Let the dimension of M be d. Let $\mathcal{A}$ be the given C^∞ atlas, and let $\mathcal{A}'$ be the family of all charts which are compatible with every chart in $\mathcal{A}$. Let us start by proving that $\mathcal{A}'$ *is a* C^∞ *atlas.* Clearly $\mathcal{A} \subset \mathcal{A}'$ and thus the charts in $\mathcal{A}'$ cover M. Thus it remains to prove that any two charts in $\mathcal{A}'$ are C^∞ compatible. Thus consider any two charts (U, x), $(V, y) \in \mathcal{A}'$; we need to prove that the map

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V) \subset \mathbb{R}^d \tag{8}$$

is C^∞. (It is clear that that map in (8) is a homeomorphism, since (U, x) and (V, y) are charts.) Take $p \in U \cap V$; it suffices to prove that there is some open neighborhood $\Omega \subset x(U \cap V)$ of $x(p)$ such that $(y \circ x^{-1})_{|\Omega}$ is C^∞. Since $\mathcal{A}$ is an atlas, there is some chart $(W, z) \in \mathcal{A}$ with $p \in W$. By assumption both (U, x) and (V, y) are compatible with (W, z); hence both the maps

$$z \circ x^{-1} : x(U \cap W) \to z(U \cap W) \tag{9}$$

and

$$y \circ z^{-1} : z(V \cap W) \to y(V \cap W) \tag{10}$$

are diffeomorphisms. Now set

$$\Omega := x(U \cap V \cap W).$$

This is an open subset of $x(U)$, since $U \cap V \cap W$ is an open subset of U and x is a homeomorphism. Restricting the diffeomorphisms in (9) and (10) to the open subsets Ω and $z(U \cap V \cap W)$, respectively, we obtain diffeomorphisms

$$z \circ x^{-1} : \Omega \to z(U \cap V \cap W) \tag{11}$$

and

$$y \circ z^{-1} : z(U \cap V \cap W) \to y(U \cap V \cap W). \tag{12}$$

It follows that the composition of these two maps is also a diffeomorphism, from Ω onto $y(U \cap V \cap W)$. But this composition equals $(y \circ x^{-1})_{|\Omega}$. Hence we have proved, in particular, that $(y \circ x^{-1})_{|\Omega}$ is C^∞. This completes the proof that $\mathcal{A}'$ is a C^∞ atlas.

It is immediate from the construction of $\mathcal{A}'$ that $\mathcal{A}'$ is C^∞ *structure*, i.e. a *maximal* C^∞ atlas. (Indeed, suppose that $\mathcal{A}''$ is any C^∞ atlas with $\mathcal{A}'' \supset \mathcal{A}'$. Let $(U, x) \in \mathcal{A}''$. By definition of "atlas", (U, x) is compatible with every chart in $\mathcal{A}''$; and $\mathcal{A} \subset \mathcal{A}' \subset \mathcal{A}''$; hence (U, x) is compatible with every chart in $\mathcal{A}$, and therefore $(U, x) \in \mathcal{A}'$, by the definition of $\mathcal{A}'$. Hence we have proved that $\mathcal{A}'' \subset \mathcal{A}'$, and so in fact $\mathcal{A}'' = \mathcal{A}'$.)

It remains to prove that $\mathcal{A}'$ is the *only* C^∞ structure on M with $\mathcal{A} \subset \mathcal{A}'$. Thus assume that $\mathcal{A}''$ is an arbitrary C^∞ structure on M with $\mathcal{A} \subset \mathcal{A}''$. Since $\mathcal{A}''$ is a C^∞ atlas, every chart $(U, x) \in \mathcal{A}''$ is compatible with every chart in $\mathcal{A}''$; in particular (U, x) is compatible with every chart in $\mathcal{A}$, and thus $(U, x) \in \mathcal{A}'$, by the definition of $\mathcal{A}'$. Hence $\mathcal{A}'' \subset \mathcal{A}'$. But this implies that $\mathcal{A}'' = \mathcal{A}'$, since $\mathcal{A}''$ is a maximal C^∞ atlas. This completes the proof. □

Problem 5:

(a) One simple way to construct such a set $\mathcal{H}$ is as follows. Given any real number $0 < t < 2$, let

$$f_t : (0,1) \to (0,1), \qquad f_t(r) := \begin{cases} tr & \text{if } r \in (0, \frac{1}{4}] \\ \frac{1}{2}(t-1) + (2-t)r & \text{if } r \in (\frac{1}{4}, \frac{1}{2}] \\ r & \text{if } r \in (\frac{1}{2}, 1). \end{cases}$$

One verifies that f_t is continuous, strictly increasing, and bijective, with inverse

$$f_t^{-1} : (0,1) \to (0,1), \qquad f_t^{-1}(r) := \begin{cases} t^{-1}r & \text{if } r \in (0, \frac{t}{4}] \\ (2-t)^{-1}(r + \frac{1}{2}(1-t)) & \text{if } r \in (\frac{t}{4}, \frac{1}{2}] \\ r & \text{if } r \in (\frac{1}{2}, 1) \end{cases}$$

which is also continuous and strictly increasing. (These facts are most easily verified by simply drawing the graph of f_t; this graph is a union of three line segments: one from $(0,0)$ to $(\frac{1}{4}, \frac{1}{4}t)$, one from $(\frac{1}{4}, \frac{1}{4}t)$ to $(\frac{1}{2}, \frac{1}{2})$, and one from $(\frac{1}{2}, \frac{1}{2})$ to $(1,1)$.) Hence f_t is a homeomorphism of $(0,1)$ onto itself.

Next define $h_t : B_1(0) \to B_1(0)$ through

$$h_t(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{f_t(\|x\|)}{\|x\|} x & \text{if } x \neq 0. \end{cases}$$

Note that

$$\|h_t(x)\| = f_t(\|x\|), \qquad \forall x \in B_1(0) \setminus \{0\};$$

and recall $f_t(r) \in (0,1)$ for all $r \in (0,1)$; hence h_t is indeed a map into $B_1(0)$. Clearly h_t is continuous in $B_1(0) \setminus \{0\}$; but we also have $\|h_t(x)\| \to 0$ (i.e. $h_t(x)$ tends to the origin) as $x \to 0$, since $f_t(r) \to 0$ as $r \to 0^+$; therefore h_t is continuous in all $B_1(0)$. Similarly one verifies that

$$\widetilde{h}_t(x) = \begin{cases} 0 & \text{if } x = 0 \\ \dfrac{f_t^{-1}(\|x\|)}{\|x\|} x & \text{if } x \neq 0. \end{cases}$$

defines a continuous map $\widetilde{h}_t : B_1(0) \to B_1(0)$ satisfying $\|\widetilde{h}_t(x)\| = f_t^{-1}(\|x\|)$ for all $x \in B_1(0) \setminus \{0\}$. Now we have

$$h_t(\widetilde{h}_t(x)) = \widetilde{h}_t(h_t(x)) = x, \qquad \forall x \in B_1(0).$$

(Proof: This is immediate for $x = 0$. Now assume $x \neq 0$. Then

$$h_t(\widetilde{h}_t(x)) = \frac{f_t(\|\widetilde{h}_t(x)\|)}{\|\widetilde{h}_t(x)\|} \widetilde{h}_t(x) = \frac{f_t(f_t^{-1}(\|x\|))}{f_t^{-1}(\|x\|)} \widetilde{h}_t(x) = \frac{\|x\|}{f_t^{-1}(\|x\|)} \widetilde{h}_t(x) = x.$$

The proof of $\widetilde{h}_t(h_t(x)) = x$ is completely similar.) Hence h_t and $\widetilde{h}_t$ are both bijections of $B_1(0)$ onto $B_1(0)$, and they are each other inverses. Since they are both continuous, it follows that h_t *is a homeomorphism of* $B_1(0)$ *onto itself.*

Note also that h_t satisfies $h_t(x) = x$ for all $x \in B_1(0) \setminus B_{1/2}(0)$, since $f_t(r) = r$ for $r \in [\frac{1}{2}, 1)$.

We now let $\mathcal{H}$ be the family of all these homeomorphisms h_t:

$$\mathcal{H} := \{h_t \,:\, t \in (0,2)\}.$$

This family clearly satisfies all the requirements, if we can only prove that for any two $t_1 \neq t_2 \in (0,2)$, the homeomorphism $h_{t_1} \circ h_{t_2}^{-1}$ is not C^∞. (Indeed, this will in particular imply that $h_{t_1} \not\equiv h_{t_2}$ for all $t_1 \neq t_2 \in (0,2)$, and so the family $\mathcal{H}$ is uncountable.)

Thus let $t_1 \neq t_2 \in (0,2)$ be given. Now for all $x \neq 0$ we have:

$$h_{t_1}(h_{t_2}^{-1}(x)) = \frac{f_{t_1}(f_{t_2}^{-1}(\|x\|))}{f_{t_2}^{-1}(\|x\|)} \cdot \frac{f_{t_2}^{-1}(\|x\|)}{\|x\|} x = \frac{f_{t_1}(f_{t_2}^{-1}(\|x\|))}{\|x\|} x. \tag{13}$$

Using the explicit formulas for f_t and f_t^{-1} given above, we compute

$$f_{t_1}(f_{t_2}^{-1}(r)) = \begin{cases} t_1 t_2^{-1} r & \text{if } \ r \in (0, \frac{1}{4}t_2] \\ \frac{1}{2}(t_1 - 1) + \frac{2-t_1}{2-t_2}(r + \frac{1-t_2}{2}) & \text{if } \ r \in (\frac{1}{4}t_2, \frac{1}{2}] \\ r & \text{if } \ r \in (\frac{1}{2}, 1). \end{cases}$$

From this we see that the (continuous) function $f_{t_1} \circ f_{t_2} : (0,1) \to (0,1)$ is not C^∞; for example at $r = \frac{1}{2}$ the function has left derivative $\frac{2-t_1}{2-t_2}$ and right derivative 1, and these are not equal, since $t_1 \neq t_2$. (Similarly the function has different left and right derivatives at $r = \frac{1}{4}t_2$.) From this it follows that the function $h_{t_1} \circ h_{t_2}^{-1} : B_1(0) \to B_1(0)$ is not C^∞. (Indeed, if $h_{t_1} \circ h_{t_2}^{-1}$ were C^∞ then, writing e_1 for the standard unit vector $(1, 0, \ldots, 0) \in \mathbb{R}^d$, it would follow that the function

$$(-1,1) \to (-1,1), \qquad r \mapsto h_{t_1}(h_{t_2}^{-1}(re_1)) \cdot e_1,$$

were C^∞; but it follows from (13) that $h_{t_1}(h_{t_2}^{-1}(re_1)) \cdot e_1 = f_{t_1}(f_{t_2}^{-1}(r))$ for $r \in (0,1)$, and so we would have a contradiction against the fact that $f_{t_1} \circ f_{t_2}$ is not C^∞.) $\square$

(b). Let $\mathcal{A}$ be a fixed C^∞ structure on M (it exists by assumption). Fix some chart $(V, y) \in \mathcal{A}$. Let y_0 be a point in $y(V)$; then since $y(V)$ is open, there is some $r > 0$ such that $B_r(y_0) \subset y(V)$. Set $U := y^{-1}(B_r(y_0))$; this is an open subset of V, and $(U, y_{|U}) \in \mathcal{A}$, since $\mathcal{A}$ is a maximal C^∞ atlas. Define the map

$$x : U \to \mathbb{R}^d, \qquad x(p) := \frac{1}{r}(y(p) - y_0).$$

Then $(U, x) \in \mathcal{A}$, since x equals $y_{|U}$ composed with a diffeomorphism of $B_r(y_0)$ onto $B_1(0)$. Note that $x(U) = B_1(0)$. From now on we keep this chart (U, x) fixed.

Now for any given homeomorphism h of $B_1(0)$ onto $B_1(0)$ satisfying $h(q) = q$ for all $q \in B_1(0) \setminus B_{1/2}(0)$, we define a function $\varphi_h : M \to M$ as follows:

$$\varphi_h(p) = \begin{cases} p & \text{if } p \notin U \\ x^{-1}(h(x(p))) & \text{if } p \in U. \end{cases}$$

We claim that φ_h is continuous. It is immediate from the definition of φ_h that the restrictions of φ_h to U and to $M \setminus U$ are both continuous. Hence since U is open (and so $M \setminus U$ is closed) it now suffices to verify that if $p_1, p_2, \ldots$ is any sequence of points in U such that $p_j \to p \in M \setminus U$ as $j \to \infty$, then $\varphi_h(p_j) \to \varphi_h(p)$. However the fact that (p_j) tends to a point outside U implies that (p_j) has only finitely many points in any fixed compact subset of U; in particular for all sufficiently large j we have $p_j \notin x^{-1}(\overline{B_{1/2}(0)})$, and thus $h(x(p_j)) = x(p_j)$ and $\varphi_h(p_j) = p_j$. Also $\varphi_h(p) = p$ since $p \notin U$, and it follows that $\varphi_h(p_j) \to \varphi_h(p)$. This completes the proof that φ_h is continuous.

Furthermore, one verifies immediately that φ_h is a bijection with inverse map equal to $\varphi_{h^{-1}} : M \to M$, and the above argument applies also to $\varphi_{h^{-1}}$, showing that $\varphi_{h^{-1}}$ is continuous. Hence φ_h *is a homeomorphism of* M *onto itself.*

Next we prove:

Lemma 1. *If $\mathcal{A}$ is a C^∞ structure on M, and φ is a homeomorphism of M onto itself, then also*

$$\mathcal{A}_\varphi := \{(\varphi^{-1}(V), y \circ \varphi) \; : \; (V, y) \in \mathcal{A}\}$$

is a C^∞ structure on M. Let us write $(M, \mathcal{A})$ for the C^∞ manifold given by $\mathcal{A}$, and $(M, \mathcal{A}_\varphi)$ for the C^∞ manifold given by $\mathcal{A}_\varphi$. Then φ is a diffeomorphism of $(M, \mathcal{A}_\varphi)$ onto $(M, \mathcal{A})$.

(Remark: The whole lemma can be seen as obvious. Namely, $(M, \mathcal{A}_\varphi)$ can be seen as "what one gets from $(M, \mathcal{A})$ after changing names on all points according to φ". Viewed in this way, φ "is identity map"!)

Proof. Let $\mathcal{T}$ be the family of all charts on the topological manifold M. Note that for any $(V, y) \in \mathcal{T}$ we have $(\varphi^{-1}(V), y \circ \varphi) \in \mathcal{T}$; hence we have a map

$$\Phi : \mathcal{T} \to \mathcal{T}, \qquad \Phi(V, y) := (\varphi^{-1}(V), y \circ \varphi).$$

In fact Φ is a bijection, with $\Phi^{-1}(V, y) = (\varphi(V), y \circ \varphi^{-1})$. Note that

$$\mathcal{A}_\varphi = \{\Phi(V, y) \,:\, (V, y) \in \mathcal{A}\}. \tag{14}$$

Next we note that for any two charts $(V, y), (W, z) \in \mathcal{T}$, we have the equivalence

$$\begin{aligned} &[(V, y) \text{ and } (W, z) \text{ are } C^\infty \text{ compatible}] \\ &\qquad \Leftrightarrow [\Phi(V, y) \text{ and } \Phi(W, z) \text{ are } C^\infty \text{ compatible}]. \end{aligned} \tag{15}$$

Indeed, by definition (V, y) and (W, z) are C^∞ compatible iff the map

$$z \circ y^{-1} : \quad y(V \cap W) \to z(V \cap W) \tag{16}$$

is a diffeomorphism (it is always a homeomorphism), and similarly $\Phi(V, y)$ and $\Phi(W, z)$ are C^∞ compatible iff the map

$$\begin{aligned} z \circ \varphi \circ (y \circ \varphi)^{-1} : \quad &(y \circ \varphi)(\varphi^{-1}(V) \cap \varphi^{-1}(W)) \\ &\to (z \circ \varphi)(\varphi^{-1}(V) \cap \varphi^{-1}(W)) \end{aligned} \tag{17}$$

is a diffeomorphism. However, $\varphi^{-1}(V) \cap \varphi^{-1}(W) = \varphi^{-1}(V \cap W)$, and now by inspection one verifies that the two maps in (16) and (17) are *the same.* Hence the equivalence in (15) holds.

Clearly the charts in $\mathcal{A}_\varphi$ cover M, since the charts in $\mathcal{A}$ cover M. Note also that any two charts in $\mathcal{A}_\varphi$ are C^∞ compatible; this follows from (14) and (15) and the fact that $\mathcal{A}$ is a C^∞ atlas. Hence $\mathcal{A}_\varphi$ is a C^∞ atlas. In fact (14) and (15) show that for *any* chart $(V, y) \in \mathcal{T}$, if (V, y) is C^∞ compatible with $\mathcal{A}_\varphi$ then $\Phi^{-1}(V, y)$ is C^∞ compatible with $\mathcal{A}$; hence $\Phi^{-1}(V, y) \in \mathcal{A}$ since $\mathcal{A}$ is a *maximal* C^∞ atlas, and so $(V, y) \in \mathcal{A}_\varphi$. Hence $\mathcal{A}_\varphi$ is a maximal C^∞ atlas on M, i.e. *$\mathcal{A}_\varphi$ is a C^∞ structure on M.*

It remains to prove that φ is a diffeomorphism of $(M, \mathcal{A}_\varphi)$ onto $(M, \mathcal{A})$. For this, our task is to verify that for any $(V, y) \in \mathcal{A}_\varphi$ and any $(W, z) \in \mathcal{A}$, the map

$$z \circ \varphi \circ y^{-1} : y(V \cap \varphi^{-1}(W)) \to z(V \cap \varphi^{-1}(W))$$

is a diffeomorphism. However $(V, y) \in \mathcal{A}_\varphi$ means that $(V, y) = \Phi(\widetilde{V}, \widetilde{y}) = (\varphi^{-1}(\widetilde{V}), \widetilde{y} \circ \varphi)$ for some $(\widetilde{V}, \widetilde{y}) \in \mathcal{A}$, and so

$$z \circ \varphi \circ y^{-1} = z \circ \varphi \circ (\widetilde{y} \circ \varphi)^{-1} = z \circ \widetilde{y}^{-1}$$

on the set

$$y(V \cap \varphi^{-1}(W)) = \widetilde{y}(\varphi(V \cap \varphi^{-1}(W))) = \widetilde{y}(\varphi(V) \cap W) = \widetilde{y}(\widetilde{V} \cap W).$$

Thus, our task is to verify that $z \circ \varphi \circ y^{-1}$ is a diffeomorphism from $\widetilde{y}(\widetilde{V} \cap W)$ onto $z(\widetilde{V} \cap W)$, and this holds since $(\widetilde{V}, y)$ and (W, z) are charts in $\mathcal{A}$. $\square$

Now take $\mathcal{H}$ as in part (a), and form the family

$$\mathcal{F} := \{\mathcal{A}_{\varphi_h} \ : \ h \in \mathcal{H}\}.$$

By Lemma 1, each $\mathcal{A}_{\varphi_h}$ in $\mathcal{F}$ is a C^∞ structure on M, and all C^∞ manifolds defined by these C^∞ structures are diffeomorphic. Hence it now only remains to prove that $\mathcal{A}_{\varphi_{h_1}} \neq \mathcal{A}_{\varphi_{h_2}}$ for any two $h_1 \neq h_2 \in \mathcal{H}$. Let us write $\mathcal{A}_1 = \mathcal{A}_{\varphi_{h_1}}$ and $\mathcal{A}_2 = \mathcal{A}_{\varphi_{h_2}}$ for short.

Recall that $(U, x) \in \mathcal{A}$; hence $(\varphi_{h_1}^{-1}(U), x \circ \varphi_{h_1}) \in \mathcal{A}_1$ and $(\varphi_{h_2}^{-1}(U), x \circ \varphi_{h_2}) \in \mathcal{A}_2$. Note that the map

$$(x \circ \varphi_{h_1}) \circ (x \circ \varphi_{h_2})^{-1} : \quad (x \circ \varphi_{h_2})(\varphi_{h_2}^{-1}(U)) \to (x \circ \varphi_{h_1})(\varphi_{h_1}^{-1}(U))$$

is the same as

$$x \circ \varphi_{h_1} \circ \varphi_{h_2}^{-1} \circ x^{-1} : \quad B_1(0) \to B_1(0),$$

and by the definition of φ_h, this is the same as

$$h_1 \circ h_2^{-1} : \quad B_1(0) \to B_1(0),$$

which is *not* C^∞. Hence the two charts $(\varphi_{h_1}^{-1}(U), x \circ \varphi_{h_1})$ and $(\varphi_{h_2}^{-1}(U), x \circ \varphi_{h_2})$ are not C^∞ compatible, and therefore $\mathcal{A}_1 \neq \mathcal{A}_2$. □

Problem 6: All this is "completely obvious" once one understands the basic machinery with (C^∞) atlases. Let us go through the details:

(a). Here we are talking about a new type of object: "a (not necessarily connected) C^∞ manifold". The definition should hopefully be obvious[7] ... Namely: A "(not necessarily connected) C^∞ manifold" is a topological space M such that every connected component of M is a C^∞ manifold!

We now solve the given problem. Note that every connected component of U is *also* an open subset of M. Hence if we can prove that every *connected* open subset of M has a natural structure of a C^∞ manifold, then it follows that U has a natural structure of a (not necessarily connected) C^∞ manifold, and so we will be done.

Thus from now on assume that U is a *connected* open subset of M. Let the dimension of M be d. We endow U with the restricted topology; then U is a connected Hausdorff space.

Let $\mathcal{A}$ be the C^∞ structure of M. Thus $\mathcal{A}$ is a maximal C^∞ atlas on M. Set

$$\mathcal{A}_{|U} := \{(V,x) \,:\, (V,x) \in \mathcal{A},\, V \subset U\}.$$

We wish to prove that $\mathcal{A}_{|U}$ is a C^∞ atlas on U. Clearly every $(V,x) \in \mathcal{A}_{|U}$ is a chart on U and these charts are pairwise C^∞ compatible, since $\mathcal{A}$ is a C^∞ atlas. Hence it remains to prove that the charts in $\mathcal{A}_{|U}$ cover U. Take $p \in U$. Then there is a chart $(V,x) \in \mathcal{A}$ with $p \in V$. Now note that also $(V \cap U, x_{|V\cap U})$ is a chart on M, and $(V \cap U, x_{|V\cap U})$ is C^∞ compatible with every chart in $\mathcal{A}$ since (V,x) is C^∞ compatible with every chart in $\mathcal{A}$. Hence, since $\mathcal{A}$ is maximal, we have $(V \cap U, x_{|V\cap U}) \in \mathcal{A}$. Thus also $(V \cap U, x_{|V\cap U}) \in \mathcal{A}_{|U}$, since $V \cap U \subset U$. Also of course $p \in V \cap U$. Hence $\mathcal{A}_{|U}$ contains a chart which contains p. Since this is true for every $p \in U$, we conclude that the charts in $\mathcal{A}_{|U}$ indeed cover U. Hence $\mathcal{A}_{|U}$ is a C^∞ atlas on U, and so determines a unique C^∞ structure on U (cf. Problem 4). [8]

It remains to prove that U is paracompact. This is equivalent to proving that U is second countable (cf. the notes to Lecture #1). However this is clear from the fact that M is second countable; indeed it is easy to prove that any open subset of a second countable topological space is second countable.

□

(b), (c) ... we leave this to the reader ... (Note that the first part of (c) is immediate from (b), since $f_{|U} = f \circ i$.)

[7]In the literature it varies whether one defines a "manifold" to always be connected. However recall that in our course, we do require every manifold to be connected!

[8](In fact one can show that $\mathcal{A}_{|U}$ is itself a C^∞ structure on U.)

Problem 7:

(a) Let $W = M \setminus \text{supp}(f)$; this is an open subset of M, and $W \cup U = M$. We have $f_{|U} \in C^\infty(U)$ by assumption. also $f_{|W} \equiv 0$; hence $f_{|W} \in C^\infty(W)$. (Cf. Problem 6 regarding the fact that U and W are C^∞ manifolds; hence the function spaces "$C^\infty(U)$" and "$C^\infty(W)$" are defined.) Hence every point in M has an open neighbourhood in which f is C^∞. Hence by Problem 6(c), $f \in C^\infty(M)$.

(b) Let $K = \text{supp}(f)$; by assumption this is a compact subset of U. We claim that $\text{supp}(\widetilde{f}) = K$; if we prove this then the desired statement $\widetilde{f} \in C^\infty(M)$ follows from part (a). Note that K is a compact subset of M (since "compactness is an absolute property"; for example, use the fact that the inclusion map $i : U \to M$ is continuous, and the image of any compact set under a continuous map is compact). Hence K is a *closed* subset of M. Also note that $\{p \in M \ : \ \widetilde{f}(p) \neq 0\} \subset K$, by the definitions of $\widetilde{f}$ and K. Hence $\text{supp}(\widetilde{f})$, being the closure of $\{p \in M \ : \ \widetilde{f}(p) \neq 0\}$ in M, is contained in K. The opposite inclusion is obvious; hence $\text{supp}(\widetilde{f}) = K$. Done! □

(c). This is a special case of part (d).

(d). Let $\{(U_\alpha, x_\alpha)\}$ be a C^∞ atlas on M. Let $(V_\beta)_{\beta \in B}$ together with $(\varphi_\beta)_{\beta \in B}$ be a partition of unity subordinate to (U_α), as in [12, Lemma 1.1.1]. For each $\beta \in F$ we write $K_\beta := \text{supp}\, \varphi_\beta$; this is a compact set contained in V_β.

Let us start by noticing that for each $\beta \in B$, there exists a C^∞ function $f_\beta : V_\beta \to [0,1]$ which has compact support contained in $V_\beta \cap U$ and which satisfies $f_{\beta | K \cap K_\beta} \equiv 1$. Indeed, since $(V_\beta)_{\beta \in B}$ is a refinement of (U_α), for our given $\beta \in F$ there exists some α such that $V_\beta \subset U_\alpha$. Using now the chart (U_α, x_α) to translate the problem into Euclidean coordinates, we are reduced to proving that for any compact set $\widetilde{K}$ and open set $\widetilde{U}$ with $\widetilde{K} \subset \widetilde{U} \subset \mathbb{R}^d$, there is a C^∞ function $f : \mathbb{R}^d \to [0,1]$ with compact support contained in $\widetilde{U}$ and which satisfies $f_{|\widetilde{K}} \equiv 1$. For this, cf., e.g., [10, Thm. 1.4.1] (one considers the convolution of the characteristic function of $\widetilde{K}$ and a "bump" function with sufficiently small support).

For any $\beta \in B$ such that $K \cap V_\beta = \emptyset$ we may of course choose the above function f_β to be identically zero; from now on we require this to hold.

Next for each $\beta \in F$ we define $\widetilde{f}_\beta : M \to \mathbb{R}$ by $\widetilde{f}_\beta \equiv f_\beta$ in V_β and $\widetilde{f}_\beta = 0$ outside V_β; by part (a) we then have $\widetilde{f}_\beta \in C^\infty(M)$. We next set

$$f := \sum_{\beta \in B} \varphi_\beta \widetilde{f}_\beta.$$

Clearly $f \in C^\infty(M)$. Also for every $p \in M$ we have $f(p) \geq 0$ and $f(p) \leq \sum_{\beta \in F} \varphi_\beta(p) \leq 1$.

Fix an arbitrary point $p \in K$. We have $\sum_{\beta \in B} \varphi_\beta(p) = 1$ by the defining property ot (φ_β). Also $\widetilde{f}_\beta(p) = 1$ for every $\beta \in B$ with $p \in K_\beta$ (since we are assuming $p \in K$), and thus $\widetilde{f}_\beta(p) = 1$ for every $\beta \in B$ with $\varphi_\beta(p) \neq 0$. Hence $f(p) = \sum_{\beta \in B} \varphi_\beta(p) \widetilde{f}_\beta(p) = \sum_{\beta \in B} \varphi_\beta(p) = 1$. We have thus proved that $f_{|K} \equiv 1$.

In order to prove that f has compact support, we will use the requirement from above that $f_\beta \equiv 0$ (and so $\widetilde{f}_\beta \equiv 0$) whenever $K \cap V_\beta = \emptyset$. This means that the sum defining f may just as well be restricted to the following subset of B:

$$F := \{\beta \in B \,:\, K \cap V_\beta \neq \emptyset\}.$$

Now since (V_β) is locally finite, F *is a finite set.* (This is a standard fact; here is a detailed proof: Since (V_β) is locally finite, for every $p \in M$ we can choose – using the axiom of choice – an open set $U_p \subset M$ such that $p \in U_p$ and $\#\{\beta \in B : V_\beta \cap U_p \neq \emptyset\} < \infty$. Since K is compact, there exists a finite subset $F' \subset K$ such that $K \subset \cup_{p \in F'} U_p$. Now for every $\beta \in F$ there is some $p \in F'$ such that $V_\beta \cap U_p \neq \emptyset$; hence $\#F \leq \sum_{p \in F'} \#\{\beta \in B \,:\, V_\beta \cap U_p \neq \emptyset\} < \infty$. Done!)

It follows from $f(p) = \sum_{\beta \in B} \varphi_\beta(p) \widetilde{f}_\beta(p) = \sum_{\beta \in F} \varphi_\beta(p) \widetilde{f}_\beta(p)$ that

$$\operatorname{supp}(f) \subset \bigcup_{\beta \in F} \operatorname{supp}(\widetilde{f}_\beta) = \bigcup_{\beta \in F} \operatorname{supp}(f_\beta)$$

(for the inclusion one uses the fact that $\cup_{\beta \in F} \operatorname{supp}(\widetilde{f}_\beta)$ is a closed subset of M; for the equality see part (b)). The last set is a finite union of compact sets, hence itself compact. Also by construction, $\operatorname{supp}(f_\beta) \subset U$ for each $\beta \in F$. Hence $\operatorname{supp}(f) \subset U$, and also since $\operatorname{supp}(f)$ is closed and contained in a compact set, $\operatorname{supp}(f)$ is itself compact.

Hence the function f has all the desired properties. □

(e) Let $g : M \to [0, 1]$ be a function as in part (d), i.e. g is C^∞, has compact support contained in U, and satisfies $g_{|K} \equiv 1$. Set

$$f_1(p) := \begin{cases} g(p)f(p) & \text{if } p \in U \\ 0 & \text{if } p \notin U. \end{cases}$$

Then clearly $f_{1|K} \equiv f_{|K}$. Also the function $p \mapsto g(p)f(p)$ is a C^∞ function $U \to \mathbb{R}$ with compact support, and f_1 is the same as "$\widetilde{f}$ in part (b), but starting from the function $p \mapsto g(p)f(p)$ on U". Hence f_1 is C^∞, by part (b). □

Problem 8.

(a). We endow $M \times N$ with the product topology (viz., a subset of $M \times N$ is open iff it can be written as a union of sets of the form $U \times V$ with $U \subset M$ and $V \subset N$). Then $M \times N$ is Hausdorff and connected. (We leave the details to the reader...) We will verify at the end that $M \times N$ is also paracompact (according to Wikipedia the product of to general paracompact topological spaces need not be paracompact; thus we need to make use of the fact that M, N have more structure).

Let the dimensions of M and N be d and d', respectively. Let $\mathcal{A}$ be a C^∞ structure on M and let $\mathcal{B}$ be a C^∞ structure on N. For any charts $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$, $U \times V$ is an open set in $M \times N$, and we write (x, y) [9] for the map

$$(x, y) : U \times V \to \mathbb{R}^d \times \mathbb{R}^{d'} = \mathbb{R}^{d+d'}, \qquad (x, y)(p, q) := (x(p), y(q)).$$

This map (x, y) is in fact a homeomorphism from $U \times V$ onto $x(U) \times y(V)$ (which is an open subset of $\mathbb{R}^d \times \mathbb{R}^{d'}$).

[Outline of proof: (x, y) is clearly a bijection from $U \times V$ onto $x(U) \times y(V)$. We leave it to the reader to verify – or recall from basic point set topology – that (x, y) is continuous. Similarly the inverse map, $(x, y)^{-1} = (x^{-1}, y^{-1})$ is continuous since x^{-1} and y^{-1} are continuous.]

Hence for any charts $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$, we have that $(U \times V, (x, y))$ is a chart on $M \times N$. Now set

$$\mathcal{C} := \{(U \times V, (x, y)) \;:\; (U, x) \in \mathcal{A},\; (V, y) \in \mathcal{B}\}.$$

This is clearly an atlas on $M \times N$.

Let us now verify that $M \times N$ is paracompact. By the notes to Lecture #1, with reference to math.stackexchange.com/questions/527642, it suffices to prove that $M \times N$ is second countable, and this is a simple consequence of the fact that M and N are second countable. Indeed, let $\mathcal{U}_M$ be a countable base for M and let $\mathcal{U}_N$ be a countable base for N, and set

$$\mathcal{U} = \{U \times V \;:\; U \in \mathcal{U}_M,\; V \in \mathcal{U}_N\}.$$

[9] Note that there is a glitch between this notation "(x, y)" and the notation "(f, g)" used in part (c). Let us discuss this carefully in the abstract setting of maps between sets: Thus if A, B_1, B_2 are three sets and $\alpha : A \to B_1$ and $\beta : A \to B_2$ are maps, then we define the map "$(\alpha, \beta) : A \to B_1 \times B_2$" by $(\alpha, \beta)(p) := (\alpha(p), \beta(p))$". This is the notation which we use in part (c), and it is also the standard notation for "category theoretical product"; cf. wikipedia. On the other hand if A_1, A_2, B_1, B_2 are sets and $\gamma : A_1 \to B_1$ and $\delta : A_2 \to B_2$ are maps then we define the map $[\gamma, \delta] : A_1 \times A_2 \to B_1 \times B_2$ by $[\gamma, \delta](p, q) := (\gamma(p), \delta(q))$. The "$(x, y)$" which we use here in our solution to part (a) is *this constructions*; we use the notation "$[\cdot, \cdot]$" in this footnote for clarity, but in practice there is no problem to use "$(\cdot, \cdot)$" for both, and it is also quite standard. Note that the two constructions are related by the simple relation $[\gamma, \delta] = (\gamma \circ \mathrm{pr}_1, \delta \circ \mathrm{pr}_2)$; indeed see part (d) of the present problem.

Then $\mathcal{U}$ is countable. We claim that $\mathcal{U}$ is a base for $M \times N$. To prove this, let W be an open set in $M \times N$, and let (p, q) be a point in W. Then, by the definition of the product topology on $M \times N$, there exist open sets $U' \subset M$ and $V' \subset N$ such that $(p,q) \in U' \times V' \subset W$. Next, since $\mathcal{U}_M$ and $\mathcal{U}_N$ are bases, there exists $U \in \mathcal{U}_M$ with $p \in U \subset U'$ and there exists $V \in \mathcal{U}_N$ with $q \in V \subset V'$. Then $U \times V \in \mathcal{U}$ and $(p,q) \in U \times V \subset W$. The fact that such a set exists in $\mathcal{U}$ for any given W, p as above proves that $\mathcal{U}$ is indeed a base for $M \times N$. Hence $M \times N$ is second countable.

Finally, we claim that $\mathcal{C}$ is a C^∞ atlas. To prove this we consider an arbitrary pair of charts in $\mathcal{C}$, say $(U \times V, (x, y))$ and $(W \times \Omega, (r, s))$, where $(U, x), (W, r)) \in \mathcal{A}$ and $(V, y), (\Omega, s) \in \mathcal{B}$. We have to prove that the map

$$(r,s) \circ (x,y)^{-1} : (x,y)(U \times V) \to \mathbb{R}^d \times \mathbb{R}^{d'} = \mathbb{R}^{d+d'}$$

is C^∞. However this map equals $(r \circ x^{-1}, s \circ y^{-1})$, and this map is C^∞ since both $r \circ x^{-1}$ and $s \circ y^{-1}$ are C^∞. (Indeed, recall that by definition a map f from an open subset $D \subset \mathbb{R}^m$ to $\mathbb{R}^n$ is C^∞ if and only if, when writing $f(z) = (f_1(z), \ldots, f_n(z))$ for $z \in D$, each "component" map $f_j : D \to \mathbb{R}$ is C^∞. When applying this to $f = (r \circ x^{-1}, s \circ y^{-1})$, each component f_j is in fact a component of either $r \circ x^{-1}$ or $s \circ y^{-1}$, hence C^∞.) Hence $\mathcal{C}$ is a C^∞ atlas on $M \times N$, and so determines a unique C^∞ structure on $M \times N$ (cf. Problem 4).

Hence $M \times N$ is a C^∞ manifold.

(b). Let the C^∞ atlases $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be as in part (a). In order to prove that pr_1 is C^∞ we have to prove that for any charts $(W, z) \in \mathcal{A}$ and $(U \times V, (x, y)) \in \mathcal{C}$ (with $(U, x) \in \mathcal{A}$ and $(V, y) \in \mathcal{B}$), the map

$$z \circ \mathrm{pr}_1 \circ (x,y)^{-1} : \quad (x,y)\Big((U \times V) \cap \mathrm{pr}_1^{-1}(W)\Big) \to z(W) \subset \mathbb{R}^d$$

is C^∞. We may here note that $(U \times V) \cap \mathrm{pr}_1^{-1}(W) = (U \cap W) \times V$. But the above map equals:

$$z \circ \mathrm{pr}_1 \circ (x,y)^{-1} = (z \circ x^{-1}) \circ p_1, \tag{18}$$

where p_1 is the projection map $\mathbb{R}^{d+d'} = \mathbb{R}^d \times \mathbb{R}^{d'} \to \mathbb{R}^d$. The map $z \circ x^{-1}$ (from $x(U \cap W)$ to $z(W)$) is C^∞ since $(U, x), (W, z) \in \mathcal{A}$ and $\mathcal{A}$ is a C^∞ chart. The map p_1 is obviously C^∞. Hence the composed map in (18) is C^∞, and we are done.

The proof that pr_2 is C^∞ is completely similar. □

(c). Let the dimensions of M, N_1, N_2 be d, d_1, d_2, respectively. Let $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$ be C^∞ structures on M, N_1, N_2, respectively, and set

$$\widetilde{\mathcal{A}} := \{(U \times V, (x, y)) \,:\, (U, x) \in \mathcal{A}_1,\ (V, y) \in \mathcal{A}_2\}.$$

By part (a), $\widetilde{\mathcal{A}}$ is a C^∞ atlas on $N_1 \times N_2$. Our task is to prove that for any charts $(W, z) \in \mathcal{A}$, $(U, x) \in \mathcal{A}_1$ and $(V, y) \in \mathcal{A}_2$, setting

$$W' := W \cap (f, g)^{-1}(U \times V),$$

the map

$$(19) \qquad (x, y) \circ (f, g) \circ z^{-1} : \quad z(W') \to (x, y)(U \times V) \subset \mathbb{R}^{d_1+d_2}$$

is C^∞. Now we compute:

$$(20) \qquad (x, y) \circ (f, g) \circ z^{-1} = (x \circ f \circ z^{-1}, y \circ f \circ z^{-1}),$$

or, in other words, for all $\alpha \in z(W') \subset \mathbb{R}^d$:

$$(x, y) \circ (f, g) \circ z^{-1}(\alpha) = \Big(x(f(z^{-1}(\alpha))), y(f(z^{-1}(\alpha)))\Big)$$

in $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2} = \mathbb{R}^{d_1+d_2}$. However the two maps $x \circ f \circ z^{-1}$ and $y \circ f \circ z^{-1}$ are C^∞ since f and g are C^∞ and $\mathcal{A}, \mathcal{A}_1, \mathcal{A}_2$ are C^∞ atlases; hence also the map in (19), (20) is C^∞ (indeed, this is a basic fact about C^∞ maps between open subsets of Euclidean spaces; cf. the argument at the end of our solution to part (a)). This completes the proof. □

(d). This is immediate from parts (b) and (c) since the map in question equals

$$(21) \qquad (f \circ \mathrm{pr}_1, g \circ \mathrm{pr}_2),$$

where we use the "$(\cdot, \cdot)$" notation from part (c), and $\mathrm{pr}_1, \mathrm{pr}_2$ are the projection maps $\mathrm{pr}_1 : M_1 \times M_2 \to M_1$ and $\mathrm{pr}_2 : M_1 \times M_2 \to M_2$.

[Detailed explanation (cf. also footnote 9 above): Write $M := M_1 \times M_2$; then $f \circ \mathrm{pr}_1$ is a C^∞ map $M \to N_1$, by part (b) and since any composition of C^∞ maps is C^∞. Similarly $g \circ \mathrm{pr}_2$ is a C^∞ map $M \to N_2$. Hence by part (c), $(f \circ \mathrm{pr}_1, g \circ \mathrm{pr}_2)$ is a C^∞ map $M \to N_1 \times N_2$. And this is indeed the map which we are interested in, since for every $(p, q) \in M_1 \times M_2 = M$ we have:

$$(f \circ \mathrm{pr}_1, g \circ \mathrm{pr}_2)(p, q) = (f \circ \mathrm{pr}_1(p, q), g \circ \mathrm{pr}_2(p, q)) = (f(p), g(q)).$$

Done!] □

Problem 9.

(a). $\sim$ is reflexive since $\text{Id} \in \Gamma$. To see that $\sim$ is symmetric, assume $p \sim q$; then there is $\gamma \in \Gamma$ s.t. $\gamma(p) = q$; but then $\gamma^{-1} \in \Gamma$ and $\gamma^{-1}(q) = p$; hence $q \sim p$. Finally let us prove that $\sim$ is transitive. Assume $p \sim q$ and $q \sim r$. Then there exist $\gamma, \gamma' \in \Gamma$ such that $\gamma(p) = q$ and $\gamma'(q) = r$. Then $\gamma'\gamma(p) = r$, and $\gamma'\gamma \in \Gamma$. Hence $\sim$ is transitive. Done! $\square$

(b). The fact that the definition gives a topology on $\Gamma\backslash M$ is immediate if we note that "π^{-1} respects intersections and unions of sets", i.e. for any family $\{U_\alpha\}$ of subsets $U_\alpha \subset \Gamma\backslash M$ we have $\pi^{-1}(\cup_\alpha U_\alpha) = \cup_\alpha \pi^{-1}(U_\alpha)$ and $\pi^{-1}(\cap_\alpha U_\alpha) = \cap_\alpha \pi^{-1}(U_\alpha)$. (This is in fact a property of the inverse of *any* map. Here we use it for finite intersections and arbitrary unions.) We also use the fact that $\pi^{-1}(\emptyset) = \emptyset$ and $\pi^{-1}(\Gamma\backslash M) = M$, both of which are open in $\Gamma\backslash M$.

Note also that it is immediate from the definition of the topology on $\Gamma\backslash M$ that *the projection map* $\pi : M \to \Gamma\backslash M$ *is continuous.*

We next prove that $\Gamma\backslash M$ is Hausdorff. This is considerably more difficult. Thus consider two distinct, arbitrary points in $\Gamma\backslash M$, say $[p]$ and $[q]$, where $p, q \in M$. Since M is locally Euclidean we can choose open sets $U, V \subset M$ such that $p \in U$, $q \in V$, and $\overline{U}$ and $\overline{V}$ are compact. Now since Γ acts properly discontinuously, the set

$$F := \{\gamma \in \Gamma \,:\, \gamma(U) \cap V \neq \emptyset\}$$

is finite. (Indeed, F is contained in $\{\gamma \in \Gamma \,:\, \gamma(K) \cap K \neq \emptyset\}$ for $K := \overline{U} \cup \overline{V}$, and K is compact.) Now for each $\gamma \in F$ we have $\gamma(p) \neq q$ (since $[p] \neq [q]$); hence since M is Hausdorff, there exist open sets V_γ, W_γ such that $q \in V_\gamma$, $\gamma(p) \in W_\gamma$, and $V_\gamma \cap W_\gamma = \emptyset$. Set $U_\gamma := \gamma^{-1}(W_\gamma)$; then $p \in U_\gamma$, and U_γ is open. Set

$$U_1 := U \,\cap\, \Big(\bigcap_{\gamma \in F} U_\gamma\Big); \qquad V_1 := V \,\cap\, \Big(\bigcap_{\gamma \in F} V_\gamma\Big).$$

Then U_1 and V_1 are open sets in M (since F is finite), and $p \in U_1$ and $q \in V_1$. We claim that

$$\forall \gamma \in \Gamma : \qquad \gamma(U_1) \cap V_1 = \emptyset. \tag{22}$$

To prove this, assume the opposite, i.e. $\gamma(U_1) \cap V_1 \neq \emptyset$ for some $\gamma \in \Gamma$. Using $U_1 \subset U$, $V_1 \subset V$, and the definition of F, it follows that $\gamma \in F$. Using $U_1 \subset U_\gamma$, $V_1 \subset V_\gamma$ it then follows that $\gamma(U_\gamma) \cap V_\gamma \neq \emptyset$, i.e. $W_\gamma \cap V_\gamma \neq \emptyset$, contradicting our choice of V_γ, W_γ. Hence (22) is proved.

Now set

$$U_2 := \pi(U_1); \qquad V_2 := \pi(V_1). \tag{23}$$

These are open subsets of $\Gamma\backslash M$! (Proof: One verifies that $\pi^{-1}(U_2) = \cup_{\gamma\in\Gamma}\gamma(U_2)$; and this is a union of open sets, hence open (in M). Therefore U_2 is open in $\Gamma\backslash M$. Similarly U_2 is open in $\Gamma\backslash M$.) Furthermore, (22) implies that $U_2 \cap V_2 = \emptyset$. Hence we have proved that $\Gamma\backslash M$ is Hausdorff.

Next we prove that $\Gamma\backslash M$ is connected; in fact we prove that $\Gamma\backslash M$ is *path-connected* (this trivially implies connectedness). Consider any two points in $\Gamma\backslash M$, say $[p]$ and $[q]$ with $p, q \in M$. Since M is path-connected (cf. Problem 1), there is a curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. But then $\pi \circ \gamma : [0,1] \to \Gamma\backslash M$ is a curve from $\pi(p) = [p]$ to $\pi(q) = [q]$. Hence $\Gamma\backslash M$ is path-connected.

Next we will prove that $\Gamma\backslash M$ is locally Euclidean. We say that a subset $U \subset M$ is injectively embedded in $\Gamma\backslash M$ if the restriction $\pi_{|U}$ is injective. Let $\mathcal{I}$ be the family of open sets in M which are injectively embedded in $\Gamma\backslash M$. Let us prove:

(24)
$$\forall U \in \mathcal{I} : \quad \big[\pi(U) \text{ is open and } \pi_{|U} \text{ is a homeomorphism from } U \text{ onto } \pi(U)\big].$$

Take $U \in \mathcal{I}$. Clearly $\pi_{|U}$ is a bijection of U onto $\pi(U)$. We have also noted that π is continuous. Furthermore π is an *open* map, i.e. maps any open subset of M to an open subset of $\Gamma\backslash M$; this is shown by the argument below (23). Using these facts it follows that $\pi(U)$ is open and $\pi_{|U}$ is a homeomorphism from U onto $\pi(U)$, i.e. (24) is proved. Next we claim:

(25)
$$\mathcal{I} \text{ cover } M; \text{ that is, } \bigcup_{U\in\mathcal{I}} U = M.$$

[Proof: Let $p \in M$. By a slight modification of the construction we used when proving that $\Gamma\backslash M$ is Hausdorff, we are going to construct an open neighborhood of p in M which is injectively embedded in $\Gamma\backslash M$. Choose an open set $U \subset M$ containing p such that $\overline{U}$ is compact. Then the set

$$F := \{\gamma \in \Gamma \ : \ \gamma(\overline{U}) \cap \overline{U} \neq \emptyset\}$$

is finite, since Γ acts properly discontinuously on M. Take any $\gamma \in F \setminus \{\text{Id}\}$. Then $\gamma(p) \neq p$, since Γ acts freely on M. Hence there exist open sets U_γ, V_γ such that $p \in U_\gamma$, $\gamma(p) \in V_\gamma$, and $U_\gamma \cap V_\gamma = \emptyset$. Set

$$U_1 := U \cap \Big(\bigcap_{\gamma\in F\setminus\{\text{Id}\}} (U_\gamma \cap \gamma^{-1}(V_\gamma)) \Big).$$

Then U_1 is an open set in M (since F is finite) and $p \in U_1$. We claim that U_1 is injectively embedded in $\Gamma\backslash M$, i.e. $U_1 \in \mathcal{I}$. Indeed, assume the opposite. Then there exist two points $q \neq q' \in U_1$ with $[q] = [q']$, i.e. $\gamma(q) = q'$ for some $\gamma \in \Gamma$. We have $\gamma \neq \text{Id}$ since $q' \neq q$. Also $q' \in \gamma(U_1) \cap U_1$, thus $\gamma(U_1) \cap U_1 \neq \emptyset$ and so (using $U_1 \subset U$) $\gamma \in F$. But now by the definition of U_1, $q \in U_1$ implies $q \in U_\gamma$, and $\gamma(q) = q' \in U_1$ implies $q \in V_\gamma$. Hence $U_\gamma \cap V_\gamma = \emptyset$, contradicting our choice of U_γ, V_γ. This proves that $U_1 \in \mathcal{I}$.]

Now to prove that $\Gamma\backslash M$ is locally Euclidean, consider an arbitrary point in $\Gamma\backslash M$, say $[p]$ with $p \in M$. By (25) there is a set $U \in \mathcal{I}$ with $p \in U$. Also, since M is a topological manifold, there is a chart (V, x) with $p \in V$. Then also $(U \cap V, x_{|U\cap V})$ is a chart on M. It follows from (24) that $\pi_{|U\cap V}$ is a homeomorphism from $U \cap V$ onto the open set $\pi(U \cap V) \subset \Gamma\backslash M$. Hence $x \circ (\pi_{|U\cap V})^{-1}$ is a homeomorphism from $\pi(U \cap V)$ onto an open subset of $\mathbb{R}^d$. The fact that every point $[p]$ in $\Gamma\backslash M$ has such an open neighborhood which is homeomorphic to an open subset of $\mathbb{R}^d$ proves that $\Gamma\backslash M$ is locally Euclidean.

It now only remains to prove that $\Gamma\backslash M$ is paracompact. Since we have proved that $\Gamma\backslash M$ is Hausdorff and locally Euclidean, it actually suffices to prove that $\Gamma\backslash M$ is second countable. (Indeed, cf. the notes to Lecture #1, with reference to math.stackexchange.com/questions/527642.) However this is quite trivial, using the fact that M is second countable, and the fact that $\pi : M \to \Gamma\backslash M$ is open and continuous. Indeed, let $\mathcal{U}$ be a countable base of M (as a topological space). Set

$$\mathcal{U}' = \{\pi(U) \,:\, U \in \mathcal{U}\}.$$

This is a countable family of open sets in $\Gamma\backslash M$, since π is open. We claim that $\mathcal{U}'$ is a base for $\Gamma\backslash M$. To prove this, take an arbitrary open set $V \subset \Gamma\backslash M$. Then $\pi^{-1}(V)$ is an open set in M, and since $\mathcal{U}$ is a base for M there is a subfamily $\mathcal{V} \subset \mathcal{U}$ such that $\pi^{-1}(V) = \cup_{U\in\mathcal{V}} U$. Applying π to each point in this set identity we obtain $\pi(\pi^{-1}(V)) = \cup_{U\in\mathcal{V}} \pi(U)$. But $\pi(\pi^{-1}(V)) = V$ (since π is surjective). Hence $V = \cup_{U\in\mathcal{V}} \pi(U)$, and this means that V is a union of certain sets in $\mathcal{U}'$. Hence we have proved that $\mathcal{U}'$ is a (countable) base for $\Gamma\backslash M$, and hence $\Gamma\backslash M$ is second countable. □

(c). Let $\mathcal{A}$ be the C^∞ structure on M. Set

$$\mathcal{A}' := \{(\pi(U), x \circ (\pi_{|U})^{-1}) \,:\, (U, x) \in \mathcal{A},\, U \in \mathcal{I}\}.$$

Using (24) we see that each element in $\mathcal{A}'$ is a chart on $\Gamma\backslash M$. We wish to prove that $\mathcal{A}'$ is an atlas on $\Gamma\backslash M$, i.e. that the charts in $\mathcal{A}'$ cover $\Gamma\backslash M$. For this consider an arbitrary point in $\Gamma\backslash M$, say $[p]$ with $p \in M$. By (25) there is some $U_1 \in \mathcal{I}$ with $p \in U_1$. Take any chart $(U_2, x) \in \mathcal{A}$ with $p \in U_2$. Then also $(U_1 \cap U_2, x_{|U_1 \cap U_2}) \in \mathcal{A}$, since $\mathcal{A}$ is a maximal C^∞ atlas. But $U_1 \cap U_2 \subset U_1$ and $U_1 \in \mathcal{I}$ implies that $U_1 \cap U_2 \in \mathcal{I}$. Hence

$$(\pi(U_1 \cap U_2), x_{|U_1 \cap U_2} \circ (\pi_{|U_1 \cap U_2})^{-1}) \in \mathcal{A}',$$

and we have $[p] \in \pi(U_1 \cap U_2)$ since $p \in U_1 \cap U_2$. This completes the proof that $\mathcal{A}'$ is an atlas on $\Gamma\backslash M$.

Next we prove that $\mathcal{A}'$ is in fact a C^∞ atlas. Consider any two charts in $\mathcal{A}'$, say

$$(\pi(U), x \circ (\pi_{|U})^{-1}) \quad \text{and} \quad (\pi(V), y \circ (\pi_{|V})^{-1}), \tag{26}$$

for some $(U, x), (V, y) \in \mathcal{A}$, $U, V \in \mathcal{I}$. We have to prove that the two charts in (26) are compatible, i.e. that the map

$$y \circ (\pi_{|V})^{-1} \circ \pi_{|U} \circ (x_{|U\cap V})^{-1} : \quad x(U \cap V) \to y(U \cap V)$$

is C^∞. However this map is equal to $y \circ (x_{|U\cap V})^{-1}$, which we know is C^∞ since $(U, x), (V, y) \in \mathcal{A}$ and $\mathcal{A}$ is a C^∞ atlas. Hence $\mathcal{A}'$ is indeed a C^∞ atlas on $\Gamma\backslash M$, and so determines a unique C^∞ structure on $\Gamma\backslash M$ (cf. Problem 4).

Finally note that for any $(U, x) \in \mathcal{A}$ with $U \in \mathcal{I}$, the map π is represented by

$$(x \circ (\pi_{|U})^{-1}) \circ \pi \circ x^{-1} \,:\, x(U) \to x(U)$$

with respect to the charts $(U, x) \in \mathcal{A}$ and $(\pi(U), x \circ (\pi_{|U})^{-1}) \in \mathcal{A}'$. But the above map is simply the identity map on $x(U)$, which of course is C^∞. This proves that the map $\pi : M \to \Gamma\backslash M$ is C^∞. $\square$

Problem 10:

(a) If V is open (in M) then $V \cap U_\alpha$ is open in U_α for every α by definition of the subspace topology of U_α.

Conversely, assume that $V \cap U_\alpha$ is open in U_α for every α. By definition of the subspace topology of U_α, this means that for each α there exists an open set $W \subset M$ such that $V \cap U_\alpha = W \cap U_\alpha$, and hence $V \cap U_\alpha$ is open as a subset of M. Now $V = \cup_\alpha (V \cap U_\alpha)$ since $M = \cup_\alpha U_\alpha$; thus V is a union of open subsets of M and therefore V is itself an open subset of M. □

(b) Let $\mathcal{T}$ be the family of all "open" sets in a given "C^∞ fold" M. We have to prove that (i) $\emptyset \in \mathcal{T}$, (ii) $M \in \mathcal{T}$, and that $\mathcal{T}$ is closed under (iii) arbitrary unions and under (iv) finite intersections:

(i) For every $\alpha \in A$ we have $x_\alpha(\emptyset \cap U_\alpha) = \emptyset$ and this is an open subset of $\mathbb{R}^d$. Hence $\emptyset \in \mathcal{T}$.

(ii) For every $\alpha \in A$ we have $x_\alpha(M \cap U_\alpha) = x_\alpha(U_\alpha)$, which is an open subset of $\mathbb{R}^d$ by our assumptions. Hence $M \in \mathcal{T}$.

(iii) Let $\{V_\beta\}_{\beta \in B}$ be an arbitrary family of sets in $\mathcal{T}$. Then for every $\alpha \in A$,

$$x_\alpha\big((\cup_{\beta \in B} V_\beta) \cap U_\alpha\big) = x_\alpha\big(\cup_{\beta \in B}\big(V_\beta \cap U_\alpha\big)\big) = \cup_{\beta \in B}\, x_\alpha\big(V_\beta \cap U_\alpha\big),$$

and here $x_\alpha\big(V_\beta \cap U_\alpha\big)$ is an open subset in $\mathbb{R}^d$ for every $\beta \in B$, since $V_\beta \in \mathcal{T}$. Hence $x_\alpha\big((\cup_{\beta \in B} V_\beta) \cap U_\alpha\big)$, being a union of open subsets of $\mathbb{R}^d$, is itself an open subset of $\mathbb{R}^d$. This is true for every $\alpha \in A$; hence $\cup_{\beta \in B} V_\beta \in \mathcal{T}$.

(iv) Let $\{V_\beta\}_{\beta \in B}$ be a finite family of sets in $\mathcal{T}$. Then for every $\alpha \in A$,

$$x_\alpha\big((\cap_{\beta \in B} V_\beta) \cap U_\alpha\big) = x_\alpha\big(\cap_{\beta \in B}\big(V_\beta \cap U_\alpha\big)\big) = \cap_{\beta \in B}\, x_\alpha\big(V_\beta \cap U_\alpha\big).$$

(The last equality holds since x_α is injective.) Here $x_\alpha\big(V_\beta \cap U_\alpha\big)$ is an open subset in $\mathbb{R}^d$ for every $\beta \in B$, since $V_\beta \in \mathcal{T}$. Hence $x_\alpha\big((\cap_{\beta \in B} V_\beta) \cap U_\alpha\big)$, being a finite intersection of open subsets of $\mathbb{R}^d$, is itself an open subset of $\mathbb{R}^d$. This is true for every $\alpha \in A$; hence $\cap_{\beta \in B} V_\beta \in \mathcal{T}$.

This completes the proof that $\mathcal{T}$ is a topology. □

Next we give examples showing that $\mathcal{T}$ is not always Hausdorff: Let $U' \subsetneq U$ be non-empty open subsets of $\mathbb{R}^d$ and let M be the set

$$M := U' \sqcup ((U \setminus U') \times \{1, 2\}).$$

M can be thought of as two copies of the set U, glued together along the set U'.

For $j = 1, 2$ we define the subset $U_j \subset M$ by

$$U_j := U' \sqcup ((U \setminus U') \times \{j\}),$$

and let $x_j : U_j \to U$ be the map defined by $x_j(p) = p$ for $p \in U'$, and $x_j((p,j)) = p$ for $p \in U \setminus U'$. Then x_j is a bijection from U_j onto U, and $M = U_1 \cup U_2$. Furthermore $x_1(U_1 \cap U_2) = x_2(U_1 \cap U_2) = U'$, an open subset of U, and both the maps $x_2 \circ x_1^{-1}$ and $x_1 \circ x_2^{-1}$ are equal to the identity map on U', which is C^∞. Hence M with the family $\{(U_1, x_1), (U_2, x_2)\}$ is a C^∞ fold.

Now fix any point $p \in U \setminus U'$ not lying in the interior of $U \setminus U'$; such a point certainly exists. (Indeed we can find such a point on any line segment between a point in U' and a point in $U \setminus U'$.) Then every open subset V of U containing p has nonempty intersection with U'. Now consider the two points $(p,1)$ and $(p,2)$ in M. Let W_1, W_2 be any two open subsets of M such that $(p,1) \in W_1$ and $(p,2) \in W_2$. Then for both $j = 1,2$ we have that $x_j(W_j \cap U_j)$ is an open subset of $x_j(U_j) = U$ containing $x_j((p,j)) = p$. Hence also $x_1(W_1 \cap U_1) \cap x_2(W_2 \cap U_2)$ is an open subset of U containing p, and as we noted above this implies that this set has nonempty intersection with U', i.e. there exists a point

$$q \in U' \cap x_1(W_1 \cap U_1) \cap x_2(W_2 \cap U_2).$$

By the definitions of x_1, x_2 it then follows that $q \in W_1 \cap W_2$. Thus we have proved that for *any* two open subsets W_1, W_2 of M subject to $(p,1) \in W_1$ and $(p,2) \in W_2$, it holds that $W_1 \cap W_2 \neq \emptyset$. Hence M is not Hausdorff. $\square$

(Compare Boothby [1, p. 59, Exc. 5]; note that the above shows that the answer to that question is NO.)

(c) Assume that the stated criterion holds. Let p, q be two distinct points in M. By assumption there is $\alpha \in A$ such that $p, q \in U_\alpha$. Now $x_\alpha(p) \neq x_\alpha(q)$ since x_α is a bijection, and hence (since $\mathbb{R}^d$ is Hausdorff) there exist two disjoint open subsets $W, W' \subset x_\alpha(U_\alpha)$ such that $x_\alpha(p) \in W$, $x_\alpha(q) \in W'$. Then $x_\alpha^{-1}(W)$ and $x_\alpha^{-1}(W')$ are disjoint, and $p \in x_\alpha^{-1}(W)$, $q \in x_\alpha^{-1}(W')$. Hence if we can prove that $x_\alpha^{-1}(W)$ and $x_\alpha^{-1}(W')$ are open in M then we are done. By definition $x_\alpha^{-1}(W)$ is open in M iff $x_\beta(x_\alpha^{-1}(W) \cap U_\beta)$ is open in $\mathbb{R}^d$ for every $\beta \in A$. But note that

$$x_\beta(x_\alpha^{-1}(W) \cap U_\beta) = \{p \in x_\beta(U_\alpha \cap U_\beta) \,:\, x_\alpha \circ x_\beta^{-1}(p) \in W\} = \varphi^{-1}(W),$$

where $\varphi := x_\alpha \circ x_\beta^{-1} \,:\, x_\beta(U_\alpha \cap U_\beta) \to \mathbb{R}^d$. (We have that φ is a bijection of $x_\beta(U_\alpha \cap U_\beta)$ onto $x_\alpha(U_\alpha \cap U_\beta)$.) By assumption φ is C^∞, in particular continuous; hence since W is open also $\varphi^{-1}(W)$ is open, and we have thus completed the proof that $x_\alpha^{-1}(W)$ is open in M. Of course the same argument shows that $x_\alpha^{-1}(W')$ is open in M. Done!

(Next we prove that the "partial converse". Thus let M be a C^∞ manifold and let $p, q \in M$. If $p = q$ then the desired statement is trivial; hence from now on we assume $p \neq q$. Then, since M is Hausdorff, there exist open sets $U_1, V_1 \subset M$ with $p \in U_1$, $q \in V_1$ and $U_1 \cap V_1 = \emptyset$. Let (U, x) and (V, y) be

C^∞ charts on M with $p \in U$ and $q \in V$. Then also $(U \cap U_1, x_{|U\cap U_1})$ and $(V \cap V_1, y_{|V\cap V_1})$ are C^∞ charts on M, and after replacing (U,x) and (V,y) with these, we have:

$$U \cap V = \emptyset.$$

We may assume that $x(p) \neq y(q)$; indeed otherwise replace y by the map $r \mapsto v + y(r)$, $V \to \mathbb{R}^d$, where v is a fixed non-zero vector in $\mathbb{R}^d$. Then we can choose open sets (e.g. open balls) U' and V' in $\mathbb{R}^d$ such that $x(p) \in U'$, $y(q) \in V'$ and $U' \cap V' = \emptyset$. Now $(x^{-1}(U'), x_{|x^{-1}(U')})$ and $(y^{-1}(V'), y_{|y^{-1}(V')})$ are C^∞ charts on M, and after replacing (U,x) and (V,y) with these, we have *both*

$$U \cap V = \emptyset \quad \text{and} \quad x(U) \cap y(V) = \emptyset.$$

Now define the map $z : U \cup V \to \mathbb{R}^d$ by:

$$z(p) := \begin{cases} x(p) & \text{if } p \in U \\ y(p) & \text{if } p \in V. \end{cases}$$

Using the fact that $x(U)$ and $y(V)$ are disjoint open subsets of $\mathbb{R}^d$ (and the fact that $x : U \to x(U)$ and $y : V \to y(V)$ are homeomorphisms) it follows that z is a homeomorphism from $U \cup V$ onto $z(U \cup V) = x(U) \cup y(V)$. Hence $(U \cup V, z)$ is a chart on M, and one easily verifies that it is a C^∞ chart. This C^∞ chart has the desired property, namely $p, q \in U \cup V$!) □

(d) It is immediate from the definitions that U_α is open in M for every $\alpha \in A$. Now the only thing that has to be verified is that for every $\alpha \in A$, the map x_α is a homeomorphism of U_α onto $x_\alpha(U_\alpha) \subset \mathbb{R}^d$. Recall that $x_\alpha(U_\alpha)$ is open in $\mathbb{R}^d$ by assumption, and also x_α is a bijection from U_α onto $x_\alpha(U_\alpha) \subset \mathbb{R}^d$. First let V be an arbitrary open subset of U_α; then by the definition of the topology on M, $x_\alpha(V) = x_\alpha(V \cap U_\alpha)$ is an open subset of $x_\alpha(U_\alpha)$. This proves that x_α is *open.* In order to prove that x_α is *continuous*, let W be an arbitrary open subset of $x_\alpha(U_\alpha)$. Then we have to prove that $x_\alpha^{-1}(W)$ is open in M. This is done by the argument in part (c). □

Problem 11.

(a). By [12, Lemma 1.1.1] there exists a locally finite refinement $\mathcal{V} = (V_\beta)_{\beta\in B}$ of $\mathcal{U}$ and C_0^∞ functions $\psi_\beta : M \to [0,1]$ with $\operatorname{supp}\psi_\beta \in V_\beta$ $(\forall \beta \in B)$ and $\sum_{\beta\in B}\psi_\beta(x) = 1$ $(\forall x \in M)$. Now since $\mathcal{V}$ is a local refinement of $\mathcal{U}$, we can choose (using the axiom of choice, in general), for each $\beta \in B$, some $\alpha(\beta) \in A$ so that $V_\beta \subset U_{\alpha(\beta)}$. Having made such a choice, we define, for each $\alpha \in A$:

$$\varphi_\alpha := \sum_{\substack{\beta\in B\\(\alpha(\beta)=\alpha)}} \psi_\beta.$$

(The sum is taken over all $\beta \in B$ which satisfy $\alpha(\beta) = \alpha$.) We claim that these functions φ_α satisfy all the requirements in the problem formulation!

To prove this, let p be an arbitrary point in M. Then there is an open neighborhood $\Omega \subset M$ of p such that the set

$$B_\Omega := \{\beta \in B \,:\, V_\beta \cap \Omega \neq \emptyset\}$$

is finite. Now for $p \in \Omega$ we have

$$\varphi_\alpha(p) = \sum_{\substack{\beta\in B_\Omega\\(\alpha(\beta)=\alpha)}} \psi_\beta(p) \qquad (p \in \Omega).$$

In other words:

$$\varphi_{\alpha|\Omega} = \sum_{\beta\in B_{\Omega,\alpha}} \psi_{\beta|\Omega} \tag{27}$$

where $B_{\Omega,\alpha} = \{\beta \in B_\Omega \,:\, \alpha(\beta) = \alpha\}$. This says that, for every $\alpha \in A$, $\varphi_{\alpha|\Omega}$ is a finite sum of C^∞ functions; hence $\varphi_{\alpha|\Omega}$ is itself a C^∞ function. Since every point $p \in M$ has such a neighborhood Ω, we conclude that φ_α is C^∞ $(\forall \alpha \in A)$.

Furthermore, from the definition of φ_α, and the fact that each ψ_β takes values in $[0,1]$ and $\sum_{\beta\in B}\psi_\beta \equiv 1$, it follows that $\varphi_\alpha(p) \in [0,1]$ for all $p \in M$. We also note that for every $p \in M$ we have

$$\sum_{\alpha\in A}\varphi_\alpha(p) = \sum_{\alpha\in A}\Bigg(\sum_{\substack{\beta\in B\\(\alpha(\beta)=\alpha)}} \psi_\beta(p)\Bigg) = \sum_{\beta\in B}\psi_\beta(p) = 1. \tag{28}$$

(The second equality follows by simply changing the order of summation; this is permitted since all the terms are nonnegative, and the total sum is convergent. In fact the sum $\sum_{\alpha\in A}\varphi_\alpha(p)$ has only finitely many nonvanishing terms; indeed for Ω as above we can have $\varphi_\alpha(p) > 0$ only if α is in the finite set $\{\alpha(\beta) \,:\, \beta \in B_\Omega\}$.)

Now it only remains to prove that $\operatorname{supp}\varphi_\alpha \subset U_\alpha$, $\forall \alpha \in A$. To prove this, fix $\alpha \in A$, and fix an arbitrary point $p \in M \setminus U_\alpha$. Take a neighborhood Ω

of p as above, i.e. so that the set B_Ω is finite. Recall the formula (27). For each $\beta \in B_{\Omega,\alpha}$ we have $\operatorname{supp} \psi_\beta \subset V_\beta \subset U_{\alpha(\beta)} = U_\alpha$. Hence also

$$F := \bigcup_{\beta \in B_{\Omega,\alpha}} \operatorname{supp} \psi_\beta \subset U_\alpha.$$

Also $\operatorname{supp} \psi_\beta$ is a closed (even compact) subset of M for every β; hence since $B_{\Omega,\alpha}$ is finite, the set F is also a closed subset of M. Hence

$$\Omega' := \Omega \setminus F$$

is an open subset of M. Note that $p \in \Omega'$ since $p \in \Omega$, $F \subset U_\alpha$ and $p \notin U_\alpha$. Also, by (27) and our definition of F, we have $\varphi_\alpha(q) = 0$ for all $q \in \Omega'$. Hence, since Ω' is open, Ω' is disjoint from $\operatorname{supp} \varphi_\alpha$, and in particular $p \notin \operatorname{supp} \varphi_\alpha$. To sum up, we have proved that every point $p \in M \setminus U_\alpha$ lies outside $\operatorname{supp} \varphi_\alpha$. Hence $\operatorname{supp} \varphi_\alpha \subset U_\alpha$, and we are done! □

(b). (We take the proof from [4, Lemma 9.5.2].)

Let us start from a partition of unity $(\varphi_\alpha)_{\alpha \in A}$ either as in [12, Lemma 1.1.1] or as in part (a). This means in particular that each φ_α is a C^∞ function $M \to [0, 1]$ and that $\sum_{\alpha \in A} \varphi_\alpha(p) = 1$ for all α. Also every point $p \in M$ has an open neighborhood Ω in M such that $\varphi_{\alpha|\Omega} \equiv 0$ for all except finitely many $\alpha \in A$ (in the case of [12, Lemma 1.1.1] this is clear from the statement, and in the case of part (a) it is a fact we noted in the proof; see the text below (28)). Now set

$$\Phi(p) = \sum_{\alpha \in A} \varphi_\alpha(p)^2 \qquad (p \in M). \tag{29}$$

Note that the "local finiteness" of the sum $\sum_{\alpha \in A} \varphi_\alpha$ mentioned above implies a similar local finiteness for the sum in (29), and in particular $\Phi \in C^\infty(M)$ (i.e. Φ is a C^∞ function $M \to \mathbb{R}$). Furthermore for every $p \in M$ we have $\Phi(p) > 0$, since $\sum_{\alpha \in A} \varphi_\alpha(p) = 1$ implies that there is at least one $\alpha \in A$ with $\varphi_\alpha(p) > 0$. Hence also $p \mapsto \Phi(p)^{-1}$ is a C^∞ function on M, and so the functions

$$\eta_\alpha := \Phi^{-1} \cdot \varphi_\alpha^2$$

are C^∞, for every $\alpha \in A$. It is also clear from the definition that each function η_α takes values in $\mathbb{R}_{\geq 0}$, and that $\operatorname{supp} \eta_\alpha = \operatorname{supp} \varphi_\alpha$. Furthermore, for every $p \in M$:

$$\sum_{\alpha \in A} \eta_\alpha(p) = \Phi(p)^{-1} \sum_{\alpha \in A} \varphi_\alpha(p)^2 = 1.$$

(Hence also $\eta_\alpha(p) \in [0, 1]$ for all $\alpha \in A$.) Hence the functions $(\eta_\alpha)_{\alpha \in A}$ satisfy all the requirements which were imposed on $(\varphi_\alpha)_{\alpha \in A}$, and furthermore $\sqrt{\eta_\alpha} = \Phi^{-1/2} \varphi_\alpha$ is a C^∞ function for every $\alpha \in A$. □

Problem 12.

(a) [We leave it to the reader to sort out certain details in the proof below, hidden in phrases such as "passing to local coordinates"; "translation and rotation"; etc; what we are doing there is creating a new C^∞ chart by composing by appropriate diffeomorphism(s)...]

Passing to local coordinates we may assume $M = \mathbb{R}^n$. After a rotation and a scaling we may also assume $\dot{c}(s) = \boldsymbol{e}_1 := (1, 0, \ldots, 0)$. Let us write $c(t) = (c_1(t), \ldots, c_n(t))$; then $c_1'(s) = 1$ and $c_j'(s) = 0$ for $j \geq 2$. It follows that there is $\varepsilon > 0$ such that c_1 restricted to $(s-\varepsilon, s+\varepsilon)$ is a diffeomorphism onto an open interval $I \subset \mathbb{R}$. Let $\alpha_1 : I \to (s-\varepsilon, s+\varepsilon)$ be the inverse diffeomorphism. Then $\alpha_1(c_1(t)) = t$ for all $t \in (s-\varepsilon, s+\varepsilon)$. Define

$$\alpha : I \times \mathbb{R}^{n-1} \to (s-\varepsilon, s+\varepsilon) \times \mathbb{R}^{n-1};$$
$$\alpha(x_1, \ldots, x_n) := (\alpha_1(x_1) - s, x_2, \ldots, x_n).$$

Then α is a diffeomorphism of $I \times \mathbb{R}^{n-1}$ onto $(-\varepsilon, +\varepsilon) \times \mathbb{R}^{n-1}$, and $\alpha(c(t)) = (t-s, *, \ldots, *)$ for all $t \in (s-\varepsilon, s+\varepsilon)$. Hence after composing our coordinate chart with α, we have $c_1(t) = t-s$ for all $t \in (s-\varepsilon, s+\varepsilon)$. Finally we consider the map

$$\beta : (-\varepsilon, \varepsilon) \times \mathbb{R}^{n-1} \to (-\varepsilon, \varepsilon) \times \mathbb{R}^{n-1}$$
$$\beta(x_1, \ldots, x_n) = (x_1, x_2 - c_2(s + x_1), \ldots, x_n - c_n(s + x_1)).$$

Note that β is a C^∞ diffeomorphism of $(-\varepsilon, \varepsilon) \times \mathbb{R}^{n-1}$ onto $(-\varepsilon, \varepsilon) \times \mathbb{R}^{n-1}$; indeed β is C^∞ and the inverse map is

$$(x_1, \ldots, x_n) \mapsto (x_1, x_2 + c_2(s + x_1), \ldots, x_n + c_n(s + x_1)),$$

which is also C^∞. Then

$$\beta(c(t)) = (t-s, 0, \ldots, 0), \qquad \forall t \in (-\varepsilon, \varepsilon).$$

Hence by composing our coordinate chart with β, we obtain a coordinate chart with the desired property! □

(b) Take $\varepsilon > 0$ and a chart (U, x) as in part (a). After possibly shrinking U, we may assume that $s + x_1(p) \in (a, b)$ for all $p \in U$. Define

$$h : U \to \mathbb{R}, \qquad h(p) := f(s + x_1(p)).$$

Then h is a C^∞ function and $h(c(t)) = f(s + x_1(c(t))) = f(t)$ for all $t \in (s-\varepsilon, s+\varepsilon)$. Now fix any open neighborhood $U_1 \subset U$ of $c(s)$ having compact closure $\overline{U}_1$ in U. Then by Problem 7(d), there exists a C^∞ function $g : M \to \mathbb{R}$ which satisfies $g_{|U_1} \equiv h_{|U_1}$. By shrinking ε, we may assume that $c(t) \in U_1$ for all $t \in (s-\varepsilon, s+\varepsilon)$. Then $g(c(t)) = h(c(t)) = f(t)$ for all $t \in (s-\varepsilon, s+\varepsilon)$, and we are done. □

Problem 13:

(a). Recall that for a given C^∞ manifold M and a point $p \in M$, we consider the set

(30)
$$S := \{(U, x, u) \,:\, (U, x) \text{ is a chart on } M \text{ with } p \in U, \text{ and } u \in T_{x(p)}(x(U))\},$$

(where $T_{x(p)}(x(U)) := \mathbb{R}^d$) and define the relation $\sim$ on S by

$$(U, x, u) \sim (V, y, v) \quad \overset{\text{def}}{\Leftrightarrow} \quad u = d(x \circ y^{-1})_{y(p)}(v).$$

We now prove that $\sim$ is an equivalence relation. For any $(U, x, u) \in S$ we have that $x \circ x^{-1}$ equals the identity map on $x(U) \subset \mathbb{R}^d$, thus the Jacobian $d(x \circ x^{-1})$ is the identity map on $T_{x(p)}(x(U)) = \mathbb{R}^d$, and so $d(x \circ x^{-1})(u) = u$. Hence $\sim$ is reflexive.

Next to prove that $\sim$ is symmetric, assume $(U, x, u) \sim (V, y, v)$, i.e. $u = d(x \circ y^{-1})_{y(p)}(v)$. Then

$$\begin{aligned} d(y \circ x^{-1})_{x(p)}(u) &= d(y \circ x^{-1})_{x(p)} \circ d(x \circ y^{-1})_{y(p)}(v) \\ &= d(y \circ x^{-1} \circ x \circ y^{-1})_{y(p)}(v) \\ &= d(1_{y(V)})_{y(p)}(v) = v. \end{aligned}$$

(Explanation: For the second equality we used the chain rule, cf. p. 3 of Lecture #2. In the last line, "$1_{y(V)}$" is the identity map on the set $y(V)$; its differential at $y(p)$ is of course the identity map on $T_{y(p)}(y(V)) = \mathbb{R}^d$.) Hence $(V, y, v) \sim (U, x, u)$. This proves that $\sim$ is symmetric.

Finally we prove that $\sim$ is transitive. Assume $(U, x, u) \sim (V, y, v)$ and $(V, y, v) \sim (W, z, w)$, i.e. $u = d(x \circ y^{-1})_{y(p)}(v)$ and $v = d(y \circ z^{-1})_{z(p)}(w)$. Then

$$\begin{aligned} u &= d(x \circ y^{-1})_{y(p)} \circ d(y \circ z^{-1})_{z(p)}(w) \\ &= d(x \circ y^{-1} \circ y \circ z^{-1})_{z(p)}(w) \\ &= d(x \circ z^{-1})_{z(p)}(w) \end{aligned}$$

(here we again used the chain rule), and thus $(U, x, u) \sim (W, z, w)$. This proves that $\sim$ is transitive.

Hence $\sim$ is an equivalence relation. $\square$

(b). Injectivity: Assume that $u, v \in \mathbb{R}^d$ give $[(U, x, u)] = [(U, x, v)]$. This means that $(U, x, u) \sim (U, x, v)$, i.e. $u = d(x \circ x^{-1})_{x(p)}(v)$. But $d(x \circ x^{-1})_{x(p)}$ is the identity map on $\mathbb{R}^d$; hence $u = v$. This proves that the given map is injective.

Surjectivity: Consider an arbitrary element in T_pM; we can always represent it as $[(V, y, v)]$ for some $(V, y, v) \in S$ (cf. (30)). Set

$$u = d(x \circ y^{-1})_{y(p)}(v) \in T_{x(p)}(x(U)) = \mathbb{R}^d.$$

Then

$$d(y \circ x^{-1})_{x(p)}(u) = d(y \circ x^{-1})_{x(p)} \circ d(x \circ y^{-1})_{y(p)}(v) = d1_{y(p)}(v) = v,$$

and thus $(U, x, u) \sim (V, y, v)$, i.e. $[(U, x, u)] = [(V, y, v)]$. In other words, the image of u under the given map equals $[(V, y, v)]$. This proves that the given map is surjective. $\square$

(c). Fix $p \in M$. Let T be a bijective linear map $V \to \mathbb{R}^d$. Then $(M, T_{|M})$ is a C^∞ chart on M. Consider the map

$$J : V \to T_pM; \qquad J(v) = [(M, T_{|M}, T(v))].$$

It follows from part (b) that J is a bijection of V onto T_pM. We claim that *J is independent of the choice of T.* To prove this, assume that also S is a bijective linear map $V \to \mathbb{R}^d$. Then we need to prove that for every $v \in V$ we have $[(M, T_{|M}, T(v))] = [(M, S_{|M}, S(v))]$ in T_pM. In other words (cf. Def. 3 in Lecture #2), we need to prove

$$S(v) = d(S \circ T^{-1})_{T(p)}(T(v)), \qquad \forall v \in V. \tag{31}$$

Now we note the following very basic fact: "The differential of a linear map is equal to the map itself". More precisely: *For any linear map $L : \mathbb{R}^d \to \mathbb{R}^n$, and any $x \in \mathbb{R}^d$, the differential $dL_x : \mathbb{R}^d \to \mathbb{R}^n$ is equal to the map L itself.* We leave it to the reader to verify this fact; it is of course just a matter of checking that the Jacobian matrix of L, exaluated at any point x, is equal to the matrix of L itself.

Applying the fact just mentioned, with $L = S \circ T^{-1} : \mathbb{R}^d \to \mathbb{R}^d$, we conclude that

$$d(S \circ T^{-1})_{T(p)}(T(v)) = S \circ T^{-1}(T(v)) = S(v),$$

i.e. we have proved (31)! Hence we have proved that our bijection $J : V \to T_pM$ is independent of the choice of linear bijection $V \to \mathbb{R}^d$, i.e. the map J is "canonically defined". Therefore we can use this map J to *identify T_pM with V.* □

(d). Recall from Definition 4 (in Lecture #2) that we assume that $f : M \to N$ is a C^∞ map between C^∞ manifolds, and $p \in M$. (Set $d = \dim M$ and $d' = \dim N$.) Then df_p is defined to be the linear map from T_pM to $T_{f(p)}N$ which wrt any chart (U, x) on M with $p \in U$ and (V, y) on N with $f(p) \in V$ is given by

$$df_p = d(y \circ f \circ x^{-1})_{x(p)} : T_{x(p)}(x(U)) \to T_{y(f(p))}(y(V)).$$

For this to make sense, recall that once the chart (U, x) is given, we can identify T_pM with $T_{x(p)}(x(U)) = \mathbb{R}^d$ via the bijection $v \mapsto [(U, x, v)]$ from $T_{x(p)}(x(U))$ onto T_pM (cf. p. 4 in Lecture #2 and part b of this problem); similarly we can identify $T_{f(p)}N$ with $T_{y(f(p))}(y(V)) = \mathbb{R}^{d'}$. Thus the above definition of df_p can be reformulated as saying that

$$df_p\big([(U, x, v)]\big) := [(V, y, d(y \circ f \circ x^{-1})_{x(p)}(v))], \qquad \forall v \in T_{x(p)}(x(U)) = \mathbb{R}^d.$$

This certainly makes $df_p(\alpha)$ defined for *every* vector $\alpha \in T_pM$ since every $\alpha \in T_pM$ can be expressed as $\alpha = [(U, x, v)]$ for some $v \in T_{x(p)}(x(U))$. The key issue is now to verify that $df_p(\alpha)$ does not depend on the above choice of the charts (U, x) and (V, y)!

Thus assume that $(\hat{U}, \hat{x})$ is also a chart on M with $p \in \hat{U}$ and that $(\hat{V}, \hat{y})$ is a chart on N with $f(p) \in \hat{V}$. Consider a fixed vector $\alpha \in T_p(M)$; assume that α is represented by $v \in \mathbb{R}^d$ wrt (U, x), and by $d(\hat{x} \circ x^{-1})_{x(p)}(v) \in \mathbb{R}^d$ wrt $(\hat{U}, \hat{x})$. Now the above definition says that $df_p(\alpha)$ is the vector in $T_{f(p)}(N)$ which is represented by

$$d(y \circ f \circ x^{-1})_{x(p)}(v) \in \mathbb{R}^d \tag{32}$$

wrt the chart (V, y), but *also* that $df_p(\alpha)$ is the vector in $T_{f(p)}(N)$ which is represented by

$$d(\hat{y} \circ f \circ \hat{x}^{-1})_{\hat{x}(p)} \circ d(\hat{x} \circ x^{-1})_{x(p)}(v) \in \mathbb{R}^d \tag{33}$$

wrt the chart $(\hat{V}, \hat{y})$. Thus we have to prove that (32) and (33) represent the *same* vector in $T_{f(p)}(N)$, i.e. that

$$d(y \circ \hat{y}^{-1})_{\hat{y}(f(p))} \circ d(\hat{y} \circ f \circ \hat{x}^{-1})_{\hat{x}(p)} \circ d(\hat{x} \circ x^{-1})_{x(p)}(v) = d(y \circ f \circ x^{-1})_{x(p)}(v).$$

However this is clear by the chain rule for the differential (for C^∞ maps between vector spaces over $\mathbb{R}$), using the fact that

$$(y \circ \hat{y}^{-1}) \circ (\hat{y} \circ f \circ \hat{x}^{-1}) \circ (\hat{x} \circ x^{-1}) = y \circ f \circ x^{-1}.$$

Done! $\square$

(e). Fix a point $p \in M_1$; then our task is to prove

$$d(g \circ f)_p = dg_{f(p)} \circ df_p \;:\; T_p(M_1) \to T_{g(f(p))}(M_3). \tag{34}$$

Fix charts (U, x) on M_1, (V, y) on M_2, and (W, z) on M_3, satisfying $p \in U$, $f(p) \in V$, $g(f(p)) \in W$. With respect to these charts, $d(g \circ f)_p$ is represented by the map

$$d(z \circ (g \circ f) \circ x^{-1})_{x(p)} \;:\; T_{x(p)}(x(U)) \to T_{z(g(f(p)))}(z(W)),$$

and df_p is represented by the map

$$d(y \circ f \circ x^{-1})_{x(p)} \;:\; T_{x(p)}(x(U)) \to T_{y(f(p))}(y(V)),$$

and $dg_{f(p)}$ is represented by the map

$$d(z \circ g \circ y^{-1})_{y(f(p))} \;:\; T_{y(f(p))}(y(V)) \to T_{z(g(f(p)))}(z(W)).$$

Hence we will have proved (34) if we can prove

$$d(z \circ (g \circ f) \circ x^{-1})_{x(p)} = d(z \circ g \circ y^{-1})_{y(f(p))} \circ d(y \circ f \circ x^{-1})_{x(p)}.$$

However this is clear by the chain rule (for maps on $\mathbb{R}^d$-spaces), since

$$z \circ (g \circ f) \circ x^{-1} = (z \circ g \circ y^{-1}) \circ (y \circ f \circ x^{-1}).$$

□

(f). Fact #1: This is proved as follows:

$$v(f) = df_p(v) = d(1_{\mathbb{R}} \circ f \circ x^{-1})_{x(p)}(v^j \frac{\partial}{\partial x^j}) = \Big(\frac{\partial f}{\partial x^1} \cdots \frac{\partial f}{\partial x^d}\Big) \begin{pmatrix} v^1 \\ \vdots \\ v^d \end{pmatrix} = v^j \frac{\partial f}{\partial x^j}.$$

(Explanation: In the first equality we use our definition of the directional derivative; "$v(f)$". In the second equality we use the definition of differential (Def. 4 in Lecture #2). In the third equality we use the definition of differential for maps between $\mathbb{R}^d$-spaces (Def. 2). The last equality is just matrix multiplication. Note that in the last two expressions "f" in fact stands for the function $f \circ x^{-1} : x(U) \to \mathbb{R}$; this is in accordance with the principle that we may *identify* U with $x(U)$ so long as the notation cannot be misunderstood. Also in the last two expressions it is understood that the partial derivatives are evaluated at the point $x = x(p)$.)

Fact #2: This is proved as follows:

$$\dot{c}(t)(f) = df_{c(t)}(\dot{c}(t)) = df_{c(t)}(dc_t(1)) = d(f \circ c)_t(1) = \frac{d}{dt}(f \circ c)(t).$$

(In the first equality we use the definition of directional derivative; in the second equality we use the definition of "tangent vector of a curve"; and in the third equality we use the chain rule, cf. part (e) of this problem. Finally the fourth equality could also be said to hold by the definition of "tangent vector of a curve"; however since $f \circ c$ is a function from $I \subset \mathbb{R}$ to $\mathbb{R}$, "$\frac{d}{dt}(f \circ c)(t)$" has a more basic meaning as derivative of a real-valued function on $\mathbb{R}$, and of course these two interpretations are really the same and give the same answer – as is easily verified by using the trivial "identity map charts" on I and $\mathbb{R}$.)

Fact #3: Using Fact #1 we have

$$v(fg) = v^j \cdot \frac{\partial(fg)}{\partial x^j}_{|x=x(p)} = v^j \cdot f(p) \cdot \frac{\partial g}{\partial x^j}_{|x=x(p)} + v^j \cdot g(p) \cdot \frac{\partial f}{\partial x^j}_{|x=x(p)}$$
$$= f(p) \cdot v(g) + g(p) \cdot v(f),$$

proving the first formula. Next note that by the definition of directional derivative the same formula can be written:

$$d(fg)_p(v) = f(p) \cdot dg_p(v) + g(p) \cdot df_p(v).$$

The fact that this holds for all $v \in T_pM$ means that

$$d(fg)_p = g(p) \cdot df_p + f(p) \cdot dg_p$$

(equality of linear maps $T_pM \to \mathbb{R}$). □

Problem 14: Once one has gotten used to the machinery which we have introduced, this problem is "completely obvious". However, as a step towards reaching such familiarity, it may be useful to work out a solution in pedantic detail.

We have defined $\dot{c}(t) := dc_t(1)$. Furthermore, for any $t \in I$ with $c(t) \in U$, the differential dc_t is, by definition, the map from $T_t(I) = \mathbb{R}$ to $T_{c(t)}M$ which with respect to the trivial chart $(I, 1_I)$ on I and the chart (U, x) on M, is represented by the linear map

$$d(x \circ c \circ 1_I^{-1})_{1_I(t)} : \mathbb{R} \to \mathbb{R}^d.$$

This map equals $d(x \circ c)_t$, and so we get that $\dot{c}(t) := dc_t(1)$ is the vector in $T_{c(t)}M$ which with respect to the chart (U, x) is represented by

$$d(x \circ c)_t(1) \in \mathbb{R}^d. \tag{35}$$

But by the definition in the problem formulation,

$$x \circ c(t) = (c^1(t), \ldots, c^d(t))$$

for all $t \in I$ with $c(t) \in U$, and hence $d(x \circ c)_t$ is the linear map given by the (Jacobi) matrix

$$\begin{pmatrix} \dfrac{\partial c^1}{\partial t} \\ \vdots \\ \dfrac{\partial c^d}{\partial t} \end{pmatrix}.$$

Of course "∂" can just as well be written "d" since each c^j depends on only one variable; i.e. the entries of the above matrix are $\frac{d}{dt}c^j(t) = \dot{c}^j(t)$ for $j = 1, \ldots, d$. Applying the above linear map to the vector $1 \in \mathbb{R}$ (so as to evaluate the expression in (35)) we find[10] that with respect to the chart (U, x), $\dot{c}(t)$ is represented by

$$(\dot{c}^1(t), \ldots, \dot{c}^d(t)) \in \mathbb{R}^d \tag{36}$$

Next we turn to the right hand side of the desired formula, i.e. "$\dot{c}^j(t)\dfrac{\partial}{\partial x^j}$". Recall that by definition, at any point $p \in U$, $\dfrac{\partial}{\partial x^j}$ is the tangent vector in T_pM which with respect to the chart (U, x) is represented by the standard unit vector $e_j = (0, \ldots, 1, \ldots, 0) \in \mathbb{R}^d$ (where the "1" is in the jth position). Hence, with respect to the chart (U, x), the tangent vector $\dot{c}^j(t)\dfrac{\partial}{\partial x^j} \in T_{c(t)}M$ is represented by

$$\dot{c}^j(t)e_j(t) = (\dot{c}^1(t), 0, \ldots, 0) + \cdots + (0, \ldots, 0, \dot{c}^d(t)) = (\dot{c}^1(t), \ldots, \dot{c}^d(t)).$$

[10] also recalling our convention that column matrices are identified with vectors

This agrees with (36), i.e. we have proved that $\dot{c}(t)$ and $\dot{c}^j(t)\frac{\partial}{\partial x^j}$ are represented by the same vector in $\mathbb{R}^d$ (wrt (U,x)); hence they are equal, i.e. we have proved the desired formula,

$$\dot{c}(t) = \dot{c}^j(t)\frac{\partial}{\partial x^j} \in T_pM.$$

Finally in order to verify that the two definitions of "tangent vector of a curve" in Lecture #2 are consistent with each other, let us apply the above to the special case $M =$ an open subset of $\mathbb{R}^d$. In this case we have the convention that T_pM is *identified* with $\mathbb{R}^d$ for every $p \in \mathbb{R}^d$, namely through the representation of tangent vectors via the *identity chart,* $(M, 1_M)$. Applying the formula which we have proved above (in the form (36)) we conclude that

$$\dot{c}(t) = (\dot{c}^1(t), \ldots, \dot{c}^d(t)) \qquad \text{in } T_{c(t)}M = \mathbb{R}^d.$$

Note that we have proved this formula starting from the general definition of "tangent vector of a curve on a manifold", and we now see that the formula agrees with the concrete definition of "tangent vector of a curve in $\mathbb{R}^d$" which we gave in Lecture #2 (p. 2). □

Problem 15:

(a). Cf., e.g., Boothby [1, Ch. 4.1] or Fieseler [5, Sec. 3]...

(b). Cf., e.g., Helgason, [8, Ch. 1.2.1]...

We here only give the easy part of the solution of part b: For any vector field $X \in \Gamma(TM)$ and any $f \in C^\infty(M)$ we define $Xf \in C^\infty(M)$ by

$$(Xf)(p) = X(p)f, \qquad \forall p \in M.$$

In other words, by definition of directional derivative:

$$(Xf)(p) = df_p(X(p)) \in T_p\mathbb{R} = \mathbb{R}.$$

By definition of the differential $df : TM \to T\mathbb{R} = \mathbb{R} \times \mathbb{R}$, the above formula can also be expressed:

$$Xf = \mathrm{pr}_2 \circ df \circ X \quad : M \to \mathbb{R}.$$

and this shows (via Problem 17(a)) that we indeed have $Xf \in C^\infty(M)$.

Let us note that for any $X \in \Gamma(TM)$, the map which we have now defined,

$$f \mapsto Xf, \qquad C^\infty(M) \to C^\infty(M),$$

is a derivation. (This is immediate from "Fact #3" on p. 9 in Lecture #2; we prove this fact in Problem 13(f), and this fact also plays a crucial role in part a of the present problem.)

Now it remains to prove that *every* derivation of $C^\infty(M)$ is obtained in this way from some $X \in \Gamma(TM)$, and that any two distinct vector fields yield distinct derivations....

Problem 16: As in the lecture, we define TM as a set to be the disjoint union of all tangent spaces T_pM ($p \in M$), and we let $\pi : TM \to M$ be the projection map; $\pi(w) = p$ for any $w \in T_pM$. Also, as a "proposed C^∞ atlas" on TM we take the set

$$\mathcal{A} := \{(TU, \varphi_x) \,:\, (U, x) \text{ any } C^\infty \text{ chart on } M\}$$

where $TU = \pi^{-1}(U) = \sqcup_{p \in U} T_pM$ and φ_x is the map

$$\varphi_x : TU \to \mathbb{R}^{2d} = \mathbb{R}^d \times \mathbb{R}^d,$$
$$\varphi_x(w) = \Big(x(\pi(w)), dx_{\pi(w)}(w) \Big).$$

Clearly for any C^∞ chart (U, x) on M, φ_x is a bijection from TU onto $x(U) \times \mathbb{R}^d$, which is an open subset of $\mathbb{R}^{2d}$; and if also (V, y) is a C^∞ chart on M then $\varphi_x(TU \cap TV) = x(U \cap V) \times \mathbb{R}^d$, which is also an open subset of $\mathbb{R}^{2d}$, and as we verify in the lecture the map $\varphi_y \circ \varphi_x^{-1} : x(U \cap V) \to \mathbb{R}^{2d}$ is C^∞. Hence all the conditions in Problem 10(b) are fulfilled, i.e. TM with the family $\mathcal{A}$ is a "C^∞ fold". In particular TM is now provided with a structure of a topological space, namely a subset $V \subset M$ is open iff $\varphi_x(V \cap TU)$ is open in $\mathbb{R}^{2d}$ for every C^∞ chart (U, x) on M.

Now it suffices to prove that TM is Hausdorff, connected and paracompact; for then it follows from Problem 10(d) that TM is a well-defined C^∞ manifold with $\mathcal{A}$ as a C^∞ atlas!

In order to prove that TM is Hausdorff, take two arbitrary points $v, w \in TM$. Then by the "partial converse" in Problem 10(c) (applied for our C^∞ manifold M) there exists a C^∞ chart (U, x) on M such that $\pi(v), \pi(w) \in U$. But then $v, w \in TU$, and so (TU, φ_x) is a 'chart' in $\mathcal{A}$ with $v, w \in TU$. The fact that $\mathcal{A}$ contains such a chart for any pair of points $v, w \in TM$ implies, by Problem 10(c), that TM is Hausdorff!

Next we prove that TM is connected: Take any $v, w \in TM$. Let $p = \pi(v)$ and $q = \pi(w)$. Let $c_1 : I \to T_pM$ ($I = [0, 1]$) be any curve in the vector space T_pM starting at $v \in T_pM$ and ending at $0 \in T_pM$. Note that the inclusion map $T_pM \to TM$ is continuous. (We leave it as an exercise to verify this fact; note that once TM has been proved to be a C^∞ manifold, the inclusion map $T_pM \to TM$ can be seen to be C^∞.) Hence c_1 is continuous also as a map from I to TM, i.e. c_1 is a curve in TM. Similarly let c_3 be any curve in $T_qM \subset TM$ going from $0 \in T_qM$ to $w \in T_qM$. Next, since M is path-connected (cf. Problem 1), there is a curve $\widetilde{c}_2 : I \to M$ going from p to q. Note that the map $f : M \to TM$ taking any $p \in M$ to the vector $0 \in T_pM$ is continuous. (Again we leave this as an exercise.) Hence $c_2 := f \circ \widetilde{c}_2 : I \to TM$ is a curve in TM, going from the 0-vector in T_pM to the 0-vector in T_qM. Now the "product curve" of c_1, c_2, c_3 [11] is a curve

[11] We will discuss this notion in Lecture #6; however it should hopefully be clear here how the curve in question is constructed; just draw a picutre!

in TM going from v to w. The fact that such a curve exists for any two $v, w \in TM$ implies that TM is path-connected, and hence connected!

Finally we prove that TM is paracompact. Note that by what we have already proved, TM is locally Euclidean and Hausdorff (cf. the solution to Problem 10(d)), and the set $\mathcal{A}$ above is an atlas on TM. By Problem 2 it suffices to prove that TM has a countable atlas. Now fix any countable atlas $\mathcal{A}'$ on M (this exists by Problem 2). Then the following subset of $\mathcal{A}$ is a countable atlas on TM:

$$\{(TU, \varphi_x) \,:\, (U, x) \in \mathcal{A}'\}.$$

This completes the proof that TM is a C^∞ manifold with $\mathcal{A}$ as a C^∞ atlas.

We now turn to the last part of the problem, i.e. to prove that the map $\pi : TM \to M$ is C^∞. For this it suffices to prove that for any C^∞ chart (V, y) on M and any $(TU, \varphi_x) \in \mathcal{A}$ (thus: (U, x) is a C^∞ chart on M), the map

$$y \circ \pi \circ \varphi_x^{-1} : \quad \varphi_x(T(U \cap V)) \to \mathbb{R}^d \tag{37}$$

is C^∞. However, the definition of φ_x says that, for any $p \in U$ and $w \in T_pU \subset TU$:

$$\varphi_x(w) = (x(p), dx_p(w)),$$

and thus

$$\pi \circ \varphi_x^{-1}\Big(x(p), dx_p(w)\Big) = \pi(w) = p.$$

Hence

$$\pi \circ \varphi_x^{-1}(z, v) = x^{-1}(z), \qquad \forall (z, v) \in \varphi_x(TU) = x(U) \times \mathbb{R}^d,$$

or, equivalently,

$$\pi \circ \varphi_x^{-1} = x^{-1} \circ \mathrm{pr} : \quad x(U) \times \mathbb{R}^d \to M,$$

where pr is the projection $\mathrm{pr} : x(U) \times \mathbb{R}^d \to x(U)$. Hence the map in (37) equals $y \circ x^{-1} \circ \mathrm{pr}$, and here $y \circ x^{-1} : x(U \cap V) \to \mathbb{R}^d$ is C^∞ since (U, x) and (V, y) are C^∞ charts on M, and pr is obviously C^∞. Hence the map in (37) is C^∞, and we are done. □

(Remark: Problem 36 gives a more general result.)

Problem 17:

(a). (When showing that df is C^∞ the key step is to verify that if $U \subset \mathbb{R}^d$ is open and $g : U \to \mathbb{R}^{d'}$ is a C^∞ map, then the map $(x, y) \mapsto (g(x), dg_x(y))$, from $U \times \mathbb{R}^d$ to $\mathbb{R}^{d'} \times \mathbb{R}^{d'}$, is C^∞.)

(b). Let $p = \pi(X) \in M$ so that $X \in T_pM$; then $df(X) \in T_{f(p)}N$. By our definition of directional derivative,

$$df(X)(\varphi) = d\varphi(df(X)) \quad \text{in } T_{\varphi(f(p))}(\mathbb{R}) = \mathbb{R}.$$

Also by the same definition,

$$X(\varphi \circ f) = d(\varphi \circ f)(X) \quad \text{in } T_{\varphi(f(p))}(\mathbb{R}) = \mathbb{R}.$$

But $d(\varphi \circ f) = d\varphi \circ df$ by the chain rule, and so the two expressions are equal. $\square$

(c). This is immediate from Problem 13(e). (Indeed, take $v \in TM_1$. Set $p := \pi(v)$; then $v \in T_pM_1$ and now

$$d(g \circ f)(v) = d(g \circ f)_p(v) = dg_{f(p)} \circ df_p(v) = dg_{f(p)}(df(v)) = dg(df(v)),$$

where we used the formula from Problem 13(e) in the second equality.) $\square$

Problem 18:

(a). Using the fact that (in the right hand side) $\langle \cdot, \cdot \rangle$ is a scalar product (viz., a positive definite symmetric bilinear form) on $T_{f(p)}N$, and df_p is a linear map from T_pM to $T_{f(p)}N$, it follows that $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form on T_pM which is positive semidefinite (viz., $\langle v, v \rangle \geq 0$ for all $v \in T_pM$). But using also the assumption that f is an *immersion,* i.e. df_p is injective for each p, it follows that $\langle \cdot, \cdot \rangle$ is in fact positive definite, i.e. a scalar product on T_pM.

It remains to prove that $\langle \cdot, \cdot \rangle$ depends smoothly on M. We leave the details of this to the reader. $\square$

(b). By definition

$$L(f \circ \gamma) = \int_a^b \|(f \circ \gamma)'(t)\| \, dt \tag{38}$$

(with the understanding that the integral has to be "splitted at each point where γ is not C^∞). But here $(f \circ \gamma)'(t) = d(f \circ \gamma)_t(1) = df_{\gamma(t)} \circ d\gamma_t(1)$ (where we used the def of directional derivative of a curve, and then the chain rule), and so

$$\begin{aligned} \|(f\circ\gamma)'(t)\| &= \sqrt{\big\langle df_{\gamma(t)}\circ d\gamma_t(1), df_{\gamma(t)}\circ d\gamma_t(1)\big\rangle} \\ &= \sqrt{\big\langle d\gamma_t(1), d\gamma_t(1)\big\rangle} \\ &= \|\gamma'(t)\|, \end{aligned} \tag{39}$$

where the second equality holds by our definition of $\langle\cdot,\cdot\rangle$ on $T_{\gamma(t)}M$. Combining (38) and (39) we obtain $L(f\circ\gamma) = L(\gamma)$. The proof of $E(f\circ\gamma) = E(\gamma)$ is completely similar. □

(c). [Remark: In the inequality which we are going to prove,

$$d(p,q) \geq d(f(p), f(q)),$$

of course "d" in the left hand side denotes the metric on M induced by the Riemannian structure on M, and "d" in the right hand side denotes the metric on N induced by the Riemannian structure on N. In the special case when f is an *inclusion map*, so that M is as a set is a *subset of* N, one should give different names to these two metrics, e.g. "d_M" and "d_N", otherwise "$d(p,q)$" for $p,q\in M$ is ambiguous!]

Let $p,q\in M$. By definition,

$$d(p,q) = \inf\big\{L(\gamma)\ :\quad \gamma:[a,b]\to M \text{ is a piecewise } C^\infty \text{ curve with } \gamma(a)=p,\ \gamma(b)=q\big\}. \tag{40}$$

and

$$d(f(p),f(q)) = \inf\big\{L(c)\ :\quad c:[a,b]\to N \text{ is a piecewise } C^\infty \text{ curve with } c(a)=f(p),\ c(b)=f(q)\big\}. \tag{41}$$

However for any curve γ satisfying the conditions in the right hand side of (40), $c := f\circ\gamma$ is a piecewise C^∞ curve with $c(a) = f(\gamma(a)) = f(p)$ and $c(b) = f(\gamma(b)) = f(q)$; thus c satisfies the conditions in the right hand side of (41). Also $L(c) = L(\gamma)$, by part (b). Hence every number $L(\gamma)$ appearing in the set in the right hand side of (40) also appears in the set in the right hand side of (41); therefore the infinimum in (40) is $\geq$ the infinimum in (41), i.e. $d(p,q) \geq d(f(p), f(q))$, qed.

Example with strict inequality: Note that this is the usual situation! For example take $M = S^{d-1}$, $N = \mathbb{R}^d$ and let f be the inclusion map. Then $d_M(p,q) > d_N(f(p), f(q))$ for *any* points $p\neq q\in M$. □

Problem 19:

(a). Let $p, q \in M$ be given. By Problem 1 there exists a (continuous) curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\gamma(1) = q$. Let $\mathcal{F}$ be the family of open subintervals $I \subset [0,1]$ [12] such that $c(I)$ is contained in some C^∞ chart on M. Note that $\mathcal{F}$ covers I. Hence since I is compact, there is a finite subfamily $\mathcal{F}_1 \subset \mathcal{F}$ which covers I. In particular some $I \in \mathcal{F}_1$ must contain 0; among all such intervals $I \in \mathcal{F}_1$ we pick the one which has the largest right end-point; it is either $[0,1]$ or $[0, t_1)$ for some $t_1 \in (0,1)$. In the latter case, the point t_1 must be contained in some interval in $\mathcal{F}_1$ not yet considered; among all intervals in $\mathcal{F}_1$ containing t_1 we pick the one which has the largest right end-point; this interval is either of the form $(t_1', 1]$ or (t_1', t_2), for some $t_1' \in (0, t_1)$ and $t_2 \in (t_1, 1)$. If it is of the form (t_1', t_2) then we consider all intervals in $\mathcal{F}_1$ which contain t_2, etc. This process must eventually finish, since $\mathcal{F}_1$ is finite, and this means that we have found a set of $n \geq 1$ intervals

$$[0, t_1),\ (t_1', t_2),\ (t_2', t_3),\ \ldots,\ (t_{n-1}', 1] \quad \text{in } \mathcal{F}_1,$$

where $0 < t_1 < t_2 < \ldots < t_{n-1}$ and $0 < t_j' < t_j$ for each $j \in \{1, \ldots, n-1\}$. (If $n = 1$ then $[0,1]$ is in $\mathcal{F}_1$ and our set consists of this single interval.) By the construction of $\mathcal{F}_1$, there exist C^∞ charts (U_j, x_j) on M such that

$$\gamma([0, t_1)) \subset U_1,\ \gamma((t_1', t_2)) \subset U_2,\ \ldots,\ \gamma((t_{n-1}', 1]) \subset U_n.$$

Now for $\varepsilon > 0$ sufficiently small, if we set $\widetilde{t}_0 = 0$, $\widetilde{t}_j = t_j - \varepsilon$ for $j \in \{1, \ldots, n-1\}$, and $\widetilde{t}_n = 1$, then

$$0 = \widetilde{t}_0 < \widetilde{t}_1 < \cdots < \widetilde{t}_{n-1} < \widetilde{t}_n = 1$$

and $t_j' < \widetilde{t}_j < t_j$ for $j \in \{1, \ldots, n-1\}$, so that

$$\gamma([\widetilde{t}_{j-1}, \widetilde{t}_j]) \subset U_j \qquad \text{for } j \in \{1, \ldots, n\}.$$

Now we can define $c : [0,1] \to M$ by letting, for each $j \in \{1, \ldots, n\}$, $c_{|[\widetilde{t}_{j-1}, \widetilde{t}_j]}$ be the curve from $\gamma(\widetilde{t}_{j-1})$ to $\gamma(\widetilde{t}_j)$ which in the chart (U_j, x_j) is represented by a straight line segment from $x_j(\gamma(\widetilde{t}_{j-1}))$ to $x_j(\gamma(\widetilde{t}_j))$ (parametrized by a constant times arc length, say). Then c is a continuous curve, and each restriction $c_{|[\widetilde{t}_{j-1}, \widetilde{t}_j]}$ is C^∞; thus c is a piecewise continuous curve, and it has $c(0) = p$ and $c(1) = q$. Done! □

[12] Here by "open" we mean wrt the topology of $[0,1]$ induced by the topology of $\mathbb{R}$; in particular $[0, x)$ and $(x, 1]$ are open subintervals of $[0,1]$ for any $x \in (0,1)$, and also $[0,1]$ itself is an open subinterval of $[0,1]$.

(b). **Outline:** With $\widetilde{t}_0, \ldots, \widetilde{t}_n$ and charts (U_j, x_j) as above, we can construct $c : [0,1] \to M$ by letting $c_{|[\widetilde{t}_0,\widetilde{t}_1]}$ be an *arbitrary* C^∞ curve from $\gamma(\widetilde{t}_0)$ to $\gamma(\widetilde{t}_1)$ (e.g., the line segment used in (a)). Using "Borel's Lemma" (cf. wikipedia) and working in the chart (U_2, x_2), one can then construct a C^∞ curve $c_{|[\widetilde{t}_1,\widetilde{t}_2]}$ from $\gamma(\widetilde{t}_1)$ to $\gamma(\widetilde{t}_2)$ which has the properties that "all derivatives at $\widetilde{t}_1$ match up; i.e. c is in fact C^∞ on all $[\widetilde{t}_0, \widetilde{t}_2]$. Then just repeat. $\square$

Problem 20:

(a). Note that we can cover the whole of H^n with one natural C^∞ chart (H^n, y), namely by letting $y(x) := (x^1, \ldots, x^n)$ for $x = (x^0, x^1, \ldots, x^n) \in H^n$. Then $y(H^n) = \mathbb{R}^n$ and the inverse map is

$$y^{-1}(x^1, \ldots, x^n) = \Big(\sqrt{1 + (x^1)^2 + \cdots + (x^n)^2}, x^1, \ldots, x^n\Big), \qquad \forall x = (x^1, \ldots, x^n) \in \mathbb{R}^n.$$

Note that this map y^{-1} gives the embedding map $i : H^n \to \mathbb{R}^{n+1}$, expressed wrt our selected chart on H^n and the standard chart on $\mathbb{R}^{n+1}$. Hence wrt these charts, for each $p = (x^0, x^1, \ldots, x^n) \in H^n$, $di_p : T_pH^n \to T_p\mathbb{R}^{n+1}$ is the linear map with matrix

$$\begin{pmatrix} \frac{\partial}{\partial x^1}\sqrt{1 + (x^1)^2 + \cdots + (x^n)^2} & \cdots & \frac{\partial}{\partial x^n}\sqrt{1 + (x^1)^2 + \cdots + (x^n)^2} \\ \frac{\partial}{\partial x^1} x^1 & \cdots & \frac{\partial}{\partial x^n} x^1 \\ \vdots & & \vdots \\ \frac{\partial}{\partial x^1} x^n & \cdots & \frac{\partial}{\partial x^n} x^n \end{pmatrix}$$
$$= \begin{pmatrix} x^1/x^0 & x^2/x^0 & \cdots & x^n/x^0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \vdots & \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

This map takes an arbitrary vector $\xi = (\xi_1, \ldots, \xi_n)$ in $\mathbb{R}^n$ to $(\sum_{j=1}^n \frac{x^j}{x^0}\xi_j, \xi_1, \ldots, \xi_n)$ in $\mathbb{R}^{n+1}$. (In other words, di_p maps $\sum_{j=1}^n \xi_j \frac{\partial}{\partial x^j}$ in T_pH^n to the vector $(\sum_{j=1}^n \frac{x^j}{x^0}\xi_j)\frac{\partial}{\partial x^0} + \sum_{j=1}^n \xi_j \frac{\partial}{\partial x^j}$ in $T_p\mathbb{R}^{n+1}$.) Hence our first task is to prove that for any $\xi \in \mathbb{R}^n$, $(\frac{x^j}{x^0}\xi_j, \xi_1, \ldots, \xi_n)$ is orthogonal to p wrt the form $\langle \cdot, \cdot \rangle$, i.e. that

$$-\sum_{j=1}^n \frac{x^j}{x^0}\xi_j \cdot x^0 + \xi_1 \cdot x^1 + \cdots + \xi_n \cdot x^n = 0.$$

This is clear by inspection!

The next task is to prove that the restriction of the form I (from [12, p. 228(top)]) to T_pH^n (or perhaps more accurately; to $di_p(T_pH^n)$) is positive definite. (At present I do not understand Jost's claim that this follows from Sylvester's theorem. Exactly which theorem is this?) Thus we have to prove that the following expression is positive, for any $p = (x^0, x^1, \ldots, x^n) \in H^n$

and $\xi \in \mathbb{R}^n \setminus \{0\}$:

$$I\Big(\Big(\sum_{j=1}^n \frac{x^j}{x^0}\xi_j\Big)\frac{\partial}{\partial x^0} + \sum_{j=1}^n \xi_j \frac{\partial}{\partial x^j},\ \Big(\sum_{j=1}^n \frac{x^j}{x^0}\xi_j\Big)\frac{\partial}{\partial x^0} + \sum_{j=1}^n \xi_j \frac{\partial}{\partial x^j}\Big)$$

$$= -\Big(\sum_{j=1}^n \frac{x^j}{x^0}\xi_j\Big)^2 + \sum_{j=1}^n \xi_j^2. \tag{42}$$

However by Cauchy-Schwarz we have

$$\Big(\sum_{j=1}^n \frac{x^j}{x^0}\xi_j\Big)^2 \le \sum_{j=1}^n \Big(\frac{x^j}{x^0}\Big)^2 \sum_{j=1}^n \xi_j^2 = \frac{(x^0)^2 - 1}{(x^0)^2}\sum_{j=1}^n \xi_j^2 < \sum_{j=1}^n \xi_j^2,$$

where we used the fact that $p = (x^0, x^1, \ldots, x^n) \in H^n$, and then used $\xi \neq 0$. This shows that the expression in (42) is positive! □

(b) By definition, $O(1,n)$ is the set of linear maps $R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ which leave $\langle \cdot, \cdot \rangle$ invariant, i.e. which satisfy

$$\langle Rx, Ry \rangle = \langle x, y \rangle, \quad \forall x, y \in \mathbb{R}^{n+1}. \tag{43}$$

In particular if $R \in O(1,n)$ and $Rx = 0$ for some $x \in \mathbb{R}^{n+1}$ then $\langle x, y \rangle = 0$ for all $y \in \mathbb{R}^{n+1}$ and this implies $x = 0$. Hence every $R \in O(1,n)$ is invertible. Setting now $x = R^{-1}x'$ and $y = R^{-1}y'$ (with arbitrary $x', y' \in \mathbb{R}^{n+1}$) in the relation (43) we get $\langle x', y' \rangle = \langle R^{-1}x', R^{-1}y' \rangle$; hence $R^{-1} \in O(1,n)$. Hence $O(1,n)$ is closed under taking inverse. The rest of the verification that $O(1,n)$ is a group is immediate.

Let us put

$$\widetilde{H}^n := \{x \in \mathbb{R}^{n+1} \ : \ \langle x, x \rangle = -1,\ x^0 < 0\},$$

so that the set

$$\{x \in \mathbb{R}^{n+1} \ : \ \langle x, x \rangle = -1\} \tag{44}$$

equals the disjoint union of H^n and $\widetilde{H}^n$. It follows directly from that definition of $O(1,n)$ that every $R \in O(1,n)$ maps the set (44) onto itself; hence since R is linear and invertible, R must map every connected component of the set (44) onto the same or another connected component, in such a way that the connected components are permuted. In other words: Every $R \in O(1,n)$ satisfies either

$$\text{“}(+)\text{”:} \quad [R(H^n) = H^n \text{ and } R(\widetilde{H}^n) = \widetilde{H}^n]$$

or

$$\text{“}(-)\text{”:} \quad [R(H^n) = \widetilde{H}^n \text{ and } R(\widetilde{H}^n) = H^n].$$

Let $O^+(1,n)$ be the set of those $R \in O(1,n)$ satisfying “(+)”. One verifies immediately that $O^+(1,n)$ is closed under multiplication and inverses; thus $O^+(1,n)$ is a subgroup of $O(1,n)$. Next note that there exists $R \in O(1,n)$

which satisfies "$(-)$"; for example $R = R_0 :=$ the diagonal matrix with diagonal entries $-1, 1, 1, \ldots, 1$. We now see that $O(1,n)$ is the disjoint union of the two cosets $O^+(1,n)$ and $R_0 \cdot O^+(1,n)$ (where the latter coset consists exactly of all $R \in O(1,n)$ satisfying "$(-)$"). Hence $O^+(1,n)$ is a subgroup of $O(1,n)$ of index 2 (and hence normal).

It now only remains to prove that each $R \in O^+(1,n)$ acts by an isometry on H^n. Thus fix $R \in O^+(1,n)$. Since H^n is an embedded submanifold of $\mathbb{R}^{n+1}$, and R is a linear (hence C^∞) map $\mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ preserving H^n, it follows that $R_{|H^n}$ is a C^∞ map $H^n \to H^n$. Considering also $R^{-1} \in O^+(1,n)$ we see that $R_{|H^n}$ is in fact a C^∞ diffeomorphism of H^n onto H^n, with inverse $= (R^{-1})_{|H^n}$. Note that for any $x \in \mathbb{R}^{n+1}$, if we identify $T_x\mathbb{R}^{n+1}$ with $\mathbb{R}^{n+1}$ in the standard way, then the bilinear form I on $T_x\mathbb{R}^{n+1}$ equals the form $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^{n+1}$. Now since R preserves the latter form, and $dR = R$ (since R is linear), it follows that dR preserves I, i.e.

$$I\big((dR)_x(v), (dR)_x(w)\big) = I\big(v, w\big), \qquad \forall x \in \mathbb{R}^{n+1},\ v, w \in T_x\mathbb{R}^{n+1}.$$

In particular this holds for all $x \in H^n$ and $v, w \in T_xH^n \subset T_x\mathbb{R}^{n+1}$; and this shows that $R_{|H^n}$ preserves the Riemannian metric on H^n. Hence R is indeed an isometry of H^n onto itself! $\square$

(c). Take $p \in H^n$ and $v \in T_pH^n$, $v \neq 0$. (Here we view T_pH^n as a linear subspace of $\mathbb{R}^{n+1}$; cf. part (a).) As we proved in part (a), we then have

$$\langle p, v \rangle = 0. \tag{45}$$

In fact p, v are linearly independent. [Proof: $p \neq 0$ since $p \in H^n$; hence we only need to prove that we cannot have $v = tp$ for some $t \in \mathbb{R}$. But $\langle p, p \rangle = -1$; hence $v = tp$ would imply $\langle p, v \rangle = -t$, and so $t = 0$ by (45), contradicting $v \neq 0$.]

Let $\Pi \subset \mathbb{R}^{n+1}$ be the 2-dimensional plane spanned by p, v. Because of (45), if $\langle \cdot, \cdot \rangle$ were a scalar product on $\mathbb{R}^{n+1}$, then

$$P : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \qquad P(x) := \frac{\langle x, p \rangle}{\langle p, p \rangle} p + \frac{\langle x, v \rangle}{\langle v, v \rangle} v \tag{46}$$

would be the orthogonal projection from $\mathbb{R}^{n+1}$ onto Π, and

$$R : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}, \qquad R(x) := 2P(x) - x \tag{47}$$

would be orthogonal reflection across Π. Now $\langle \cdot, \cdot \rangle$ is NOT a scalar product on $\mathbb{R}^{n+1}$ (since it is not positive definite); however we still see that P and R, as defined in (46) and (47), are well-defined linear maps on $\mathbb{R}^{n+1}$ (indeed recall that $\langle p, p \rangle = -1$ and $\langle v, v \rangle > 0$ by part (a)). Furthermore $P(x) \in \Pi$ for all $x \in \mathbb{R}^{n+1}$ and $P(x) = x$ for every $x \in \Pi$ (for the last claim it suffices to verify $P(p) = p$ and $P(v) = v$). Hence also $R(x) = x$ for all $x \in \Pi$, while $R(x) \neq x$ for $x \notin \Pi$ (indeed $R(x) = x \Rightarrow x = P(x) \in \Pi$). In other words, the set of fixed points of R equals Π.

It remains to prove $R \in O^+(1,n)$. Let us first note that for any $x \in \mathbb{R}^{n+1}$, using (46) and (45) we have

$$\langle P(x) - x, p \rangle = \frac{\langle x, p \rangle}{\langle p, p \rangle} \langle p, p \rangle + 0 - \langle x, p \rangle = 0$$

and similarly $\langle P(x) - x, v \rangle$; hence $P(x) - x$ is orthogonal to p and v and hence to all $\Pi = \mathrm{Span}_{\mathbb{R}}\{p, v\}$. In particular $\langle P(x) - x, P(x) \rangle = 0$, since $P(x) \in \Pi$. Therefore,

$$\begin{aligned}
\langle Rx, Rx \rangle &= \langle P(x) + (P(x) - x), P(x) + (P(x) - x) \rangle \\
&= \langle P(x), P(x) \rangle + \langle P(x) - x, P(x) - x \rangle \\
&= \langle P(x) - (P(x) - x), P(x) - (P(x) - x) \rangle \\
&= \langle x, x \rangle.
\end{aligned}$$

This holds for all $x \in \mathbb{R}^{n+1}$; hence $R \in O(1,n)$. Finally note that $R(p) = p$, since $p \in \Pi$, and $p \in H^n$; hence in the notation of part (c) R cannot satisfy "$(-)$" and so it must satisfy "$(+)$", i.e. $R \in O^+(1,n)$. □

(d). Consider arbitrary p, v as in (c), but now also assume $\|v\| = 1$ (i.e. $\langle v, v \rangle = 1$). Let $c : \mathbb{R} \to H^n$ be the geodesic with $c(0) = p$, $\dot{c}(0) = v$. (The fact that c is defined on all $\mathbb{R}$ follows from the Theorem of Hopf-Rinow, since H^n is complete.) Take $R \in O^+(1,n)$ as in part (c); this is an isometry of H^n onto H^n by part (b); hence also $R \circ c : \mathbb{R} \to H^n$ is a geodesic on H^n. But R preserves p and v; hence $R(c(0)) = p$ and $\frac{d}{dt} R(c(t))_{|t=0} = v$, and so by [12, Thm. 1.4.2], $R(c(t)) = c(t)$ for all $t \in \mathbb{R}$. Also the set of fixed points of R is Π; hence

$$c(t) \in \Pi, \qquad \forall t \in \mathbb{R}.$$

Hence there are (uniquely determined) C^∞ functions $x : \mathbb{R} \to \mathbb{R}$ and $y : \mathbb{R} \to \mathbb{R}$ such that

$$c(t) = x(t)p + y(t)v, \qquad \forall t \in \mathbb{R}.$$

It follows from $c(0) = p$ and $\dot{c}(0) = v$ that

$$x(0) = 1, \ \dot{x}(0) = 0; \qquad y(0) = 0, \ \dot{y}(0) = 1.$$

We also have $c(t) \in H^n$ and thus $\langle c(t), c(t) \rangle = -1$, for all $t \in \mathbb{R}$. Using $\langle p, p \rangle = -1$, $\langle p, v \rangle = 0$ and $\langle v, v \rangle = 1$, this translates into:

$$-x(t)^2 + y(t)^2 = -1, \qquad \forall t \in \mathbb{R}. \tag{48}$$

We also have $\|\dot{c}(t)\| = 1$, i.e. $\langle \dot{c}(t), \dot{c}(t) \rangle = 1$ for all $t \in \mathbb{R}$, by [12, Lemma 1.4.5]; and this similarly translates into:

$$-\dot{x}(t)^2 + \dot{y}(t)^2 = 1, \qquad \forall t \in \mathbb{R}. \tag{49}$$

Equation (48) implies $|x(t)| \geq 1$, and since $x(0) = 1$ and x is continuous, it follows that $x(t) \geq 1$ and $x(t) = \sqrt{y(t)^2 + 1}$ for all $t \in \mathbb{R}$. Hence

$\dot{x}(t) = \dfrac{y(t)\dot{y}(t)}{\sqrt{y(t)^2+1}}$, and inserting this in (49) and simplifying we get

$$\dot{y}(t) = \sqrt{1+y(t)^2} \qquad \forall t \in \mathbb{R}.$$

Separating variables etc, this implies $y(t) = \sinh(C+t)$, $\forall t \in \mathbb{R}$, where C is a fixed real constant, and in fact $C = 0$ since $y(0) = 0$. Hence $y(t) = \sinh t$ and so $x(t) = \sqrt{y(t)^2+1} = \cosh t$, i.e.

$$c(t) = (\cosh t)p + (\sinh t)v, \qquad \forall t \in \mathbb{R}.$$

□

Problem 21:

Remark: The results which we prove here are special cases of corresponding results on (maximal) integral curves of a vector field; cf. [1, Thm. IV.4.5]. Indeed, the geodesics are simply projections of the integral curves of a certain vector field on TM; cf. [1, Thm. 7.1] as well as [12, Thm. 2.2.3 and Def. 2.2.3].

(a). Let $p \in M$ and $v \in T_pM$ be given. Let $I, J \subset \mathbb{R}$ be any two open intervals containing 0 and let $f : I \to M$ and $g : J \to M$ be two geodesics both satisfying $f(0) = p$, $\dot{f}(0) = v$, $g(0) = p$, $\dot{g}(0) = v$. Let

$$I^* = \{t \in I \cap J \;:\; f(t) = g(t) \text{ and } \dot{f}(t) = \dot{g}(t)\}.$$

We claim that $I^* = I \cap J$. Note that this implies that f and g together define a geodesic on the interval $I \cup J$!

[Proof of $I^* = I \cap J$: Take any $s \in I^*$. Using $I^* \subset I \cap J$ and the fact that $I \cap J$ is open, it follows that there exists some $\delta > 0$ such that $(s - \delta, s + \delta) \subset I \cap J$, and so we have two well-defined geodesics

$$f_1, g_1 : I_\delta \to M; \quad f_1(t) := f(s + t), \quad g_1(t) := g(s + t).$$

(Here $I_\delta := (-\delta, \delta)$.) These satisfy $f_1(0) = f(s) = g(s) = g_1(0)$ and $\dot{f}_1(0) = \dot{f}(s) = \dot{g}(s) = \dot{g}_1(0)$. Hence by the local uniqueness theorem for geodesics (Theorem 1' on p. 2 in Lecture #4), there is some $\delta' \in (0, \delta]$ such that $f_1(t) = g_1(t)$ for all $t \in I_{\delta'}$, and so $f(t) = g(t)$ and $\dot{f}(t) = \dot{g}(t)$ for all $t \in (s - \delta', s + \delta')$, i.e. $(s - \delta', s + \delta') \subset I^*$. The fact that I^* contains such a neighborhood around every point $s \in I^*$ implies that I^* is *open.* But also I^* is a *closed* subset of $I \cap J$; this follows from the definition of I^* and the fact that $f, \dot{f}, g, \dot{g}$ are continuous. Hence I^* is either empty or a connected component of $I \cap J$, i.e. $I^* = \emptyset$ or $I^* = I \cap J$. However $0 \in I^*$, i.e. I^* is non-empty. Therefore $I^* = I \cap J$.]

Using the property just proved, it follows that if we let I be the union of the domains of *all* geodesic curves such as f and g above, then there is a well-defined geodesic $c_v : I \to M$ with $c_v(0) = p$, $\dot{c}_v(0) = v$, and it has the desired property! □

(b). Let $v \in TM$ and $s \in I_v$. First note that

$$\theta(0, v) = \dot{c}_v(0) = v. \tag{50}$$

Set

$$w := \theta(s, v) \in TM.$$

Since $c_v : I_v \to M$ is a geodesic with $\dot{c}_v(s) = \dot{c}_w(0)$, it follows that the curve $\gamma : I_v - s \to M$, $\gamma(t) := c_v(s+t)$ is a geodesic with $\dot{\gamma}(0) = \dot{c}_w(0)$; hence by the defining property of the maximal geodesic c_w we have $I_v - s \subset I_w$ and $\gamma(t) = c_w(t)$ for all $t \in I_v - s$. In other words:

$$c_v(s+t) = c_w(t), \qquad \forall t \in I_v - s. \tag{51}$$

Hence also

$$\dot{c}_v(s+t) = \dot{c}_w(t), \qquad \forall t \in I_v - s, \tag{52}$$

and applying this for $t = -s$ we get $\dot{c}_v(0) = \dot{c}_w(-s)$. Hence the curve $\eta : I_w + s \to M$, $\eta(t) := c_w(s - t)$ is a geodesic with $\dot{\eta}(0) = \dot{c}_v(0)$; and so by the defining property of the maximal geodesic c_v we have $I_w + s \subset I_v$ (and $\eta(t) = c_v(t)$, $\forall t \in I_w + s$). Now $I_w + s \subset I_v$ and $I_v - s \subset I_w$ together imply $I_w = I_v - s$. Hence we have proved all desired relations; indeed cf. (50) and (52), and note that (52) can be expressed as $\theta(s+t, v) = \theta(w, t)$, $\forall t \in I_v - s = I_w$. □

(c). For any $v \in TM$, let $c_v : I_v \to M$ be the unique maximal geodesic starting at v. Now define $\mathcal{D}$ as follows:

$$\mathcal{D} := \{v \in TM \ : \ 1 \in I_v\}.$$

Then define the map $\exp : \mathcal{D} \to M$ by

$$\exp(v) := c_v(1).$$

Note that $\exp(v)$ is well-defined, since $v \in \mathcal{D}$ implies $1 \in I_v$.

In order to prove that $\mathcal{D}$ and exp have the desired properties, let us first note a basic scaling property. As we noted in the lecture, if $t \mapsto c(t)$ is any geodesic then so is $t \mapsto c(\lambda t)$ for any constant $\lambda \in \mathbb{R}$, and $\frac{d}{dt} c(\lambda t) = \lambda \dot{c}(\lambda t)$ everywhere. Using this fact one easily derives the following scaling formula for the maximal geodesics: For any $v \in TM$ and $\lambda \in \mathbb{R}$,

$$I_{\lambda v} = \lambda^{-1} I_v \quad \text{and} \quad c_{\lambda v}(t) = c_v(\lambda t), \ \forall t \in I_{\lambda v}. \tag{53}$$

(Explanation of notation: "$\lambda^{-1} I_v$ denotes the open interval $\{\lambda^{-1} t \, : \, t \in I_v\}$; in the special case $\lambda = 0$ the formula should of course be interpreted to say $I_{0v} = \mathbb{R}$.)

Now for any $v \in TM$ and $t \in \mathbb{R}$, note that $t \in I_v$ holds iff $1 \in t^{-1} I_v$, and by (53) this holds iff $1 \in I_{tv}$, i.e. iff $tv \in \mathcal{D}$ (note that with the appropriate interpretation this discussion is correct also when $t = 0$; in particular note that $\mathcal{D}$ contains the zero vector from every tangent space T_pM). We have thus proved that for every $v \in TM$,

$$I_v = \{t \in \mathbb{R} \,:\, tv \in \mathcal{D}\}. \tag{54}$$

Also by (53),

$$c_v(t) = c_{tv}(1) = \exp(tv), \qquad \forall t \in I_v.$$

Hence it "only" remains to prove that $\mathcal{D}$ is open and that our map exp is C^∞.

A crucial ingredient for the remaining part of the proof is to translate the formula "$\theta(\theta(s,v),t) = \theta(t+s,v)$" from part (b) into a composition formula for exp. Thus take any $v \in TM$ and $s \in I_v$, and write $q = c_v(s) = \exp(sv)$ and $w := \theta(s,v) = \dot{c}_v(s) \in T_q(v)$. By (54), the formula $I_w = I_v - s$ proved in part (b) can be equivalently expressed as

$$\forall v \in TM \,:\, \forall s \in I_v \,:\, \forall t \in \mathbb{R} : \quad \big[t \cdot \dot{c}_v(s) \in \mathcal{D} \Leftrightarrow (t+s)v \in \mathcal{D}\big]. \tag{55}$$

For any s, t satisfying the condition in (55), by part (b) we have $\theta(w,t) = \theta(t+s,v)$, i.e. $\dot{c}_w(t) = \dot{c}_v(t+s)$. Applying the projection $TM \to M$ this implies $c_w(t) = c_v(t+s)$, i.e. $\exp(tw) = \exp((t+s)v)$. Hence:

$$\forall v \in TM \,:\, \forall s \in I_v \,:\, \forall t \in I_v - s : \quad \exp(t \cdot \dot{c}_v(s)) = \exp((t+s)v). \tag{56}$$

Let $\mathcal{D}'$ be the set of all $v \in TM$ with the property that v has an open neighborhood $\Omega \subset \mathcal{D}$ and exp is C^∞ on Ω. Clearly $\mathcal{D}'$ is an open subset of the interior of $\mathcal{D}$, and $\exp_{|\mathcal{D}'}$ is C^∞. Our task is to prove that $\mathcal{D}' = \mathcal{D}$! The local existence theorem for geodesics (Theorem 1 in Lecture #4) implies that $\mathcal{D}'$ *contains the zero section in TM,* i.e. $\mathcal{D}'$ contains the zero vector from every tangent space T_pM ($p \in M$).

Next we claim that, as a consequence of (56), $\mathcal{D}'$ has the following property:

$$\forall v \in TM \,:\, \forall s \in I_v \,:\, \forall t \in I_v - s :$$
$$\big[\text{If } sv \in \mathcal{D}' \text{ and } t \cdot \dot{c}_v(s) \in \mathcal{D}' \text{ then } (t+s)v \in \mathcal{D}'\big]. \tag{57}$$

[Proof: Fix $s \in I_v$, $t \in I_v - s$ and assume $sv \in \mathcal{D}'$ and $t \cdot \dot{c}_v(s) \in \mathcal{D}'$. Since exp is C^∞ on $\mathcal{D}'$, the function

$$u \mapsto \dot{c}_u(s) = \Big(\frac{d}{dt}\exp(tu)\Big)_{|t=s}$$

is C^∞ on the set $\mathcal{U} := \{u \in TM \,:\, su \in \mathcal{D}'\}$ (verify this claim as an exercise!), and $\mathcal{U}$ is an open subset of TM; also $v \in \mathcal{U}$. In particular $u \mapsto t \cdot \dot{c}_u(s)$ is *continuous* on $\mathcal{U}$, and so

$$\mathcal{U}' := \{u \in \mathcal{U} \,:\, \dot{c}_u(s) \in \mathcal{D}'\} = \{u \in TM \,:\, su \in \mathcal{D}' \text{ and } t \cdot \dot{c}_u(s) \in \mathcal{D}'\}$$

is also an open subset of TM. Note that $v \in \mathcal{U}'$ by our assumptions. Now for any $u \in \mathcal{U}'$ we have $su \in \mathcal{D}' \subset \mathcal{D}$, thus $s \in I_u$ (cf. (54)) and also

$t \cdot \dot{c}_u(s) \in \mathcal{D}' \subset \mathcal{D}$, which by (55) implies $(t+s)u \in \mathcal{D}$, i.e. $t \in I_u - s$; hence by (56) we conclude:

$$\forall u \in \mathcal{U}' : \quad \exp((t+s)u) = \exp(t \cdot \dot{c}_u(s)).$$

Here the right hand side is the composition of the function $u \mapsto t \cdot \dot{c}_u(s)$ and exp, and both these functions are C^∞ when $u \in \mathcal{U}'$ (by the discussion above regarding $u \mapsto \dot{c}_u(s)$, and since $t \cdot \dot{c}_u(s) \in \mathcal{D}'$, and by the definition of $\mathcal{D}'$). Hence $u \mapsto \exp((t+s)u)$ is C^∞ on $\mathcal{U}'$, and since $\mathcal{U}'$ is an open set containing v, this proves that $(t+s)v \in \mathcal{D}'$.]

Assume now that there is some $v \in \mathcal{D} \setminus \mathcal{D}'$ (we will prove that this leads to a contradiction). Let $p = \pi(v)$, so that $v \in T_pM$. Then $\{t \in [0,1] : tv \notin \mathcal{D}'\}$ is a closed subset of $[0,1]$ which contains 1 but not 0; this implies that there is a *minimal* $t_1 \in (0,1]$ with $t_1 v \notin \mathcal{D}'$. Note that $t_1 \in I_v$, since $1 \in I_v$. Set $q = \exp(t_1 v)$, and let 0_q be the zero vector in T_qM. Then $0_q \in \mathcal{D}'$, since $\mathcal{D}'$ contains the zero section; also $\mathcal{D}'$ is open and $\varepsilon \cdot \dot{c}_v(t_1 - \varepsilon)$ tends to 0_q in TM as $\varepsilon \to 0$; hence there is some $\varepsilon \in (0, t_1)$ such that

$$\varepsilon \cdot \dot{c}_v(t_1 - \varepsilon) \in \mathcal{D}'.$$

Now set $s = t_1 - \varepsilon \in (0, t_1)$. Note that $sv \in \mathcal{D}'$, by our choice of t_1. Hence (57) applies with our s and $t = \varepsilon$, and implies that $t_1 v \in \mathcal{D}'$. This is a contradiction, since we constructed t_1 so that $t_1 v \notin \mathcal{D}'$! The conclusion is that the does not exist any $v \in \mathcal{D} \setminus \mathcal{D}'$; in other words $\mathcal{D}' = \mathcal{D}$, and we are done! □

(d). (This is now more or less straightforward; cf., e.g., [1, Thm. 3.12].)

Problem 22:

Fix a chart (U, x) on M containing p; then (TU, dx) is a C^∞ chart on TU (cf. Problem 16 and the end of Lecture #2); dx maps TU onto $x(U) \times \mathbb{R}^d$. Set $\mathcal{D}_U := TU \cap \mathcal{D}$ and $V = dx(\mathcal{D}_U)$; this is an open subset of $x(U) \times \mathbb{R}^d$. Of course, $(U \times U, (x, x))$ is a chart on $M \times M$; cf. Problem 8.

Let us write $x_0 := x(p)$; then 0_p is represented by $(x_0, 0) \in V \subset \mathbb{R}^d \times \mathbb{R}^d$ in our chart (TU, dx).

In the local coordinates described above, the function

$$G := (\pi, \exp) : \mathcal{D}_U \to M \times M$$

takes the form

$$V \to \mathbb{R}^d \times \mathbb{R}^d; \qquad (x, v) \mapsto (x, F(x, v)), \tag{58}$$

where $F : V \to \mathbb{R}^d$ is the function exp composed with the appropriate chart maps. Let us write $F(x, v) = (F^1(x, v), \ldots, F^d(x, v))$. Then the Jacobian matrix of the map in (58) is given by

$$\begin{pmatrix} 1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \cdots & 0 \\ & & & & & & \\ 0 & 0 & \cdots & 1 & 0 & \cdots & 0 \\ \frac{\partial F^1}{\partial x^1} & \frac{\partial F^1}{\partial x^2} & \cdots & \frac{\partial F^1}{\partial x^d} & \frac{\partial F^1}{\partial v^1} & \cdots & \frac{\partial F^1}{\partial v^d} \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ \frac{\partial F^d}{\partial x^1} & \frac{\partial F^d}{\partial x^2} & \cdots & \frac{\partial F^d}{\partial x^d} & \frac{\partial F^d}{\partial v^1} & \cdots & \frac{\partial F^d}{\partial v^d} \end{pmatrix}.$$

Note that this matrix has the structure of a 2×2 block matrix where each block is a $d \times d$ matrix. It follows from the basic formula $(d\exp_p)_0 = 1_{T_p(M)}$ (cf. the proof of Theorem 3 in Lecture #4 = Jost [12, Thm. 1.4.3]) that *at* $(x_0, 0)$, *the lower bottom block is the* $d \times d$ *identity matrix.* Hence at $(x_0, 0)$ the above $2d \times 2d$ matrix is non-singular (in fact the determinant equals 1).

Hence we have proved that the differential $dG_{0_p} : T_{0_p}(TM) \to T_pM \times T_pM$ is non-singular. Therefore, by the Inverse Function Theorem, there is a open neigborhood $\Omega \subset \mathcal{D}$ of 0_p such that G restricted to Ω is a diffeomorphism onto an open subset $G(\Omega)$ of $M \times M$. After shrinking Ω if necessary, we may assume that Ω has the following form, for some $r > 0$ and some open neighborhood $\mathcal{U}$ of p in M:

$$\Omega = \sqcup_{q \in \mathcal{U}} B_r(0_q) = \{v \in TM \,:\, \pi(v) \in \mathcal{U} \text{ and } \|v\| < r\}. \tag{59}$$

(Here we used the facts that any set of the form (59) is open in TM, and these sets form a neighborhood basis of 0_p; we leave the verification of these as an exercise.)

Now G^{-1} is a C^∞ map from $G(\Omega)$ onto Ω. For each $q \in \mathcal{U}$, set

$$W_q := \{u \in M \;:\; (q,u) \in G(\Omega)\};$$

this is an open subset of M. Also for each $q \in \mathcal{U}$ define the map

$$H_q : W_q \to TM, \qquad H_q(u) = G^{-1}(q,u).$$

Then H_q is a C^∞ map, and using $G \circ G^{-1} = 1$ and $G = (\pi, \exp)$ we have $\pi(H_q(u)) = q$ and $\exp(H_q(u)) = u$ for all $u \in W_q$, i.e. H_q in fact maps W_q onto $T_qM \cap \Omega = B_r(0_q)$, and

$$\exp \circ H_q = 1_{W_q} \quad \text{and} \quad H_q \circ \exp_{|B_r(0_q)} = 1_{B_r(0_q)}.$$

Hence for every $q \in \mathcal{U}$, $\exp_{|B_r(0_q)}$ is a diffeomorphism onto an open set (namely W_q) in M, i.e. we have proved Theorem 3' in Lecture #4! □

(b). Set

$$U' := \bigsqcup_{p \in U} B_r(0_p) \subset TM;$$

this is an open subset of TM (as is easy to verify from the definition of the topology of TM; cf. Problem 16), and by assumption we have $U' \subset \mathcal{D}$. As in part (a) let us consider the C^∞ map $G := (\pi, \exp)$, but this time with U' as domain of definition:

$$G := (\pi, \exp) : U' \to M \times M.$$

Note that $G(U') = V$ and it follows from our assumptions that G is a bijection from U' to V and $G^{-1} : V \to U'$ is exactly the map $(p,q) \mapsto \exp_p^{-1}(q)$ which we are interested in. Hence if we can prove that G is a diffeomorphism onto an open subset of $M \times M$ then we are done; and in fact it suffices to prove that every point in U' has an open neighbourhood in U' on which G restricts to a diffeomorphism onto an open subset of $M \times M$. By the Inverse Function Theorem, this will be ensured if we can prove that dG is non-singular at every point in U'.

Thus consider an arbitrary point $v \in U'$; set $q := \pi(v) \in U$ so that $v \in B_r(0_q) \subset T_qM$. Working with local coordinates of the same type as in part (a)[13], dG_v is expressed by a $2d \times 2d$ matrix which again is naturally viewed as a 2×2 block matrix where each block is a $d \times d$ matrix: The upper left block is the $d \times d$ identity matrix and the upper right block is the $d \times d$ zero matrix; also the lower right block is a non-singular $d \times d$ matrix, since $(d\exp_q)_v : T_vT_qM = T_qM \to T_{\exp(v)}M$ is non-singular (this holds since $v \in B_r(0_q)$ and $\exp_{q|B_r(0_q)}$ is a diffeomorphism by assumption). This implies that dG_v is non-singular, and we are done. □

[13] we now leave some details to the reader...

Problem 23: By definition of the exponential map, for any vector $v \in T_pM$ the curve

$$x(t) = tv \qquad (\text{for } |t| < \frac{r}{\|v\|}) \tag{60}$$

represents a geodesic in M wrt the chart (U, x). Assuming now $\|v\| = 1$ and $v \in V$, the ($t > 0$ part of the) curve (60) takes the following form in the chart $(U', y \circ x)$:

$$y(t) = (t, \varphi(v)) \qquad (0 < t < r). \tag{61}$$

Now, wrt the chart $(U', y \circ x)$, let $\Gamma^i_{jk}(y)$ be the Christoffel symbols and $(h_{ij}(y))$ be the matrix representing the Riemannian metric; then by [11, Lemma 1.4.4] we have

$$\ddot{y}^i(t) + \Gamma^i_{jk}(y(t))\dot{y}^j(t)\dot{y}^k(t) = 0, \qquad \forall t \in (0, r), i \in \{1, \ldots, d\}, \tag{62}$$

and

$$\Gamma^i_{jk}(y) = \frac{1}{2} h^{i\ell}(y)\big(h_{j\ell,k}(y) + h_{k\ell,j}(y) - h_{jk,\ell}(y)\big) \tag{63}$$

for all y in the coordinate range. We now follow the discussion in [11, p. 22(mid)–23(top)]. Inserting (61) in (62) gives

$$0 + \Gamma^i_{jk}(y(t))\delta_{j,1}\delta_{k,1} = 0,$$

i.e.

$$\Gamma^i_{11}(y(t)) = 0, \qquad \forall t \in (0, r), i \in \{1, \ldots, d\}.$$

Let us write Ω for the coordinate range for y, i.e. $\Omega = (0, r) \times \varphi(V) \subset \mathbb{R}^d$. Note that the previous argument applies to any fixed $v \in V$; this means that $y(t) = (t, \varphi(v))$ can take any value in Ω. Hence

$$\Gamma^i_{11}(y) = 0, \qquad \forall y \in \Omega, i \in \{1, \ldots, d\}.$$

By (63), this means that

$$h^{i\ell}(y)\big(2h_{1\ell,1}(y) - h_{11,\ell}(y)\big) = 0, \qquad \forall y \in \Omega, i \in \{1, \ldots, d\}.$$

Multiplying this by $h_{ki}(y)$ and adding over i (using $\sum_{i=1}^d h_{ki}(y)h^{i\ell}(y) = \delta_{k\ell}$), we get:

$$2h_{1k,1}(y) - h_{11,k}(y) = 0, \qquad \forall y \in \Omega, k \in \{1, \ldots, d\}. \tag{64}$$

In particular for $k = 1$ this implies $h_{11,1}(y) \equiv 0$, i.e.

$$\frac{\partial}{\partial y^1} h_{11}(y) = 0, \qquad \forall y \in \Omega. \tag{65}$$

However, by the transformation rule for the Riemannian metric expressed wrt the two charts $(U', y \circ x)$ and (U, x), we have

$$h_{11}(y) = \frac{\partial x^k}{\partial y^1}\frac{\partial x^\ell}{\partial y^1} g_{k\ell}(x) \qquad (\forall y \in \Omega), \tag{66}$$

and recalling

$$(y^1, \ldots, y^d) = \Big(\|x\|, \varphi\Big(\frac{x}{\|x\|}\Big)\Big); \quad \text{thus} \quad x = y^1 \cdot \varphi^{-1}(y^2, \ldots, y^d), \tag{67}$$

and writing

$$z = \varphi^{-1}(y^2, \ldots, y^d) \in S^1 \subset \mathbb{R}^d,$$

we get:

$$h_{11}(y) = z^k z^\ell g_{k\ell}(x) \qquad (\forall y \in \Omega). \tag{68}$$

Now for any fixed $(y^2, \ldots, y^d) \in \varphi(V)$, if we let $y^1 \to 0^+$ then $x \to 0$ in $\mathbb{R}^d$ and thus $g_{k\ell}(x) \to \delta_{k\ell}$ by part (a) of this problem; meanwhile $z = \varphi^{-1}(y^2, \ldots, y^d)$ is fixed; hence from (68) we get

$$\lim_{y^1 \to 0^+} h_{11}(y) = z^k z^\ell \lim_{x \to 0} g_{k\ell}(x) = \sum_{k=1}^d (z^k)^2 = 1. \tag{69}$$

But (65) implies that $h_{11}(y)$ equals a constant as y^1 varies in $(0, r)$ while $(y^2, \ldots, y^d)$ is kept fixed; now (69) says that this constant must be 1, and so we have proved

$$h_{11}(y) = 1, \qquad \forall y \in \Omega. \tag{70}$$

Inserting this in (64), for $k = j \geq 2$, we get

$$\frac{\partial}{\partial y^1} h_{1j}(y) = 0, \qquad \forall y \in \Omega, j \in \{2, \ldots, d\}. \tag{71}$$

Next, using (67) and the analogue of (66) for $h_{1j}(y)$ ($j \geq 2$), we have

$$h_{1j}(y) = z^k y^1 w^\ell \cdot g_{k\ell}(x), \qquad \text{with } w = \frac{\partial}{\partial y^j} \varphi^{-1}(y^2, \ldots, y^d) \in \mathbb{R}^d.$$

If we fix $(y^2, \ldots, y^d) \in \varphi(V)$ and let $y^1 \to 0^+$ then z, w are fixed while $g_{k\ell}(x) \to \delta_{k\ell}$; hence

$$\lim_{y^1 \to 0^+} h_{1j}(y) = 0, \qquad \forall (y^2, \ldots, y^d) \in \varphi(V) \text{ (fixed)}, \ j \in \{2, \ldots, d\}.$$

Combining this with (71) gives

$$h_{1j}(y) = 0, \qquad \forall y \in \Omega, \ j \in \{2, \ldots, d\}. \tag{72}$$

From (70), (72) and the symmetry $h_{k\ell} \equiv h_{\ell k}$, we see that

$$(h_{ij}(y)) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & h_{22}(y) & \cdots & h_{2d}(y) \\ \vdots & \vdots & & \vdots \\ 0 & h_{d2}(y) & \cdots & h_{dd}(y) \end{pmatrix}, \qquad \forall y \in \Omega.$$

Note also that since we know that $(h_{ij}(y))$ is positive definite for every $y \in \Omega$, it follows that the $(d-1) \times (d-1)$ matrix

$$\begin{pmatrix} h_{22}(y) & \cdots & h_{2d}(y) \\ \vdots & & \vdots \\ h_{d2}(y) & \cdots & h_{dd}(y) \end{pmatrix}$$

is positive definite for every $y \in \Omega$. $\square$

Problem 24: Assume that γ is *not* a geodesic. Then there is some $t_0 \in (a, b)$ such that either $\gamma(t)$ is not C^∞ at $t = t_0$ or γ does not satisfy the geodesic ODE at $t = t_0$. Then for *any* $t_1 \in [a, t_0)$ and $t_2 \in (t_0, b]$ the restricted curve $\gamma_{[t_1,t_2]}$ fails to be a geodesic. Set

$$p = \gamma(t_0).$$

By Theorem 4:3' (viz., Theorem 3' in Lecture #4; cf. Problem 22(a)) there exists $r > 0$ such that for every $q \in B_r(p)$ we have $B_r(0_q) \subset \mathcal{D}_q$ and $\exp_{q|B_r(0_q)}$ is a diffeomorphism onto an open set in M. Then by Theorem 4:4 we have, for every $q \in B_r(p)$:

$$\exp_q(B_r(0_q)) = B_q(r), \tag{73}$$

and for every $v \in B_r(0_q)$, any pw C^∞ curve in M from q to $\exp_q(v)$ which is not a reparametrization of the curve $c(t) = \exp(tv)$, $t \in [0, 1]$, has length strictly larger than $\|v\| = d(q, \exp_q(v))$.

Now choose $t_1 \in [a, t_0)$ and $t_2 \in (t_0, b]$ so that $|t_1 - t_0| < r/2$ and $|t_2 - t_0| < r/2$. Then $d(\gamma(t_1), \gamma(t_2)) \leq L(\gamma_{[t_1,t_2]}) = t_2 - t_1 < r$, since γ is parametrized by arc length. Similarly $d(\gamma(t_1), p) < r/2$, so that $\gamma(t_1) \in B_r(p)$. Set $q = \gamma(t_1)$; we have just noted that $d(q, \gamma(t_2)) < r$; hence by (73) there is a (unique) $v \in B_r(0_q)$ such that $\gamma(t_2) = \exp_q(v)$. Also note that $\gamma_{[t_1,t_2]}$ cannot be a reparametrization of the geodesic $c(t) = \exp_q(tv)$, $t \in [0, 1]$, since we have from above that $\gamma_{[t_1,t_2]}$ is *not* a geodesic (also γ is parametrized by arc length). Hence by the property mentioned just below (73), $L(\gamma_{[t_1,t_2]}) > d(\gamma(t_1), \gamma(t_2))$, and so we get a shorter curve from $\gamma(a)$ to $\gamma(b)$ by forming the product path of $\gamma_{[a,t_1]}$ and c and $\gamma_{[t_2,b]}$. This contradicts $L(\gamma) = d(\gamma(a), \gamma(b))$.

Hence γ is a geodesic. $\square$

Problem 25: The standard Riemannian metric on $\mathbb{R}^d$ is an example of a complete metric.

In order to give a non-complete metric, consider e.g. any non-surjective embedding of $\mathbb{R}^d$ in $\mathbb{R}^d$, that is, an injective (C^∞) immersion $i : \mathbb{R}^d \to \mathbb{R}^d$ which is not surjective. (Such immersions certainly exist; one example is $i(x) = (1 + \|x\|^2)^{-1/2}x$.) Let $U = i(\mathbb{R}^d)$; the Inverse Function Theorem implies that U is an open subset of $\mathbb{R}^d$. We provide U with its C^∞ manifold structure as an open submanifold of $\mathbb{R}^d$; also equip U with the Riemannian metric induced by the standard Riemannian metric on $\mathbb{R}^d$; we denote this by $\langle \cdot, \cdot \rangle$ as usual. The Inverse Function Theorem also implies that i is a C^∞ diffeomorphism of $\mathbb{R}^d$ onto U. Now *equip* $\mathbb{R}^d$ *with the Riemannian metric which makes* i *an isometry;* let us denote this metric by $[\cdot, \cdot]$. Thus for any $p \in \mathbb{R}^d$ and $v, w \in \mathbb{R}^d$,

$$[v, w] := \langle di_p(v), di_p(w) \rangle.$$

(In other words, $[\cdot, \cdot]$ is the Riemannian metric on $\mathbb{R}^d$ coming from identifying $\mathbb{R}^d$ with the (open) submanifold U of $\mathbb{R}^d$, cf. Problem 18; here the *latter* "$\mathbb{R}^d$" is equipped with the standard Riemannian metric $\langle \cdot, \cdot \rangle$.) We know that U with the Riemannian metric $\langle \cdot, \cdot \rangle$ is not complete; hence since i is a surjective isometry, $\mathbb{R}^d$ with the Riemannian metric $[\cdot, \cdot]$ is not complete. Done! □

Alternative (for the non-complete example): Equip $\mathbb{R}^d$ with any explicit, sufficiently rapidly decaying Riemannian metric, for example $(g_{ij}(x))$ with

$$g_{ij}(x) = \delta_{ij} e^{-2\|x\|^2} \qquad (\text{with } \|x\|^2 = (x^1)^2 + \cdots + (x^d)^2). \tag{74}$$

Note that each matrix entry $g_{ij}(x)$ is a C^∞ function of $x \in \mathbb{R}^d$ and also the matrix $(g_{ij}(x))$ is positive definite for every $x \in \mathbb{R}^d$ (since it is a positive multiple of the identity matrix); hence the formula indeed gives a Riemannian metric on $\mathbb{R}^d$. To prove that this metric is not complete, consider the sequence of points $p_1, p_2, \ldots$ with $p_j := (j, 0, \cdots, 0)$. For any integers $1 \leq j \leq k$ we have

$$d(p_j, p_k) \leq e^{-j}. \tag{75}$$

Indeed, consider the C^∞ curve $c : [j, k] \to M$, $c(t) := (t, 0, \ldots, 0)$. This is a curve from p_j to p_k and its length with respect to the Riemannian metric (74) is

$$L(c) = \int_j^k \sqrt{\langle \dot{c}(t), \dot{c}(t) \rangle}\, dt = \int_j^k \sqrt{e^{-2t^2}}\, dt = \int_j^k e^{-t^2}\, dt < \int_j^\infty e^{-t}\, dt = e^{-j}.$$

(Here we used the fact that $t^2 > t$ for all $t > j \geq 1$.) This proves that (75) holds, and (75) in turn implies that $p_1, p_2, \ldots$ *is a Cauchy sequence in* $\mathbb{R}^d$ *equipped with the Riemannian metric in* (74). On the other hand the sequence $p_1, p_2, \ldots$ does not converge to any point in $\mathbb{R}^d$. (Recall here that "convergence" is a topological notion, i.e. it depends only on the topology of $\mathbb{R}^d$ and not on the metric metrizing it; cf. here also Lemma 2 in Lecture #3.) Hence $\mathbb{R}^d$ equipped with the Riemannian metric in (74) is not complete. □

Problem 26:

(a). Let us first note that this is *not* an immediate consequence of the following fact from basic point set topology: "Every closed subset of a complete metric space is itself a complete metric space". Namely, in that statement it is understood that the subset is endowed with the metric which is simply the *restriction* of the metric on the larger space. This is *not* the case in our situation; we typically have $d_M(p,q) \neq d_N(p,q)$ for $p,q \in M$; cf. Problem 18(c)!

However, as we'll see, the completeness of M is still easy to prove...

Let $p_1, p_2, \ldots$ be a Cauchy sequence in (M, d_M). By Problem 18(c) we have $d_N(p_j, p_k) \leq d_M(p_j, p_k)$ for any j,k; hence $p_1, p_2, \ldots$ is also a Cauchy sequence in (N, d_N). Therefore, since N is complete there exists a (unique) limit point $p := \lim_{j\to\infty} p_j$ in N. Recall that the last limit relation by definition means that

$$\lim_{j\to\infty} d_N(p_j, p) = 0. \tag{76}$$

Since M is closed in N we have $p \in M$. But since M is an *embedded* submanifold of N, the topology of M equals the subspace topology of M as a subset of N; therefore $\lim_{j\to\infty} d_N(p_j,p) = 0$ implies $\lim_{j\to\infty} d_M(p_j,p) = 0$, [14] i.e. $p = \lim_{j\to\infty} p_j$ *in* M.

This proves that M is complete. □

(b). ((E.g. there is an appropriate "isometric immersion" of $(0,\infty)$ into $\mathbb{R}^2$ with closed image; easy to draw a picture...))

[14] Some further explanation: Here we are using the fact that the relation "$p = \lim_{j\to\infty} p_j$" ($\Leftrightarrow$ "$p_j \to p$") only depends on the *topology* of the space which we are working in, and *not* on the choice of metric metrizing that topology. For example, $p_j \to p$ in N holds iff for every open neighborhood $U \subset N$ of p, there exists $J \in \mathbb{Z}^+$ such that $p_j \in U$ for all $j \geq J$ (and this is equivalent to (76) holding for *any* metric metrizing N's topology). Now in our situation we wish to prove that $\lim_{j\to\infty} d_M(p_j,p) = 0$, or equivalently that for every open set U in M with $p \in U$, there exists $J \in \mathbb{Z}^+$ such that $p_j \in U$ for all $j \geq J$. Let such an open set $U \subset M$ be given. Since M has the subspace topology as a subset of N, there is an open set V in N such that $U = M \cap V$. Next, since $p_j \to p$ in N there exists $J \geq \mathbb{Z}^+$ such that $p_j \in V$ for all $j \geq J$. But the points $p_1, p_2, \ldots$ all lie in M; hence $p_j \in V \cap M = U$ for all $j \geq J$. Therefore $p_j \to p$ in M.

Problem 27:

(a). This continuity is clear from

$$(77)\qquad |d(p,q)-d(p',q')| \leq d(p,p')+d(q,q'), \qquad \forall p,p',q,q' \in X.$$

[Proof of (77): By the triangle inequality we have

$$|d(p,q)-d(p',q)| \leq d(p,p') \quad \text{and} \quad |d(p',q)-d(p',q')| \leq d(q,q').$$

Hence

$$|d(p,q)-d(p',q')| \leq |d(p,q)-d(p',q)|+|d(p',q)-d(p',q')| \leq d(p,p')+d(q,q').$$

Done!]

(b). Take any $q \in X$. If $d(p,q) < r$ then setting $s = r-d(p,q) > 0$ we have $B_s(q) \subset B_r(p)$ (by the triangle inequality) and so $q \notin \partial B_r(p)$. If $d(p,q) > r$ then setting $s = d(p,q)-r > 0$ we have $B_s(q) \cap B_r(p) = \emptyset$ (again by the triangle inequality) and so $q \notin \partial B_r(p)$. This proves the stated inclusion. The set $\partial B_r(p)$ is closed since the boundary of *any* set (in any topological space) is closed. The fact that the set $\{q \in X \,:\, d(p,q) = r\}$ is closed is an immediate consequence of the fact that the metric d is continuous.

Next assume that (X,d) is a Riemannian manifold. Take any $q \in X$ with $d(p,q) = r$. By the definition of "d", there exists a sequence of pw C^∞ curves $\gamma_1, \gamma_2, \ldots$ on X such that each γ_j starts at p and ends at q, and $\ell_j := L(\gamma_j) < r + j^{-1}$; also $\ell_j \geq r$. We may assume that each γ_j is parametrized by arc length, and has domain $[0, \ell_j]$ where $\ell_j = L(\gamma_j)$. Take $J \in \mathbb{Z}^+$ so large that $J^{-1} < r$, and for each $j \geq J$ set

$$q_j := \gamma_j(r-j^{-1}).$$

Note that $\gamma_{j|[0,r-j^{-1}]}$ is a curve of length $r-j^{-1}$ from p to q_j; hence $d(p,q_j) \leq r-j^{-1}$ and $q_j \in B_r(p)$. On the other hand $\gamma_{j|[r-j^{-1},\ell_j]}$ is a curve of length

$$\leq \ell_j - (r-j^{-1}) < (r+j^{-1}) - (r-j^{-1}) = 2j^{-1}$$

from q_j to q; hence $d(q_j, q) < 2j^{-1}$. Hence $q_j \to q$ as $j \to \infty$. This shows that q is in the closure of the set $B_r(p)$. Also $q \notin B_r(p)$ since $d(p,q) = r$. Hence $q \in \partial B_r(p)$. We have thus proved that $\{q \in X \,:\, d(p,q) = r\} \subset B_r(p)$, and we are done. $\square$

(Remark: If r is so small that there exists some $r' > r$ such that $B_{r'}(0_p) \subset \mathcal{D}_p$ and $\exp_{|B_{r'}(0_p)}$ is a diffeomorphism onto an open set, then the identity $\partial B_r(p) = \{q \in X \,:\, d(p,q) = r\}$ is a trivial consequence of Theorem 4 in Lecture #4. This is the only situation which occurs in the proof of the Hopf-Rinow Theorem.)

(c). (The same argument appears on [12, p. 36].) We have

$$d(p,q) \leq d(p,p_0) + d(p_0,q)$$

by the triangle inequality; hence it now suffices to prove the opposite inequality. Let $\gamma : [a,b] \to X$ be any pw C^∞ curve with $\gamma(a) = p$ and $\gamma(b) = q$. Since $t \to d(p,\gamma(t))$ is a continuous function of t, and $d(p,\gamma(a)) = 0$, $d(p,\gamma(b)) = d(p,q) > r$, there must exist some $t_0 \in (a,b)$ such that $d(p,\gamma(t_0)) = r$. By part (b) we then have $\gamma(t_0) \in \partial B_r(p)$, and hence because of the way p_0 was chosen,

$$d(\gamma(t_0),q) \geq d(p_0,q).$$

Therefore,

$$L(\gamma) = L(\gamma_{[a,t_0]}) + L(\gamma_{[t_0,b]}) \geq d(p,\gamma(t_0)) + d(\gamma(t_0),q) \geq r + d(p_0,q).$$

Since this is true for every pw C^∞ curve from p to q, we have

$$d(p,q) \geq r + d(p_0,q) = d(p,p_0) + d(p_0,q),$$

and the proof is complete. □

Problem 28: The fact that every distance $d(p,q) < R$ is realized by a geodesic is proved by more or less exactly the same proof as the "key fact" in the proof of the Hopf-Rinow Theorem; cf. Lecture #5, pp. 9–11. Indeed, assume $q \in M$ and $r := d(p,q) < R$. Now the proof in Lecture #5, pp. 9–11 applies to our situation, word by word. The only difference is that now the geodesic

$$c(t) := \exp_p(tV)$$

(introduced in the last line of p. 9) is not guaranteed to be defined for all $t \in \mathbb{R}$, but it is certainly defined for all t with $|t| < R$, since $B_R(p) \subset \mathcal{D}_p$; in particular $c(t)$ is defined for all $t \in [0,r]$, and these are the only t-values which are ever considered in the proof.

Finally, $B_R(p) = \exp_p(B_R(0_p))$ is indeed an immediate consequence of the above fact. Indeed, the above fact implies $B_R(p) \subset \exp_p(B_R(0_p))$. On the other hand for every $v \in B_R(0_p)$, the geodesic $t \mapsto \exp(tv)$, $t \in [0,1]$, is a curve of length $\|v\|$ from p to $\exp_p(v)$, so that $d(p,\exp_p(v)) \leq \|v\| < R$, i.e. $\exp_p(v) \in B_R(p)$. Hence also the opposite inclusion, $\exp_p(B_R(0_p)) \subset B_R(p)$, holds. Done! □

Problem 29:

(Note that Jost mentions this generalization in the beginning of his proof of [12, Thm. 5.8.1].)

The proof in the two cases (fixed endpoints versus closed curves) is very similar, and we treat here only the first case. Thus let $c : I \to M$ be a curve. Set $p = c(0)$ and $q = c(1)$, and let $\mathcal{F}$ be the family of all pw C^∞ curves homotopic to c. Then pick a minimizing sequence (γ_n) for arc length in $\mathcal{F}$ Thus each γ_n is a pw C^∞ curve, and

$$\lim_{n\to\infty} L(\gamma_n) = L_0 := \inf_{c_0 \in \mathcal{F}} L(c_0).$$

Wlog, assume also

$$L(\gamma_n) \geq L_0 + 1, \qquad \forall n.$$

Set $R := L_0 + 2$ and

$$K := \overline{B_R(p)}.$$

Note that by construction, all the curves $\gamma_1, \gamma_2, \ldots$ are contained in K. By the Hopf-Rinow Theorem (Theorem 5:3, (1) $\Rightarrow$ (2)), K is compact. Hence the proof of Cor. 4.1 (an immediate application of Thm. 4.3') extends to show that there exists some $r_0 > 0$ such that for every point $p' \in K$, $\exp_{p'|B_{r_0}(0_{p'})}$ is a diffeomorphism onto an open set in M. By shrinking r_0 if necessary, we may assume $r_0 < 1$.

We have already remarked that all curves $\gamma_1, \gamma_2, \ldots$ are contained in K; in fact they are even contained in the smaller ball $\overline{B_{L_0+1}(p)}$, and hence since $r_0 < 1$ and $R = L_0 + 2$, it follows that the whole neighborhood $B_{r_0}(p')$ is contained in K, for all $p' \in \gamma_n$, any n. Hence all points ever considered in the proof of Theorem 1 in Lecture #5 lie in K, and so the proof carries over to our situation! $\square$

Problem 30:

(a). Consider the C^∞ map

$$f : \mathbb{R}^3 \to \mathbb{R}, \qquad f(x,y,z) = y^2 + z^2 - e^{2x}.$$

One verifies immediately that $S = f^{-1}(0)$. Furthermore

$$df_{(x,y,z)} = \begin{pmatrix} -2e^{2x} & 2y & 2z \end{pmatrix},$$

which has rank 1 for all $(x,y,z) \in \mathbb{R}^3$. Hence by [12, Lemma 1.3.2] (with the slightly more precise formulation in the notes to Lecture #2; note that this is what Jost's proof actually gives), every connected component of S in $\mathbb{R}^3$ is a closed differentiable submanifold of $\mathbb{R}^3$. Hence if we prove that S is connected then it follows that S itself is a differentiable submanifold of $\mathbb{R}^3$. However the connectedness is clear from the parametrization of S given in the statement of the problem. Indeed, set

$$g(x,\alpha) := (x, e^x \cos\alpha, e^x \sin\alpha).$$

Then g is a continuous (even C^∞) function from $\mathbb{R}^2$ to S. [15] Consider two arbitrary points in S; these can be expressed as $g(x,\alpha)$ and $g(x',\beta)$ for some $x, x', \alpha, \beta \in \mathbb{R}$. Then

$$c : [0,1] \to S, \qquad c(t) = g((1-t)x + tx', (1-t)\alpha + t\beta)$$

is a curve in S from $g(x,\alpha)$ to $g(x',\beta)$. This proves that S is path-connected, and thus connected. This completes the proof that S is a differentiable submanifold of $\mathbb{R}^3$.

Note also that S is closed; for example this follows from $S = f^{-1}(0)$ and the fact that f is continuous. $\square$

[15] Indeed, by inspection g is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}^3$, and $g(x,\alpha) \in S$ for all $(x,\alpha) \in \mathbb{R}^2$. Hence since S is a (disconnected) union of differentiable submanifolds of $\mathbb{R}^3$ and thus the topology of S agrees with the subset topology from $S \subset \mathbb{R}^3$, it follows that g is continuous also as a function from $\mathbb{R}^2$ to S.

(b). Set $p_1 := (x_0, -e^{x_0}, 0) \in S$, and let γ be the following C^∞ curve (half circle) on S:

$$\gamma : [0, \pi] \to S, \qquad \gamma(t) = (x_0, e^{x_0} \cos t, e^{x_0} \sin t).$$

This is a curve from p_0 to p_1, and $L(\gamma) = \pi e^{x_0}$ (by Problem 18(b)). Hence $d(p_0, p_1) \le \pi e^{x_0}$. By the Hopf-Rinow Theorem there exists a geodesic c from p_0 to p_1 with

$$L(c) = d(p_0, p_1) \le \pi e^{x_0}.$$

We can take c to be parametrized by arc length; thus $\|\dot{c}\| \equiv 1$ and the domain of c is the interval $[0, L(c)]$. Now let R be the reflection map $(x, y, z) \mapsto (x, y, -z)$. This is an isometry of $\mathbb{R}^3$ onto itself, and $R(S) = S$; hence R is also an isometry of S onto itself. Therefore R maps any geodesic in S to a geodesic in S, and in particular the curve

$$\widetilde{c} := R \circ c : [0, L(c)] \to S$$

is a geodesic in S, with $\|\dot{\widetilde{c}}\| \equiv 1$ and $L(\widetilde{c}) = L(c)$. We have $R(p_0) = p_0$ and $R(p_1) = p_1$; hence $\widetilde{c}$ is a geodesic from p_0 to p_1, just like c. Take $v, \widetilde{v} \in T_{p_0}S$ so that $c(t) = \exp_{p_0}(tv)$ and $\widetilde{c}(t) = \exp_{p_0}(t\widetilde{v})$ for $t \in [0, L(c)]$. Note that

$$\|v\| = \|\widetilde{v}\| = 1,$$

since $\|c\| \equiv \|\dot{\widetilde{c}}\| \equiv 1$. Furthermore,

$$v \neq \widetilde{v},$$

since $c \not\equiv \widetilde{c}$. [Proof of $c \not\equiv \widetilde{c}$: The second coordinate of $c(t)$ is a continuous function of t starting at e^{x_0} and ending at e^{-x_0}; hence for some $t \in (0, L(c))$ this coordinate must equal 0. For this t we have $c(t) = (x, 0, z) \in S$ for some $x, z \in \mathbb{R}$, and from the definition of S it follows that $z \neq 0$ and so $R(c(t)) \neq c(t)$, i.e. $\widetilde{c}(t) \neq c(t)$ for this t.] Now

$$\exp_{p_0}(L(c)v) = c(L(c)) = p_1 = \widetilde{c}(L(c)) = \exp_{p_0}(L(c)\widetilde{v}),$$

and $L(c)v \neq L(c)\widetilde{v}$, $\|L(c)v\| = \|L(c)\widetilde{v}\| = L(c) \le \pi e^{x_0}$. This proves that for every $r > \pi e^{x_0}$, the function $\exp_{p_0}$ is non-injective on the open ball $B_r(0_p) \subset T_p(S)$. Hence $i(p_0) \le \pi e^{x_0}$. □

(See also alternative solution on the next page!)

Alternative: Set $p_1 := (x_0, -e^{x_0}, 0) \in S$. Let γ_1, γ_2 be the following two curves (half circles) on S:

$$\gamma_1, \gamma_2 : [0, \pi] \to S, \qquad \gamma_1(t) = (x_0, e^{x_0} \cos t, e^{x_0} \sin t); \\ \gamma_2(t) = (x_0, e^{x_0} \cos t, -e^{x_0} \sin t).$$

Both these are curves from p_0 to p_1, and

$$L(\gamma_1) = L(\gamma_2) = \pi e^{x_0}$$

(by Problem 18(b)). (Hence $d(p_0, p_1) \leq \pi e^{x_0}$.)

Now by Theorem 1 in Lecture #5, generalized to complete manifolds (cf. Problem 29), there exist geodesics c_1 and c_2 homotopic to γ_1 and to γ_2, respectively, and from the proof of that theorem we see that we can take $L(c_j)$ to be smaller than or equal to the length of *any* pw C^∞ curve in the homotopy class of γ_j; in particular

$$L(c_j) \leq L(\gamma_j) = \pi e^{x_0} \qquad (j = 1, 2).$$

We can take c_1, c_2 to be parametrized by arc length; then there exist two unit vectors $v_1, v_2 \in T_{p_0} S$ such that

$$c_j(t) = \exp_{p_0}(t v_j), \qquad t \in [0, L(c_j)].$$

In particular

$$\exp_{p_0}(L(c_j) v_j) = p_1 \qquad \text{for } \ j = 1, 2,$$

It follows that *if we can only prove that* $v_1 \neq v_2$, then

$$i(p_0) \leq \max(L(c_1), L(c_2)) \leq \pi e^{x_0},$$

and the proof will be complete.

In order to prove $v_1 \neq v_2$, let us assume the opposite, $v_1 = v_2$. This means that $c_1 \equiv c_2$, and so γ_1 and γ_2 are homotopic. However this is "obviously" not the case! (Details: $\gamma_1 \simeq \gamma_2$ would imply $\gamma_1 \cdot \overline{\gamma}_2 \simeq \gamma_2 \cdot \overline{\gamma}_2 \simeq p_0$, the constant curve at p_0. Now note that the map

$$F : S \to S^1, \quad (x, e^x \cos \alpha, e^x \sin \alpha) \mapsto (\cos \alpha, \sin \alpha)$$

is well-defined and continuous, and it maps the loop $\gamma_1 \cdot \overline{\gamma}_2$ to the loop $t \mapsto (\cos t, \sin t)$, $[0, 2\pi] \to S^1$. Hence, composing any homotopy showing $\gamma_1 \cdot \overline{\gamma}_2 \simeq p_0$ with F, we obtain that the loop $t \mapsto (\cos t, \sin t)$ in S^1 represents the identity element in $\pi_1(S^1)$. However this is not the case, as we discussed in Lecture #6, and as is carefully proved in Hatcher, [7, Thm. 1.7].) $\square$

Problem 31:

The following solution is sketchy and leaves out several details.

By [7, Prop. 1.5] we are free to choose the basepoint x_0. Let us choose x_0 so that it does not lie on any line between two points in $\{p_1, \ldots, p_n\}$. Let $\widetilde{r}_j$ be the ray starting at x_0 and going through p_j; it follows from our choice of x_0 that the n rays $\widetilde{r}_1, \ldots, \widetilde{r}_n$ are distinct. After renaming the points $p_1, \ldots, p_n$ we may assume that the rays $\widetilde{r}_1, \ldots, \widetilde{r}_n$ are ordered in positive direction. Now choose rays $r_1, \ldots, r_n$ with startpoint x_0 such that r_1 lies between $\widetilde{r}_n$ and $\widetilde{r}_1$, and r_j for $j \in \{2, \ldots, n\}$ lies between $\widetilde{r}_{j-1}$ and $\widetilde{r}_j$. Let $\widetilde{A}_n$ be the infinite open wedge between r_n and r_1 coontaining $\widetilde{r}_n \setminus \{x_0\}$; similarly for $j \in \{1, \ldots, n-1\}$ let $\widetilde{A}_j$ be the infinite open wedge between r_j and r_{j+1} containing $\widetilde{r}_j$. Take $\varepsilon > 0$ small. (Specifically, ε should be smaller than the distance between x_0 and r_j for each j.) Let A_j be the open ε-neighborhood of $\widetilde{A}_j$ (viz., the set of points in $\mathbb{R}^2$ which have distance $< \varepsilon$ to some point in A_j), but with the point p_j removed. The reader is adviced to draw a picture of the situation!

Now $A_1, \ldots, A_n$ are open and path-connected subsets of

$$X := \mathbb{R}^2 \setminus \{p_1, \ldots, p_n\}$$

with $X = \cup_{j=1}^n A_j$, and $A_j \cap A_k$ is path-connected for all $j, k \in \{1, \ldots, n\}$; hence van Kampen's theorem can be applied with $A_1, \ldots, A_n$; in particular the natural homomorphism

$$\Phi : \pi_1(A_1, x_0) * \cdots * \pi_1(A_n, x_0) \to \pi_1(X, x_0)$$

is *surjective.* Note also that for any $j \neq k \in \{1, \ldots, n\}$, the set $A_j \cap A_k$ is simply connected (proof?) i.e. $\pi_1(A_j \cap A_k) = \{e\}$. Hence van Kampen's theorem implies that Φ *is an isomorphism.* Finally each A_j is homotopy equivalent with S^1 (proof?); hence $\pi_1(A_j) \cong \mathbb{Z}$, and so we conclude that

$$\pi_1(X, x_0) \textit{ is a free group with } n \textit{ generators.}$$

Generators: $[\gamma_1], \ldots, [\gamma_n]$, where γ_j is a loop that is contained in A_j and goes one time around p_j. $\square$

Problem 32:

(a). Let us first prove that $\widetilde{M}$ is Hausdorff. Let p and q be two distinct points in $\widetilde{M}$. Then $\pi(p), \pi(q) \in M$. If $\pi(p) \neq \pi(q)$, then since M is Hausdorff, there exist disjoint open sets $U, V \subset M$ with $\pi(p) \in U$ and $\pi(q) \in V$. Then $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint open sets in $\widetilde{M}$, and $p \in \pi^{-1}(U)$ and $q \in \pi^{-1}(V)$. On the other hand if $\pi(p) = \pi(q)$, then by the definition of "covering space" there is an open neighborhood U of $\pi(p) = \pi(q)$ in M such that $\pi^{-1}(U)$ can be written as a union $\pi^{-1}(U) = \sqcup_{j \in J} U_j$, where for each $j \in J$, U_j is an open set in $\widetilde{M}$ and $\pi_{|U_j}$ is a homeomorphism of U_j onto U, and the sets U_j $(j \in J)$ are pairwise disjoint. Now $p, q \in \pi^{-1}(U)$ and hence there are unique $i, j \in J$ such that $p \in U_i$ and $q \in U_j$. If $i = j$ then $\pi(p) = \pi(q)$ and the fact that $\pi_{|U_i}$ is injective imply $p \neq q$, contrary to our assumption. Therefore $i \neq j$, and now U_i and U_j are two disjoint open sets in $\widetilde{M}$ with $p \in U_i$ and $q \in U_j$. This proves that $\widetilde{M}$ is Hausdorff.

Next we prove that $\widetilde{M}$ is locally Euclidean (of dimension d). Let p be an arbitrary point in $\widetilde{M}$. Then $\pi(p) \in M$, and since $\pi : \widetilde{M} \to M$ is a covering space, $\pi(p)$ has an open neighborhood U in M such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{M}$, each of which is mapped homeomorphically onto U by π. Exactly one of these open sets in $\widetilde{M}$ contains p; call this open set $\widetilde{U} \subset \widetilde{M}$. Thus $\pi_{|\widetilde{U}}$ is a homeomorphism of $\widetilde{U}$ onto U. Furthermore, since M is a d-dimensional topological manifold, $\pi(p)$ has an open neighborhood V in M which is homeomorphic to an open subset of $\mathbb{R}^d$. It follows that also $W := U \cap V$ is homeomorphic to an open subset of $\mathbb{R}^d$; let $\varphi : W \to \mathbb{R}^d$ be one such homeomorphism. Now $\widetilde{W} := (\pi_{|\widetilde{U}})^{-1}(W)$ is an open subset of $\widetilde{U}$ containing p, and $\pi_{|\widetilde{W}}$ is a homeomorphism of $\widetilde{W}$ onto W. It follows that $\varphi \circ \pi_{|\widetilde{W}}$ is a homeomorphism of $\widetilde{W}$ onto an open subset of $\mathbb{R}^d$. The fact that every point $p \in \widetilde{M}$ has such an open neighborhood $\widetilde{W}$ in $\widetilde{M}$ proves that $\widetilde{M}$ is locally Euclidean.

$\widetilde{M}$ is connected and second countable by assumption; hence also paracompact (cf. the notes to Lecture #1).

Hence $\widetilde{M}$ is a topological manifold of dimension d. □

Remark: In fact the assumption that $\widetilde{M}$ is second countable is redundant; any connected covering space $\widetilde{M}$ of a topological manifold is *automatically* second countable. This is a consequence of the fact that the fundamental group $\pi_1(M)$ of any topological manifold M is *countable;* cf., e.g., Lee, [15, Prop. 1.16]. (Once we know that $\pi_1(M)$ is countable, the fact that $\widetilde{M}$ is second countable is proved by fairly simple arguments using the theory developed in [7, Ch. 1.3]; cf. also Problem 2(a) above.)

(In this connection, here's an issue which for a moment had me confused: One might think that the "obvious" map π from the Long Line L (cf. wikipedia) to the circle $S^1 \simeq \mathbb{R}/\mathbb{Z}$ makes L a covering space of S^1; but L is *not* second countable! The resolution to this seeming paradox is that the map $\pi : L \to S^1$ is in fact not continuous, and hence not a covering map; this is discussed here.)

(b). By part (a), $\widetilde{M}$ is a topological manifold and $\dim \widetilde{M} = \dim M = d$, say. Let $\mathcal{A}$ be the C^∞ structure on M, and set

$$\widetilde{\mathcal{A}} := \{(\widetilde{U}, x \circ \pi) \,:\, (U, x) \in \mathcal{A} \text{ and } \widetilde{U} \text{ is an open subset of } \widetilde{M} \text{ which is mapped homeomorphically onto } U \text{ by } \pi\}$$

Note that for every $(\widetilde{U}, x \circ \pi) \in \widetilde{\mathcal{A}}$, $x \circ \pi$ is a homeomorphism of $\widetilde{U}$ onto an open subset of $\mathbb{R}^d$. Furthermore for every $p \in \widetilde{M}$ there is some $(\widetilde{U}, x \circ \pi) \in \widetilde{\mathcal{A}}$ such that $p \in \widetilde{U}$ (this is proved by the same argument which we used to prove that $\widetilde{M}$ is locally Euclidean in part (a)). Hence $\widetilde{\mathcal{A}}$ is a topological atlas on M. We claim that $\widetilde{\mathcal{A}}$ is in fact a C^∞ atlas on $\widetilde{M}$. To show this it remains to prove C^∞ compatibility between the charts in $\widetilde{\mathcal{A}}$. Thus let $(\widetilde{U}, x \circ \pi)$ and $(\widetilde{V}, y \circ \pi)$ be two arbitrary elements in $\widetilde{\mathcal{A}}$; set $U := \pi(\widetilde{U})$ and $V := \pi(\widetilde{V})$ so that $(U, x), (V, y) \in \mathcal{A}$. We have to prove that the map

$$(y \circ \pi) \circ (x \circ \pi)^{-1} : x \circ \pi(\widetilde{U} \cap \widetilde{V}) \to y \circ \pi(\widetilde{U} \cap \widetilde{V})$$

is C^∞. But note that $\pi(\widetilde{U} \cap \widetilde{V}) = U \cap V$ and the above map equals

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V),$$

which is C^∞ since $(U, x), (V, y) \in \mathcal{A}$. Hence we have proved that $\widetilde{\mathcal{A}}$ is a C^∞ atlas on $\widetilde{M}$.

By Problem 4, $\widetilde{\mathcal{A}}$ determines a (unique) C^∞ structure on $\widetilde{M}$. Let us prove that $\widetilde{M}$ equipped with this C^∞ structure has the desired properties. First we prove that π is C^∞. Given $p \in \widetilde{M}$, take $(\widetilde{U}, x \circ \pi) \in \widetilde{\mathcal{A}}$ with $p \in \widetilde{U}$; also set $U := \pi(U)$, so that $(U, x) \in \mathcal{A}$. Then wrt the charts $(\widetilde{U}, x \circ \pi)$ and (U, x), the map π is represented by

$$x \circ \pi \circ (x \circ \pi)^{-1} : x \circ \pi(\widetilde{U}) \to x(U).$$

But $\pi(\widetilde{U}) = U$, and we see that the last map is simply the identity map on $x(U) \subset \mathbb{R}^d$, which of course is a C^∞ map. Hence π is C^∞ locally near p, and since this is true for all $p \in \widetilde{M}$, the map π is C^∞.

Next we prove that every point $p \in M$ has an open neighborhood with the stated property. Let $p \in M$ be given. We know that p has an open neighborhood U in M such that $\pi^{-1}(U)$ is a union of disjoint open sets in $\widetilde{M}$, each of which is mapped *homeomorphically* onto U by π. We will prove that *any* such U is in fact ok for us. Thus let $\widetilde{U}$ be any one of the open sets in $\widetilde{M}$ with the property that $\pi_{|\widetilde{U}}$ is a homeomorphism of $\widetilde{U}$ onto U. We claim that $\pi_{|\widetilde{U}}$ is in fact a *diffeomorphism* of $\widetilde{U}$ onto U. We proved above that π is C^∞; hence $\pi_{|\widetilde{U}}$ is C^∞, and it remains to prove that $(\pi_{|\widetilde{U}})^{-1} : U \to \widetilde{U}$ is C^∞. Take any point $q \in U$ and set $\widetilde{q} := (\pi_{|\widetilde{U}})^{-1}(q) \in \widetilde{U}$. Take $(\widetilde{V}, y \circ \pi) \in \widetilde{\mathcal{A}}$ with $\widetilde{q} \in \widetilde{V}$, and set $V := \pi(\widetilde{V})$ so that $(V, y) \in \mathcal{A}$. Set $W := U \cap V$; then

$(W, y_{|W}) \in \mathcal{A}$ (since $\mathcal{A}$ is a maximal C^∞ atlas on M); also $\widetilde{W} := (\pi_{|\widetilde{U}})^{-1}(W)$ is mapped homeomorphically onto W by π and so $(\widetilde{W}, y_{|W} \circ \pi) \in \widetilde{\mathcal{A}}$. Now wrt the charts $(W, y_{|W})$ and $(\widetilde{W}, y_{|W} \circ \pi)$, the map $(\pi_{|\widetilde{U}})^{-1}$ is represented by

$$(y \circ \pi) \circ (\pi_{|\widetilde{U}})^{-1} \circ (y_{|W})^{-1} : y(W) \to y \circ \pi(\widetilde{W}).$$

One verifies that this is simply the identity map on $y(W) \subset \mathbb{R}^d$, which of course is a C^∞ map. Note also that $q \in W$. The fact that every point $q \in U$ has such an open neighborhood W in which $(\pi_{|\widetilde{U}})^{-1}$ is C^∞, implies that $(\pi_{|\widetilde{U}})^{-1}$ is C^∞. This completes the proof that our C^∞ structure on $\widetilde{M}$ has all the desired properties.

Finally we prove that the above C^∞ structure on $\widetilde{M}$ is uniquely determined by the stated requirements. (This is more or less obvious, but it becomes somewhat technical to write out the details – at least in the way I've done it. I think it is the least important part of this problem...) Thus let $\mathcal{B}$ be *any* C^∞ structure on the topological manifold $\widetilde{M}$ which satisfies the stated requirements; our task is then to prove that $\mathcal{B}$ is compatible with the C^∞ atlas $\widetilde{M}$. Let $(\widetilde{U}, \varphi)$ be any chart in $\mathcal{B}$ and let $(\widetilde{V}, y \circ \pi)$ be any chart in $\widetilde{\mathcal{A}}$; then our task is to prove that the map

$$(y \circ \pi) \circ \varphi^{-1} : \varphi(\widetilde{U} \cap \widetilde{V}) \to y \circ \pi(\widetilde{U} \cap \widetilde{V})$$

is a diffeomorphism,

Set $V := \pi(\widetilde{V})$ so that $(V, y) \in \mathcal{A}$ and $\pi_{|\widetilde{V}}$ is a homeomorphism of $\widetilde{V}$ onto V. Set $\widetilde{W} = \widetilde{U} \cap \widetilde{V}$ and $W = \pi(\widetilde{W})$; then $\widetilde{W}$ is an open subset of $\widetilde{V}$, W is an open subset of V, and $\pi_{|\widetilde{W}}$ is a homeomorphism of $\widetilde{W}$ onto W. (For nontriviality, assume $W \neq \emptyset$.) Also $(\widetilde{W}, \varphi_{|\widetilde{W}}) \in \mathcal{B}$ and $(W, y_{|W}) \in \mathcal{A}$ (since $\mathcal{B}$ and $\mathcal{A}$ are maximal); hence also $(\widetilde{W}, \pi \circ y_{|W}) \in \widetilde{\mathcal{A}}$. Our task is to prove that the map

$$(y \circ \pi) \circ \varphi^{-1} : \varphi(\widetilde{W}) \to y(W) \tag{78}$$

is a diffeomorphism, i.e. that both the map (78) and its inverse,

$$\varphi \circ (y \circ \pi)^{-1} : y(W) \to \varphi(\widetilde{W}), \tag{79}$$

are C^∞.

Let $p \in W$ and set $\widetilde{p} := (\pi_{|\widetilde{W}})^{-1}(p)$. By the requirement which we have imposed on $\mathcal{B}$, there is an open neighborhood Ω' of p in M such that $\pi^{-1}(\Omega')$ is a union of disjoint open sets in $\widetilde{M}$, each of which is mapped diffeomorphically (wrt $\mathcal{B}$) onto Ω' by π. Among these open sets in $\widetilde{M}$, let $\widetilde{\Omega}'$ be the one which contains $\widetilde{p}$. Then set $\widetilde{\Omega} := \widetilde{\Omega}' \cap \widetilde{W}$; it follows that $\widetilde{\Omega}$ is an open subset of $\widetilde{W}$ which contains $\widetilde{p}$ and which is mapped diffeomorphically (wrt $\mathcal{B}$) by π onto $\Omega := \pi(\widetilde{\Omega}' \cap \widetilde{W})$ which is an open subset of W containing p. The last statement (together with $(\widetilde{W}, \varphi_{|\widetilde{W}}) \in \mathcal{B}$ and $(W, y_{|W}) \in \mathcal{A}$) implies that both the maps $y \circ \pi \circ \varphi^{-1} : \varphi(\widetilde{\Omega}) \to y(\Omega)$ and $\varphi \circ \pi^{-1} \circ y^{-1} : y(\Omega) \to \varphi(\widetilde{\Omega})$ are C^∞. Those maps are restrictions of the maps (78) and (79), and the fact that any point p has such a neighborhood Ω in W now implies that the two maps (78) and (79) are C^∞, and we are done. $\square$

(c). By part (b), $\widetilde{M}$ has a C^∞ manifold structure which is uniquely determined by the stated requirements (since any isometry is a diffeomorphism). Note that $\pi : \widetilde{M} \to M$ is an immersion (since locally it is a diffeomorphism); hence by Problem 18(a) there is a unique Riemannian structure on $\widetilde{M}$ such that $\langle v, w\rangle = \langle d\pi(v), d\pi(w)\rangle$ for any $p \in \widetilde{M}$ and $v, w \in T_p\widetilde{M}$. It is clear from part (b) that this Riemannian structure has the desired properties. □

(d). This is easily deduced by inspecting the solution to Problem 9(b). Indeed, there we saw that $\Gamma\backslash M$ is a topological manifold and $\pi : M \to \Gamma\backslash M$ is a continuous map. Now consider an arbitrary point in $\Gamma\backslash M$, say $[p]$ with $p \in M$. By (25) there is some $U \in \mathcal{I}$ (viz., an open set in M which is injectively embedded in $\Gamma\backslash M$) with $p \in U$. By (24), $\pi(U)$ is an open set in $\Gamma\backslash M$ and $\pi_{|U}$ is a homeomorphism from U onto $\pi(U)$. Now for every point $q \in M$, the equivalence class $[q] = \pi^{-1}(\pi(q))$ consists exactly of the points $\gamma(q)$ $(\gamma \in \Gamma)$, and these are pairwise distinct (since Γ acts freely on M). Hence $\pi^{-1}(\pi(U))$ is a disjoint union of the sets $\gamma(U)$ $(\gamma \in \Gamma)$:

$$\pi^{-1}(\pi(U)) = \bigsqcup_{\gamma\in\Gamma} \gamma(U). \tag{80}$$

Here for each $\gamma \in \Gamma$, $\gamma(U)$ is open in M, since γ is a homeomorphism; furthermore $\pi_{|\gamma(U)}$ is a homeomorphism of $\gamma(U)$ onto $\pi(U)$, since $\pi_{|\gamma(U)} = \pi_{|U} \circ (\gamma_{|U})^{-1}$, i.e. a composition of two homeomorphisms. Hence (80) expresses $\pi^{-1}(\pi(U))$ as a union of disjoint open sets in M, each of which is mapped homeomorphically onto $\pi(U)$ by π. The fact that each point $[p]$ in $\Gamma\backslash M$ has such an open neighborhood $\pi(U)$ proves that $\pi : M \to \Gamma\backslash M$ is a covering space of $\Gamma\backslash M$. □

Comments to part (d): Hatcher in [7, Prop. 1.40] proves a stronger result under a weaker assumption. Indeed, note that our assumption that Γ acts freely and properly discontinuously on M implies that the action is a "covering space action" in the terminology of [7, p. 72]; this implication is seen in the solution to Problem 9(b); indeed it is equivalent to (25). (This is also the content of [7, p. 81, Problem 23].) It is worth pointing out that the condition that $\Gamma < \text{Homeo}(M)$ acts by a "covering space action" on M does not guarantee $\Gamma\backslash M$ to be Hausdorff; cf. [7, p. 81, Problem 25].

Problem 33:

WLOG we assume $U = M$. (To see that this is really no loss of generality, note that if we prove (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c) in the special case $U = M$, then the general case follows by applying that statement to the vector bundle $(E_{|U}, \pi, U)$.)

Thus our task is to prove that the following statements are equivalent:

(a) E is trivial;

(b) there is some φ such that (M, φ) is a bundle chart for E;

(c) there is a *basis of sections* in ΓE, i.e. sections $s_1, \ldots, s_n \in \Gamma E$ such that $s_1(p), \ldots, s_n(p)$ is a basis of E_p for every $p \in M$.

Here (a) $\Leftrightarrow$ (b) is immediate by inspecting the definitions. Indeed, by definition E is trivial iff there is a bundle isomorphism $\varphi : E \to M \times \mathbb{R}^n$, i.e. a C^∞ diffeomorphism $\varphi : E \to M \times \mathbb{R}^n$ with $\text{pr}_1 \circ \varphi = \pi$ such that $\varphi_x = \varphi_{|E_x}$ is a vector space isomorphism $E_x \to \{x\} \times \mathbb{R}^n$ for each $x \in M$. But this is the same as saying that (M, φ) is a bundle chart for E.

(b) $\Rightarrow$ (c): Let φ be such that (M, φ) is a bundle chart for E. Let $e_1, \ldots, e_n$ be the standard basis of $\mathbb{R}^n$. For each $j \in \{1, \ldots, n\}$ we define the function $s_j : M \to E$ by $s_j(x) = \varphi^{-1}(x, e_j)$; then s_j is C^∞ and $\pi \circ s_j = 1_M$; hence $s_j \in \Gamma(E)$. Now for each $x \in M$, since $e_1, \ldots, e_n$ is a basis for $\mathbb{R}^n$ and φ_x^{-1} is a vector space isomorphism $\{x\} \times \mathbb{R}^n \to E_x$, it follows that $s_1(x), \ldots, s_n(x)$ is a basis of E_x. Hence $s_1, \ldots, s_n$ form a basis of sections of E.

(c) $\Rightarrow$ (b): Assume that $s_1, \ldots, s_n$ is a basis of sections of E. Let us define the map $\psi : M \times \mathbb{R}^n \to E$ by

$$\psi(x, (c_1, \ldots, c_n)) := \sum_{j=1}^{n} c_j \cdot s_j(x) \in E_x.$$

Clearly ψ is C^∞ and $\pi \circ \psi = 1_M$. Furthermore $\psi(x, \cdot)$ is a vector space isomorphism $\mathbb{R}^n \to E_x$ for each $x \in M$, since $s_1(x), \ldots, s_n(x)$ is a basis of E_x. It follows that ψ is a bijection of $M \times \mathbb{R}^n$ onto E. Let

$$\varphi = \psi^{-1} : E \to M \times \mathbb{R}^n.$$

It follows that $\varphi_x := \varphi_{|E_x}$ is a vector space isomorphism $E_x \to \{x\} \times \mathbb{R}^n$ for each $x \in M$. It remains to prove that φ is a diffeomorphism. We already know that $\varphi^{-1} = \psi$ is C^∞, so it suffices to prove that φ is C^∞, and for this it suffices to prove that every point in E has an open neighborhood in E in which φ is C^∞.

Thus let $p_0 \in E$ be given. Set $x_0 = \pi(p_0)$. Choose a bundle chart $(U, \widetilde{\varphi})$ for E with $x_0 \in U$ and also a chart (V, α) for M with $x_0 \in V$. In fact we may assume $V = U$, since otherwise we may replace $(U, \widetilde{\varphi})$ with $(U \cap V, \widetilde{\varphi}_{|U \cap V})$

and replace (V, α) with $(U \cap V, \alpha_{|U\cap V})$. Thus from now on $(U, \widetilde{\varphi})$ is a bundle chart for E, (U, α) is a chart for M, and $x_0 \in U$.

Let us write "1" for the identity map on $\mathbb{R}^n$; then $(U \times \mathbb{R}^n, (\alpha, 1))$ is a chart on $M \times \mathbb{R}^n$ [16] and $(\pi^{-1}(U), (\alpha, 1) \circ \widetilde{\varphi})$ is a chart on E. With respect to these two charts, the map ψ is represented by the map

$$(\alpha, 1) \circ \widetilde{\varphi} \circ \psi \circ (\alpha, 1)^{-1} : \alpha(U) \times \mathbb{R}^n \to \alpha(U) \times \mathbb{R}^n,$$

and we compute that this map equals

$$(y, (c_1, \ldots, c_n)) \mapsto \Big(y, \sum_{j=1}^{n} c_j \cdot \mathrm{pr}_2(\widetilde{\varphi}(s_j(\alpha^{-1}(y))))\Big), \tag{81}$$

where pr_2 is the projection map $U \times \mathbb{R}^n \to \mathbb{R}^n$. Now for each $y \in \alpha(U)$, the vectors

$$\mathrm{pr}_2(\widetilde{\varphi}(s_1(\alpha^{-1}(y)))), \ldots, \mathrm{pr}_2(\widetilde{\varphi}(s_n(\alpha^{-1}(y)))) \tag{82}$$

form a basis of $\mathbb{R}^n$, since $s_1(\alpha^{-1}(y)), \ldots, s_n(\alpha^{-1}(y))$ form a basis of $E_{\alpha^{-1}(y)}$. Let $T(y)$ be the real $n \times n$ matrix formed by the columns of the vectors in (82). It follows that this matrix is invertible for every $y \in \alpha(U)$, and that the map in (81) is given by

$$(y, c) \mapsto \big(y, T(y) \cdot c\big). \tag{83}$$

(Remember that we represent vectors in $\mathbb{R}^n$ as column matrices.) It follows that φ, which is the inverse of ψ, with respect to the two charts above is represented by the *inverse* of (83), i.e. by the map

$$\alpha(U) \times \mathbb{R}^n \to \alpha(U) \times \mathbb{R}^n, \qquad (y, c) \mapsto (y, T(y)^{-1} \cdot c).$$

Our task is to prove that this map is C^∞ (it is a map from an open subset of $\mathbb{R}^d \times \mathbb{R}^n = \mathbb{R}^{d+n}$ to $\mathbb{R}^{d+n}$). However this is clear from the formula of the inverse matrix $T(y)^{-1}$ in terms of the adjunct of $T(y)$; cf. here. (Indeed, every entry of $T(y)$ is a C^∞ function of y since it is a composition of C^∞ functions, and using the formula for $T(y)^{-1}$ we see that each of the n coordinates of

$$T(y)^{-1} \cdot c$$

equals a certain polynomial in the entries of $T(y)$ and the coordinates of c, divided by $\det(T(y))$, and $\det(T(y))$ is a nowhere vanishing, C^∞ function of $y \in \alpha(U)$.)

This completes the proof that (U, φ) is a bundle chart for E, and thus of the implication (c) $\Rightarrow$ (b). □

[16] Of course, "$(\alpha, 1)$" here stands for the map $U \times \mathbb{R}^n \to \alpha(U) \times \mathbb{R}^n$, $(x, v) \mapsto (\alpha(x), v)$. Cf. footnote 9 above – in the (pedantic) language of that footnote, we would write "$[\alpha, 1]$" in place of "$(\alpha, 1)$".

Problem 34:

The fact that there exist unique *functions* $\alpha^1, \ldots, \alpha^n \in C^\infty(U)$ satisfying $s = \alpha^j s_j$ is clear from the fact that $s_1(p), \ldots, s_n(p)$ is a basis of E_p for every $p \in U$. The fact that each function α^j is C^∞ is clear from the proof of "(c) $\Rightarrow$ (b)" in Problem 33. $\square$

Problem 35:

(a) In fact we can achieve this for *any* open set $V \subset U$ whose closure in U is compact. (This clearly suffices for us, since every point $p \in U$ is contained in such a set V.) Indeed, take any such set V. Let K be the closure of V in U; thus K is compact by our assumption. Now by Problem 7(d), there exists a C^∞ function $f : M \to [0, 1]$ which has compact support contained in U, and which satisfies $f_{|K} \equiv 1$. Let us define the function $s' : M \to E$ by

$$s'(p) = \begin{cases} f(p)s(p)\ (\in E_p) & \text{if } p \in U \\ 0\ (\in E_p) & \text{if } p \notin U. \end{cases}$$

Then $\pi \circ s' = 1_M$ by construction. Also s' is C^∞. [Proof: Let C be the support of f; this is a compact set contained in U. Let $U' = M \setminus C$; this is an open set. Now $s'_{|U} = f_{|U} \cdot s$ is C^∞, and $s'_{|U'}$ is C^∞, since it is identically zero. Also $U \cup U' = M$. Hence every point $p \in M$ has an open neighbourhood in which s' is C^∞; this implies that s' is C^∞ throughout M.] Hence $s' \in \Gamma(E)$. Also for every $p \in V$ we have $s'(p) = f(p)s(p) = s(p)$; hence $s'_{|V} = s_{|V}$. Done! $\square$

(b) Let $p \in M$ be given. Let (U, φ) be a bundle chart with $p \in U$, and let $\widetilde{b}_1, \ldots, \widetilde{b}_n$ be the corresponding basis of sections of $E_{|U}$ (cf. Problem 33; thus $\widetilde{b}_j(y) = \varphi^{-1}(y, e_j)$, $\forall y \in U$). Now by part (a) there exist an open subset $V \subset U$ with $p \in V$ and global sections $b_1, \ldots, b_n \in \Gamma(E)$ such that $b_{j|V} = \widetilde{b}_{j|V}$ for $j = 1, \ldots, n$. In other words $b_j(y) = \widetilde{b}_j(y)$ for all $y \in V$, $j = 1, \ldots, n$, and thus $b_1(y), \ldots, b_n(y)$ is a basis of E_y for each $y \in V$. Hence $b_{1|V}, \ldots, b_{n|V}$ form a basis of sections of $E_{|V}$. $\square$

(c) Given p, we choose $V, b_1, \ldots, b_n$ as in part (b). Now also let $v \in E_p$ be given. Since $b_1(p), \ldots, b_n(p)$ is a basis of E_p, there exist (unique) $c^1, \ldots, c^n \in \mathbb{R}$ such that $v = c^j \cdot b_j(p)$. Set $s = c^j b_j \in \Gamma E$. Then $s(p) = v$. $\square$

Problem 36:

Cf., e.g., Lee, [15, Lemma 10.6].

If (V, x) is any chart for M and $\alpha \in A$ is such that $U_\alpha \cap V \neq \emptyset$, then we let $\sigma_{x,\alpha}$ be the map

$$\begin{aligned}\sigma_{x,\alpha} \,:\, \pi^{-1}(U_\alpha \cap V) \to \mathbb{R}^d \times \mathbb{R}^n,\\ \sigma_{x,\alpha}(v) = (x_\alpha \circ \mathrm{pr}_1 \circ \varphi_\alpha(v), \mathrm{pr}_2 \circ \varphi_\alpha(v))\\ = (x_\alpha \circ \pi(v), \mathrm{pr}_2 \circ \varphi_\alpha(v))\end{aligned}$$

This is a bijection from $\pi^{-1}(U_\alpha \cap V)$ onto $x(U_\alpha \cap V) \times \mathbb{R}^n$, which is an open subset of $\mathbb{R}^d \times \mathbb{R}^n$. Clearly E can be covered by sets of the form $\pi^{-1}(U_\alpha \cap V)$ as above. Hence we see from Problem 10 (parts b and d) that the family of all $(\pi^{-1}(U_\alpha \cap V), \sigma_{x,\alpha})$ as above generate a (unique!) C^∞ manifold structure on E, [17] provided that we can only prove (1) C^∞ compatibility and (2) that the topology generated by the family of all $(\pi^{-1}(U_\alpha \cap V), \sigma_{x,\alpha})$ is Hausdorff, connected and paracompact.

We first prove C^∞ compatibility. Specifically, we have to prove that for any charts (V, x) and (W, y) for M and any $\alpha, \beta \in A$ subject to $U_\alpha \cap U_\beta \cap V \cap W \neq \emptyset$,

$$\sigma_{x,\alpha}(\pi^{-1}(U_\alpha \cap U_\beta \cap V \cap W)) \tag{84}$$

is an open subset of $\mathbb{R}^d \times \mathbb{R}^n$, and the map $\sigma_{y,\beta} \circ \sigma_{x,\alpha}^{-1}$ from the set (84) to $\mathbb{R}^d \times \mathbb{R}^n$ is C^∞. However by parsing the definitions we see that the set in (84) equals

$$x_\alpha(U_\alpha \cap U_\beta \cap V \cap W) \times \mathbb{R}^n,$$

which is indeed an open subset of $\mathbb{R}^d \times \mathbb{R}^n$. Also the map $\sigma_{x,\alpha}^{-1}$ on this set is given by

$$\sigma_{x,\alpha}^{-1}(z, w) = \varphi_\alpha^{-1}(x_\alpha^{-1}(z), w),$$

and hence

$$\begin{aligned}\sigma_{y,\beta} \circ \sigma_{x,\alpha}^{-1}(z, w) &= \Big(x_\beta \circ \mathrm{pr}_1 \circ \varphi_\beta \circ \varphi_\alpha^{-1}(x_\alpha^{-1}(z), w), \mathrm{pr}_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1}(x_\alpha^{-1}(z), w)\Big)\\ &= \Big(x_\beta \circ x_\alpha^{-1}(z), \mathrm{pr}_2 \circ \varphi_\beta \circ \varphi_\alpha^{-1}(x_\alpha^{-1}(z), w)\Big),\end{aligned}$$

which is C^∞ by inspection (in particular using our assumption that $\varphi_\beta \circ \varphi_\alpha^{-1}$ is C^∞).

Now it is easy to prove that the induced topology on E is Hausdorff. (See Problem 10(b) for the definition of the topology on E.) Indeed, let $p, q \in E$, $p \neq q$. Note that $\pi^{-1}(U)$ is open in E for every open set $U \subset M$; hence if $\pi(p) \neq \pi(q)$ then we can use the fact that M is Hausdorff to find disjoint

[17] And this choice of C^∞ manifold structure is clearly *forced* on us, from the requirements that (E, π, M) is a vector bundle of rank n, and $(U_\alpha, \varphi_\alpha)$ is a bundle chart for every $\alpha \in A$.

open sets $U_1, U_2 \subset M$ with $p \in U_1$, $q \in U_2$; then $\pi^{-1}(U_1)$ and $\pi^{-1}(U_2)$ are disjoint open sets in E, containing p resp. q, and we are done. It remains to treat the case $\pi(p) = \pi(q)$. Then choose a chart (V, x) for M and $\alpha \in A$ such that $\pi(p) = \pi(q) \in U_\alpha \cap V$. Now $\sigma_{x,\alpha}(p) \neq \sigma_{x,\alpha}(q)$ and hence there are disjoint open subsets $U_1, U_2 \subset x(U_\alpha \cap V) \times \mathbb{R}^n$ which contain $\sigma_{x,\alpha}(p)$ resp. $\sigma_{x,\alpha}(q)$. By the argument in the solution to Problem 10(c), $\sigma_{x,\alpha}^{-1}(U_1)$ and $\sigma_{x,\alpha}^{-1}(U_2)$ are open subsets of E; they are clearly disjoint and contain p resp. q. Done!

Next we verify that E is *connected.* Let A be any subset of E which is both open and closed. This means that for any chart (V, x) for M and any $\alpha \in A$, $\sigma_{x,\alpha}(A \cap \pi^{-1}(U_\alpha \cap V))$ is both open and closed in $x_\alpha(U_\alpha \cap V) \times \mathbb{R}^n$. This implies that for every $p \in U_\alpha \cap V$, the set

$$\{w \in \mathbb{R}^n \,:\, (x_\alpha(p), w) \in \sigma_{x,\alpha}(A \cap \pi^{-1}(U_\alpha \cap V))\} \tag{85}$$

is both open and closed in $\mathbb{R}^n$, and since $\mathbb{R}^n$ is connected, the set in (85) equals either $\emptyset$ or $\mathbb{R}^n$. In view of the definition of $\sigma_{x,\alpha}$ and the fact that $(\varphi_\alpha)_{|E_p}$ is a bijection from E_p onto $\{p\} \times \mathbb{R}^n$, this implies that

$$A \cap E_p = \emptyset \text{ or } E_p \subset A. \tag{86}$$

Since every point $p \in M$ is contained in some set of the form $U_\alpha \cap V$, the dichotomy (86) holds for *every* $p \in M$. Set

$$A_M := \{p \in M \,:\, E_p \subset A\} = \{p \in M \,:\, 0_p \in A\}.$$

Here 0_p denotes the zero vector in E_p, and the last equality holds because of (86). Note that for any (V, x) and α as above, the fact that $\sigma_{x,\alpha}(A \cap \pi^{-1}(U_\alpha \cap V))$ is both open and closed in $x_\alpha(U_\alpha \cap V) \times \mathbb{R}^n$ implies that the set

$$\{z \in x_\alpha(U_\alpha \cap V) \,:\, (z, 0) \in \sigma_{x,\alpha}(A \cap \pi^{-1}(U_\alpha \cap V))\}$$

is both open and closed in $x_\alpha(U_\alpha \cap V)$. Using here the fact that x_α is a homeomorphism, and the definition of $\sigma_{x,\alpha}$ and the fact that $\varphi_\alpha(0_p) = (p, 0)$ $\forall p \in U_\alpha \cap V$ (and φ_α is a bijection), it follows that the set

$$\{p \in U_\alpha \cap V \,:\, 0_p \in A\}$$

is both open and closed in $U_\alpha \cap V$. But that set equals $A_M \cap U_\alpha \cap V$, and the fact that this set is both open and closed in $U_\alpha \cap V$, for any (V, x) and α as above, implies that A_M is both open and closed in M. But M is connected, hence $A_M = \emptyset$ or $A_M = M$. In view of (86) this implies that either $A = M$ or $A = \emptyset$. Hence we have proved that E is connected.

Next we verify that E is paracompact. Note that by what we have already verified, E is connected, Hausdorff, and locally Euclidean, and hence by Problem 2(b) it suffices to prove that E a countable (topological) atlas. Let $\mathcal{U}$ be a countable base for the topology of M. Let $\mathcal{U}'$ be the subset of those $\Omega \in \mathcal{U}$ for which there exists a chart (V, x) for M and some $\alpha \in A$ such that $\Omega \subset U_\alpha \cap V$. Then $\mathcal{U}'$ covers M (by the same argument as in the second half of the solution to Problem 2(a)). Now for each $\Omega \in \mathcal{U}'$ we choose *one* chart (V, x) for M and *one* $\alpha \in A$ such that $\Omega \subset U_\alpha \cap V$, and then set $\psi_\Omega := (\sigma_{x,\alpha})_{|\pi^{-1}(\Omega)}$. It follows from what we have proved above, and (the solution to) Problem 10, that $\sigma_{x,\alpha}$ is a homeomorphism of $\pi^{-1}(U_\alpha \cap V)$ onto $x(U \cap V) \times \mathbb{R}^n$, and hence also ψ_Ω is a homeomorphism of $\pi^{-1}(\Omega)$ onto an open subset of $\mathbb{R}^d \times \mathbb{R}^n$. Hence

$$\{(\pi^{-1}(\Omega), \psi_\Omega) \ : \ \Omega \in \mathcal{U}'\}$$

is a (topological) atlas for E. This atlas is countable, since $\mathcal{U}' \subset \mathcal{U}$ and $\mathcal{U}$ is countable. Hence we have proved that E is paracompact!

Now we have verified all conditions necessary for Problem 10 (parts b and d) to apply. Hence we have now provided E *with a structure of a* C^∞ *manifold.*

Now the map $\pi : E \to M$ is immediately verified to be C^∞. Indeed, for any (V, x) and α as above, using the chart $(\pi^{-1}(U_\alpha \cap V), \sigma_{x,\alpha})$ for E and the chart (V, x) for M, the map π is represented by the identity map on $x_\alpha(U_\alpha \cap V)$.

Now it only remains to verify that for every $\alpha \in A$, the bijection φ_α from $\pi^{-1}(U_\alpha)$ onto $U_\alpha \times \mathbb{R}^n$ is in fact a diffeomorphism. For this, it suffices to verify that for every chart (V, x) for M (with $U_\alpha \cap V \neq \emptyset$), the restriction of φ_α to $\pi^{-1}(U_\alpha \cap V)$ is a diffeomorphism onto $(U_\alpha \cap V) \times \mathbb{R}^n$. However this is clear since $\varphi_{\alpha|\pi^{-1}(U_\alpha \cap V)}$ equals the composition of $\sigma_{x,\alpha}$ with the diffeomorphism $(z, v) \mapsto (x^{-1}(z), v)$ from $x(U_\alpha \cap V) \times \mathbb{R}^n$ onto $(U_\alpha \cap V) \times \mathbb{R}^n$, and $\sigma_{x,\alpha}$ is a diffeomorphism since $(\pi^{-1}(U_\alpha \cap V), \sigma_{x,\alpha})$ is a chart in our C^∞ atlas for E (by Problem 10(d)). Done! $\square$

Problem 37:

(a). Let E be the Möbius bundle over S^1, defined as in Lecture #7, p. 2. *Assume that E is trivial,* i.e. there is a bundle chart (E, φ). (This will lead to a contradiction.) Then by Problem 33 there is a global basis of sections $s \in \Gamma E$, in other words a section $s \in \Gamma E$ which is everywhere non-zero. Let us identify S^1 with $[0,1]/\approx$ (where $\approx$ stands for identifying the points 0 and 1 in $[0,1]$) in the standard way, i.e. by mapping $x \in [0,1]/\approx$ to $(\cos(2\pi x), \sin(2\pi x))$. By definition s is a C^∞ function from $[0,1]/\approx$ to $E = [0,1] \times \mathbb{R}/\sim$ such that $\text{pr}_1(s(x)) = x$ for all $x \in [0,1]/\approx$. In particular there is some $y \in \mathbb{R}$ such that

$$s(0) = (0, y) = (1, -y) \qquad \text{in } E.$$

Note that $y \neq 0$, since s is everywhere non-zero. Now $x \mapsto \text{pr}_2(s(x))$ is a C^∞ function from $(0,1)$ to $\mathbb{R}$ satisfying $\lim_{x \to 0^+} \text{pr}_2(s(x)) = y$ and $\lim_{x \to 1^-} \text{pr}_2(s(x)) = -y$; hence by the intermediate value theorem there is some $x \in (0,1)$ for which $\text{pr}_2(s(x)) = 0$, contradicting the fact that s is everywhere non-zero.

Hence E is *not* trivial. $\square$

(b). Let us write $E_n = S^1 \times \mathbb{R}^n$, the trivial vector bundle over S^1 of rank n. Also let $\widetilde{E}_1 := \widetilde{E}$ be the Möbius bundle over S^1, and set for $n \geq 2$: $\widetilde{E}_n := \widetilde{E} \oplus E_{n-1}$. We claim that the desired classification is as follows: The vector bundles

$$E_1, \widetilde{E}_1, E_2, \widetilde{E}_2, E_3, \widetilde{E}_3, \ldots, \tag{87}$$

are pairwise non-isomorphic, and *every* vector bundle over S^1 is isomorphic to one of these!

We start by proving that the vector bundles in (87) are pairwise non-isomorphic. Since isomorphisms of vector bundles preserve the rank, we only need to prove that for each n the two vector bundles E_n and $\widetilde{E}_n$ are non-isomorphic, or equivalently that $\widetilde{E}_n$ is not trivial. We have already proved this for $n = 1$ in part (a); hence we may here assume $n \geq 2$. Thus *assume that $\widetilde{E}_n$ is trivial.* (This will lead to a contradiction.)

It seems convenient to use a slightly different model for the Möbius bundle $\widetilde{E}_1$ than that used in part (a): We view $\widetilde{E}_1$ as $\mathbb{R}^2/\sim$, where $\sim$ is the equivalence relation

$$(x, y) \sim (x', y') \quad \overset{\text{def}}{\Longleftrightarrow} \big[x' - x \in \mathbb{Z} \text{ and } y' = (-1)^{x'-x} y\big].$$

(For a precise description of the C^∞ manifold structure, see Problem 9(c), and note that this quotient $\mathbb{R}^2/\sim$ is the same as $\Gamma \backslash \mathbb{R}^2$, where Γ is the group of diffeomorphisms of $\mathbb{R}^2$ of the form $(x, y) \mapsto (x + n, (-1)^n y)$, for $n \in \mathbb{Z}$.) We also view S^1 as $\mathbb{R}/\sim$ where $x \sim x' \overset{\text{def}}{\Longleftrightarrow} x' - x \in \mathbb{Z}$; then the projection map $\pi : \widetilde{E} \to S^1$ is given simply by projection onto the first coordinate; $[(x,y)] \mapsto [x]$. In a similar vein we also represent $\widetilde{E}_n$ as the quotient space $(\mathbb{R} \times \mathbb{R}^n)/\!\sim$ where

$$(x,y) \sim (x',y') \quad \overset{\text{def}}{\Longleftrightarrow} \big[x' - x \in \mathbb{Z} \text{ and } y' = (J_n)^{x'-x} \cdot y\big],$$

where

$$J_n := \begin{pmatrix} -1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ & \vdots & & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \end{pmatrix} \in \mathrm{GL}_n(\mathbb{R}).$$

(Note that $(J_n)^m = I$ for all even m and $(J_n)^m = J_n$ for all odd m.) The projection $\pi : \widetilde{E}_n \to S^1$ is again given by $[(x,y)] \mapsto [x]$.

Recall that we are assuming that $\widetilde{E}_n$ is trivial. Then by Problem 33 there is a global basis of sections $s_1, \ldots, s_n \in \Gamma E$. Each s_j is a C^∞ map from $S^1 = \mathbb{R}/\!\sim$ to $\widetilde{E}_n = (\mathbb{R} \times \mathbb{R}^n)/\!\sim$; composing s_j with the projection $\mathbb{R} \to \mathbb{R}/\!\sim$ we obtain a C^∞ map $\widetilde{s}_j : \mathbb{R} \to (\mathbb{R} \times \mathbb{R}^n)/\!\sim$ such that for every $x \in \mathbb{R}$ we have $\widetilde{s}_j(x) = [(x, f_j(x))]$ for some (unique) $f_j(x) \in \mathbb{R}^n$. One verifies that f_j is a C^∞ map $\mathbb{R} \to \mathbb{R}^n$; furthermore for each $x \in \mathbb{R}$ we have $[x] = [x+1]$ in S^1; hence $\widetilde{s}_j(x) = \widetilde{s}_{j+1}(x)$, i.e.

$$f_j(x+1) = J_n \cdot f_j(x), \qquad \forall x \in \mathbb{R}. \tag{88}$$

Let $F(x)$ be the real $n \times n$ matrix whose columns equal $f_1(x), \ldots, f_n(x)$, in this order. Then $F(x) \in \mathrm{GL}_n(\mathbb{R})$ for each $x \in \mathbb{R}$, since $s_1([x]), \ldots, s_n([x])$ is a basis of the fiber $\widetilde{E}_{n,[x]}$. Hence F is a C^∞ map $\mathbb{R} \to M_n(\mathbb{R})$. Furthermore (88) implies

$$F(x+1) = J_n \cdot F(x), \qquad \forall x \in \mathbb{R}.$$

However $\det J_n = -1$; hence the above implies $\det F(x+1) = -\det F(x)$. By continuity this implies that for any fixed $x \in \mathbb{R}$ there is some $x' \in [x, x+1]$ such that $\det F(x') = 0$, contradicting the fact that $F(x') \in \mathrm{GL}_n(\mathbb{R})$.

We have seen that the assumption that $\widetilde{E}_n$ is trivial leads to a contradiction. Hence $\widetilde{E}_n$ is *not* trivial.

It remains to prove that *every* vector bundle over S^1 is isomorphic to one of the vector bundles in (87). Thus let E be an arbitrary vector bundle over S^1. Set $n = \operatorname{rank} E$. By definition of vector bundle, every point in S^1 is contained in some bundle chart (U, φ) for E, and by shrinking U if necessary we can assume U to be an open arc on S^1. Hence since S^1 is compact, there is a *finite* family of bundle charts (U_j, φ_j) for E, $j = 1, \ldots, k$, such that $S^1 = U_1 \cup \cdots \cup U_k$ and each U_j is an open arc.

Now we have:

Lemma 2. *If (U, φ) and (V, ψ) are bundle charts for E such that U and V are open arcs on S^1 and also $U \cup V$ is an open arc, then there exists a bundle chart for E of the form $(U \cup V, \eta)$.*

Proof. After a rotation we may assume $U = (0, u)$ for some $0 < u \leq 1$. [18] If $V \subset U$ or $U \subset V$ then there is nothing to prove; hence let us assume that $V \not\subset U$ and $U \not\subset V$. Then the assumptions imply that V contains either $[0]$ or $[u]$; after a reflection we may assume that V contains $[u]$, and then we must have $V = (v_1, v_2)$ for some $0 < v_1 < u < v_2 \leq 1$. Then $U \cap V = (v_1, u)$. Let $T : (v_1, u) \to \mathrm{GL}_n(\mathbb{R})$ be the transition map between (U, φ) and (V, ψ) (cf. [12, p. 42]), so that

$$\varphi \circ \psi^{-1}(p, x) = (p, T(p) \cdot x), \qquad \forall p \in (v_1, u),\ x \in \mathbb{R}^n. \tag{89}$$

Note that T is a C^∞ curve on the manifold $\mathrm{GL}_n(\mathbb{R})$. Fix any real number $u' \in (v_1, u)$; thus we now have

$$0 < v_1 < u' < u < v_2 \leq 1.$$

Using the same technique as in Problem 19(b), one shows that there exists a C^∞ curve $\widetilde{T} : (v_1, v_2) \to \mathrm{GL}_n(\mathbb{R})$ such that

$$\widetilde{T}(p) = T(p), \qquad \forall p \in (v_1, u'). \tag{90}$$

Note that

$$U \cup V = (0, v_2).$$

Let us now define the map

$$\eta : \pi^{-1}(U \cup V) \to (U \cup V) \times \mathbb{R}^n$$

as follows:

$$\eta(w) := \begin{cases} \varphi(w) & \text{if } \pi(w) \in (0, u') \\ \Big(\pi(w), \widetilde{T}(\pi(w)) \cdot \mathrm{pr}_2(\psi(w))\Big) & \text{if } \pi(w) \in (v_1, v_2). \end{cases}$$

Note that this map is "over-defined", since *both* options apply whenever $\pi(w) \in (v_1, u')$; however using (89) and (90) one verifies that in this case both options give the same value for $\eta(w)$. It follows from this that η is C^∞, since φ is C^∞ on $\pi^{-1}((0, u'))$ and $w \mapsto (\pi(w), \widetilde{T}(\pi(w)) \cdot \mathrm{pr}_2(\psi(w)))$ is C^∞ on $\pi^{-1}((v_1, v_2))$. Also, by inspection, $\mathrm{pr}_1 \circ \eta = \pi$, and for every $p \in (0, v_2)$, $\eta_p := \mathrm{pr}_2 \circ \eta_{|E_p}$ is a linear bijection from E_p onto $\mathbb{R}^n$. Furthermore one verifies that η is a bijection from $\pi^{-1}(U \cup V)$ onto $(U \cup V) \times \mathbb{R}^n$, with inverse given by:

$$\eta^{-1}(p, x) = \begin{cases} \varphi^{-1}(p, x) & \text{if } p \in (0, u') \\ \psi^{-1}\big(p, \widetilde{T}(p)^{-1} \cdot x\big) & \text{if } p \in (v_1, v_2). \end{cases}$$

[18] Of course, "$(0, u)$" here stands for the arc $\{[x] \,:\, x \in (0, u)\}$ in $S^1 = \mathbb{R}/\sim$. We will employ this type of mild abuse of notation several times in the following...

As above one verifies that this map is well-defined although it is "over-defined", and that it is C^∞. Hence η is a C^∞ *diffeomorphism* from $\pi^{-1}(U \cup V)$ onto $(U \cup V) \times \mathbb{R}^n$, and therefore $(U \cup V, \eta)$ is a bundle chart for E. □

Applying the above lemma a finite number of times to our family of arcs $U_1, \ldots, U_k$, [19] we reduce to the case $k = 2$! Let us then write $(U, \varphi) := (U_1, \varphi_1)$ and $(V, \psi) := (U_2, \varphi_2)$. Thus now U, V are open arcs which cover S^1, and (U, φ) and (V, ψ) are bundle charts for E. With notation as in the proof of Lemma 2, we may now assume $U = (0, u)$ and $V = (v_1, v_2)$ where

$$0 < v_2 - 1 < v_1 < u < 1 < v_2 < v_1 + 1.$$

Thus $U \cap V$ is the union of the two disjoint open arcs (v_1, u) and $(1, v_2)$. As in the proof of Lemma 2, let $T : U \cap V \to \mathrm{GL}_n(\mathbb{R})$ be the transition map between (U, φ) and (V, ψ), so that

(91)
$$\varphi \circ \psi^{-1}(p, x) = (p, T(p) \cdot x), \qquad \forall p \in U \cap V = (v_1, u) \cup (1, v_2),\ x \in \mathbb{R}^n.$$

Note that both $T_{|(v_1,u)}$ and $T_{|(1,v_2)}$ are C^∞ curves on the manifold $\mathrm{GL}_n(\mathbb{R})$. Note that $\mathrm{GL}_n(\mathbb{R})$ has two connected components, namely

$$\mathrm{GL}_n^+(\mathbb{R}) := \{B \in \mathrm{GL}_n(\mathbb{R}) \ : \ \det B > 0\}$$

and

$$\mathrm{GL}_n^-(\mathbb{R}) := \{B \in \mathrm{GL}_n(\mathbb{R}) \ : \ \det B < 0\}.$$

Of course each of $T_{|(v_1,u)}$ and $T_{|(1,v_2)}$ is contained in a single connected component.

Case I: $T_{|(v_1,u)}$ and $T_{|(1,v_2)}$ lie in the *same* connected component. Then using the same technique as in Problem 19(b), one shows that, given any $\varepsilon > 0$ so small that

$$0 < \varepsilon < v_2 - 1 < v_1 < u - \varepsilon < u < 1 < 1 + \varepsilon < v_2 < v_1 + 1,$$

there exists a C^∞ curve $\widetilde{T} : (v_1, v_2) \to \mathrm{GL}_n(\mathbb{R})$ such that

$$\widetilde{T}(p) = T(p), \qquad \forall p \in (v_1, u - \varepsilon) \cup (1 + \varepsilon, v_2).$$

We can then define a map

$$\eta : E \to S^1 \times \mathbb{R}^n$$

through:

$$\eta(w) := \begin{cases} \varphi(w) & \text{if } \ \pi(w) \in (\varepsilon, u - \varepsilon) \\ \Big(\pi(w), \widetilde{T}(\pi(w)) \cdot \mathrm{pr}_2(\psi(w))\Big) & \text{if } \ \pi(w) \in (v_1, v_2). \end{cases}$$

[19] each application reduces the number of arcs by one, and we stop whenever we find two arcs which together cover S^1

As in the proof of Lemma 2 one verifies that this map is well-defined, and is an *isomorphism of vector bundles over* S^1. Hence we conclude that E is isomorphic to the trivial vector bundle $E_n = S^1 \times \mathbb{R}^n$!

Case II: $T_{|(v_1,u)}$ and $T_{|(1,v_2)}$ lie in the *different* connected component. Then the curve $p \mapsto J_n \cdot T(p)$ for $p \in (1, v_2)$ lies in the *same* connected component as $T_{|(v_1,u)}$, and hence, using the same technique as in Problem 19(b), one shows that, given any $\varepsilon > 0$ so small that

$$0 < \varepsilon < v_2 - 1 < v_1 < u - \varepsilon < u < 1 < 1 + \varepsilon < v_2 < v_1 + 1,$$

there exists a C^∞ curve $\widetilde{T} : (v_1, v_2) \to \mathrm{GL}_n(\mathbb{R})$ such that

$$\widetilde{T}(p) = \begin{cases} T(p) & \forall p \in (v_1, u - \varepsilon) \\ J_n \cdot T(p) & \forall p \in (1 + \varepsilon, v_2). \end{cases}$$

Then $\widetilde{T}$ can be used to define an isomorphism of vector bundles $E \xrightarrow{\sim} \widetilde{E}_n$. We leave out the details.

□

Problem 38: See Conlon, [3, Thm. 7.5.16].

Problem 39:

(a) Recall that as a *set*, we defined $E_1 \otimes E_2$ to be

$$E_1 \otimes E_2 = \bigsqcup_{p \in M} \big(E_{1,p} \otimes E_{2,p}\big),$$

with projection map $\pi : E_1 \otimes E_2 \to M$ defined by $\pi(v) = p$ if $v \in E_{1,p} \otimes E_{2,p}$ (for any $p \in M$). Also, if (U, φ_1) is a bundle chart for E_1 and (U, φ_2) is a bundle chart for E_2 [20] then we postulated that if we define

$$\begin{aligned} \tau\ &: \pi^{-1}(U) \to U \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}) \\ \tau(v) &:= \big(p, (\varphi_{1,p} \otimes \varphi_{2,p})(v)\big), \qquad \forall p \in U,\ v \in E_{1,p} \otimes E_{2,p}, \end{aligned} \tag{92}$$

then (U, τ) is a bundle chart for $E_1 \otimes E_2$. In other words, $\tau_p = \varphi_{1,p} \otimes \varphi_{2,p}$ for each $p \in M$.

In order to verify that the above indeed gives a vector bundle

$$(E_1 \otimes E_2, \pi, M),$$

we apply Problem 36, with the family of proposed bundle charts taken to be the family of all (U, τ) constructed as above, as $\langle (U, \varphi_1), (U, \varphi_2) \rangle$ varies through all pairs of bundle charts for E_1, E_2 with "same U". Most of the conditions in Problem 36 are immediately verified to hold. For example, $\tau_p = \varphi_{1,p} \otimes \varphi_{2,p}$ is a linear isomorphism of $(E_1 \otimes E_2)_p = E_{1,p} \otimes E_{2,p}$ onto $\{p\} \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ – which we identify with $\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$, since $\varphi_{j,p}$ is a linear isomorphism for $E_{j,p}$ onto $\mathbb{R}^{n_j}$ for $j = 1, 2$. Furthermore the sets U cover M; cf. footnote 20. The only condition which is not (completely) immediate is the C^∞ compatibility of the proposed bundle charts.

Thus we need to verify that if both (U, φ_j) $(j = 1, 2)$ and (V, ψ_j) $(j = 1, 2)$, are bundle charts for E_1 and E_2 respectively, and if τ is defined as in (92) and $\widetilde{\tau} : \pi^{-1}(V) \to V \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ is similarly defined using (V, ψ_1) and (V, ψ_2) (that is, $\widetilde{\tau}(v) := (p, (\psi_{1,p} \otimes \psi_{2,p})(v))$ for all $p \in V$ and $v \in E_{1,p} \otimes E_{2,p}$), then (if also $U \cap V \neq \emptyset$) the map $\widetilde{\tau} \circ \tau^{-1}$ from $(U \cap V) \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ to itself is C^∞. Now for any $(p, v) \in (U \cap V) \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2})$ we have

$$\widetilde{\tau} \circ \tau^{-1}(p, v) = (p, \widetilde{\tau}_p(\tau_p^{-1}(v)));$$

hence (using Problem 8(c)) it suffices to verify that the map

$$(p, v) \mapsto \widetilde{\tau}_p(\tau_p^{-1}(v)), \qquad (U \cap V) \times (\mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}) \to \mathbb{R}^{n_1} \otimes \mathbb{R}^{n_2}$$

[20]– with the same U! Note that the family of such open sets U certainly cover M, i.e. for each $p \in M$ there exist U, φ_1, φ_2 such that $p \in U$ and (U, φ_j) is a bundle chart for E_j for $j = 1, 2$! (Proof?)

is C^∞. But

(93)
$$\widetilde{\tau}_p \circ \tau_p^{-1} = (\psi_{1,p} \otimes \psi_{2,p}) \circ (\varphi_{1,p} \otimes \varphi_{2,p})^{-1} = (\psi_{1,p} \circ \varphi_{1,p}^{-1}) \otimes (\psi_{2,p} \circ \varphi_{2,p}^{-1}),$$

where the last equality holds since $\otimes$ is a bifunctor which is covariant in both arguments (cf. Sec. 7.2 of the lecture notes). Furthermore we know that the two maps

$$(p, v) \mapsto \psi_{1,p} \circ \varphi_{1,p}^{-1}(v), \qquad (U \cap V) \times \mathbb{R}^m \to \mathbb{R}^m$$

and

$$(p, v) \mapsto \psi_{2,p} \circ \varphi_{2,p}^{-1}(v), \qquad (U \cap V) \times \mathbb{R}^n \to \mathbb{R}^n$$

are C^∞. By using charts on $U \cap V$ we see that we will be done if we can prove the following: Given any open set $\Omega \subset \mathbb{R}^d$ and maps $\alpha : \Omega \to M_m(\mathbb{R})$ and $\beta : \Omega \to M_n(\mathbb{R})$ [21] such that the two maps

$$(x, v) \mapsto \alpha(x) \cdot v, \qquad \Omega \times \mathbb{R}^m \to \mathbb{R}^m$$

and

$$(x, v) \mapsto \beta(x) \cdot v, \qquad \Omega \times \mathbb{R}^n \to \mathbb{R}^n$$

are C^∞, then also the map

(94)
$$(x, v) \mapsto (\alpha(x) \otimes \beta(x)) \cdot v, \qquad \Omega \times (\mathbb{R}^m \otimes \mathbb{R}^n) \to (\mathbb{R}^m \otimes \mathbb{R}^n)$$

is C^∞. However the assumption about α and β is easily seen to be equivalent to the statement that each matrix entry of $\alpha(x)$ is a C^∞ function of $x \in \Omega$, and similarly for β. Now in (94), by a (hopefully) obvious abuse of notation, $\alpha(x) \otimes \beta(x)$ stands for the matrix of the linear map from $\mathbb{R}^m \otimes \mathbb{R}^n = \mathbb{R}^{mn}$ to itself which is the "tensor product" of the two linear maps $v \mapsto \alpha(x) \cdot v$ and $v \mapsto \beta(x) \cdot v$. This matrix is the *Kronecker product* of the matrices $\alpha(x)$ and $\beta(x)$; cf. wikipedia, and from the explicit formula for the Kronecker product we immediately see that each of the $(nm)^2$ matrix entries of $\alpha(x) \otimes \beta(x)$ is a C^∞ function of $x \in \Omega$; hence it follows that the map in (94) is C^∞, and we are done!

(b) This is very similar to part (a) and we here only describe the set-up: We define

$$\mathrm{Hom}(E_1, E_2) := \sqcup_{p \in M} \mathrm{Hom}(E_{1,p} \otimes E_{2,p}),$$

and if (U, φ_1) is a bundle chart for E_1 and (U, φ_2) is a bundle chart for E_2 then we postulate a corresponding bundle chart for $\mathrm{Hom}(E_1, E_2)$ to be (U, τ), where τ is given by (analogue of (92)):

$$\tau \; : \pi^{-1}(U) \to U \times \mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$$

(95)
$$\tau(v) := \big(p, \mathrm{Hom}(\varphi_{1,p}^{-1}, \varphi_{2,p})(v)\big), \qquad \forall p \in U, \; v \in \mathrm{Hom}(E_{1,p}, E_{2,p}),$$

[21] Here $M_m(\mathbb{R})$ is the space of real $m \times m$ matrices.

Cf. (96) below regarding the def of "$\mathrm{Hom}(\varphi_{1,p}^{-1}, \varphi_{2,p})$"; thus for each $p \in M$ we get $\tau_p = \mathrm{Hom}(\varphi_{1,p}^{-1}, \varphi_{2,p})$; this is an $\mathbb{R}$-linear map from $\mathrm{Hom}(E_1, E_2)_p = \mathrm{Hom}(E_{1,p}, E_{2,p})$ to $\mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. The reason that we have to use "$\varphi_{1,p}^{-1}$" is that Hom is contravariant in its first argument; cf. the discussion below, especially footnote 23.

Instead of giving further details, we discuss the problem from a more general point of view. The key property that is used in parts (a), (b), (c) is that both "$\otimes$" and "Hom" and "dual" are *smooth (=C^∞) functors* on the category $\mathcal{C}$ of finite dimensional vector spaces over $\mathbb{R}$. More specifically, $\otimes$ is a bifunctor covariant in both arguments (cf. Sec. 7.2 of the lecture notes); Hom is a bifunctor which is contravariant in the first argument and covariant in the second argument, and "dual" is a contravariant functor of one variable. It is a general fact that given *any* smooth functor $\mathcal{F}$ of k variables on $\mathcal{C}$ [22] then for any vector bundles $E_1, \ldots, E_k$ over M one can define in a natural way a vector bundle "$\mathcal{F}(E_1, \ldots, E_k)$" over M. Cf. [16, 1.34 – 1.39]. Each of (a), (b), (c) is a special case of this fact.

Let us explain in some detail what it means to say that Hom is a "smooth bifunctor on $\mathcal{C}$". Recall that $\mathcal{C}$ is the category of finite dimensional vector spaces over $\mathbb{R}$; thus "$A \in \mathrm{ob}(\mathcal{C})$" means that A is a finite dimensional vector space over $\mathbb{R}$. For any two $A, B \in \mathrm{ob}(\mathcal{C})$, $\mathrm{Hom}(A, B) \in \mathrm{ob}(\mathcal{C})$ is the vector space of $\mathbb{R}$-linear maps $A \to B$. Furthermore given any $A, A', B, B' \in \mathrm{ob}(\mathcal{C})$ and $\mathbb{R}$-linear maps $h : A' \to A$ and $f : B \to B'$ we define an $\mathbb{R}$-linear map "$\mathrm{Hom}(h, f)$":

(96)
$$\mathrm{Hom}(h, f) : \mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B'); \qquad \big[\mathrm{Hom}(h, f)\big](g) := f \circ g \circ h.$$

One immediately verifies that $\mathrm{Hom}(1_A, 1_B) = 1_{\mathrm{Hom}(A,B)}$ for all $A, B \in \mathrm{ob}(\mathcal{C})$, and that for any $A, A', A'', B, B', B'' \in \mathrm{ob}(\mathcal{C})$ and any $\mathbb{R}$-linear maps

$$A'' \xrightarrow{h'} A' \xrightarrow{h} A \quad \text{and} \quad B \xrightarrow{f} B' \xrightarrow{f'} B''$$

we have:

$$\mathrm{Hom}(h', f') \circ \mathrm{Hom}(h, f) = \mathrm{Hom}(h \circ h', f' \circ f). \tag{97}$$

The relations which we have here pointed out, mean exactly that *Hom is a bifunctor $\mathcal{C} \times \mathcal{C} \to \mathcal{C}$, contravariant in the first argument*[23] *and covariant in the second argument.*

Next, the *smoothness* of the bifunctor Hom consists in the following: Note that for any $A, A', B, B' \in \mathrm{ob}(\mathcal{C})$, the operation of taking any pair of linear maps $h : A' \to A$ and $f : B \to B'$ to the linear map $\mathrm{Hom}(f, g)$:

[22] Thus for "$\otimes$" and "Hom" we have $k = 2$, and for "dual" we have $k = 1$.

[23] *contra*variant – because of the switch of order between A and A' in (96) versus in "$h : A' \to A$", and the corresponding switch of order between h' and h in the left versus the right hand side of (97).

$\mathrm{Hom}(A, B) \to \mathrm{Hom}(A', B')$, is itself a map

$$\text{(98)} \qquad \mathrm{Hom}(A', A) \times \mathrm{Hom}(B, B') \to \mathrm{Hom}(\mathrm{Hom}(A, B), \mathrm{Hom}(A', B'));$$
$$(h, f) \mapsto \mathrm{Hom}(h, f).$$

Here both $\mathrm{Hom}(A', A) \times \mathrm{Hom}(B, B')$ and $\mathrm{Hom}(\mathrm{Hom}(A, B), \mathrm{Hom}(A', B'))$ are C^∞ manifolds (since each "Hom" space is a finite dimensional vector space over $\mathbb{R}$), and hence it makes sense to claim that the map in (98) is C^∞. This is exactly what we mean by saying that the bifunctor Hom is smooth. (Exercise: *Prove* this smoothness!)

(The corresponding smoothness of the bifunctor $\otimes$ is the statement that for any $A, A', B, B' \in \mathrm{ob}(\mathcal{C})$, the map

$$\mathrm{Hom}(A, A') \times \mathrm{Hom}(B, B') \to \mathrm{Hom}(A \otimes B, A', \otimes B')$$
$$(f, g) \mapsto f \otimes g$$

is C^∞. This is proved by choosing bases for A, A', B, B'; then $\mathrm{Hom}(A, A')$ and $\mathrm{Hom}(B, B')$ and $\mathrm{Hom}(A \otimes B, A', \otimes B')$ become spaces of (real) *matrices*, and $f \otimes g$ is given by the Kronecker product of the matrices f and g, and the smoothness is clear by inspection in the explicit formula for the Kronecker product. Cf. the discussion at the end of the solution of part (a).)

(c) This is also covered by the general discussion above; viz., it is a special case of [16, 1.38].

(In fact (c) can be obtained as a special case of (b); namely we have $E_1^* = \mathrm{Hom}(E_1, E_2)$ when E_2 is the trivial vector bundle $E_2 = M \times \mathbb{R}$. But alternatively one could also deduce (b) as a consequence of (a) and (c), namley for any vector bundles E_1 and E_2 we can identify $\mathrm{Hom}(E_1, E_2) = E_1^* \otimes E_2$.)

Problem 40: Let us write $n_j = \operatorname{rank} E_j$ $(j = 1, 2)$ and let π be the projection map $\pi : \operatorname{Hom}(E_1, E_2) \to M$.

Let $s \in \Gamma(\operatorname{Hom}(E_1, E_2))$. Then for each $p \in M$, $s(p) \in \operatorname{Hom}(E_1, E_2)_p = \operatorname{Hom}(E_{1,p}, E_{2,p})$, and so s gives rise to a map

$$f : E_1 \to E_2, \qquad f(x) := s(\pi_1(x))(x) \quad (x \in E_1).$$

By construction this map f satisfies $\pi_2 \circ f = \pi_1$, and furthermore for each $p \in M$,

$$f_p := f_{|E_{1,p}} = s(p) \in \operatorname{Hom}(E_{1,p}, E_{2,p}).$$

Hence if we can only prove that f is C^∞ then f is a bundle homomorphism $E_1 \to E_2$.

To prove that f is C^∞ is a local problem: Thus we may pass to bundle charts for E_1, E_2 and a chart for M (suitably adapted), after which the problem becomes$^{(*)}$: Given any open set $\Omega \subset \mathbb{R}^d$ and any C^∞ function

$$T : \Omega \to \operatorname{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}), \tag{99}$$

show that the map

$$\Omega \times \mathbb{R}^{n_1} \to \Omega \times \mathbb{R}^{n_2}, \qquad (x, v) \mapsto (x, T(x)(v)) \tag{100}$$

is C^∞. This, however, is trivial: Recall that $\operatorname{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$ can be identified with the space of real $n_2 \times n_1$ matrices, and to say that T is C^∞ means that each matrix entry of $T(x)$ is a smooth function of $x \in \Omega$; then the smoothness of the map (100) is clear from the explicit formula for the matrix product $T(x) \cdot v$. This completes the proof that f is C^∞, and hence that f is a bundle homomorphism $E_1 \to E_2$.

[(*) Let us give a few more details on the reduction to the Euclidean version of the problem stated in (99), (100). It is remarkable how much more complicated this is to actually spell out than it is to just "think it through in ones head"![24]

Given $x \in E_1$ it suffices to prove that there exists some open set $V \subset E_1$ containing x such that $f_{|V}$ is C^∞. Let us choose $V = \pi_1^{-1}(U)$ where U is an open set in M with $\pi_1(x) \in U$ for which there exist φ_1, φ_2 such that (U, φ_1) is a bundle chart for E_1 and (U, φ_2) is a bundle chart for E_2; thus our task is to prove that $f_{|\pi_1^{-1}(U)}$ is C^∞. Recall from Problem 39(b) that (U, φ_1) and (U, φ_2) give rise to a bundle chart (U, τ) for $\operatorname{Hom}(E_1, E_2)$ such that

$$\tau_p = \operatorname{Hom}(\varphi_{1,p}^{-1}, \varphi_{2,p}) \; : \; \operatorname{Hom}(E_{1,p}, E_{2,p}) \to \operatorname{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}),$$

for all $p \in U$, and that, by the definition of the bifunctor "Hom", this means that

$$\tau_p(\alpha) = \varphi_{2,p} \circ \alpha \circ \varphi_{1,p}^{-1} \in \operatorname{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}), \qquad \forall \alpha \in \operatorname{Hom}(E_{1,p}, E_{2,p}).$$

[24] In many situations in mathematics, such a phenomenon is a clear warning sign that one does not really have a complete proof – and so there is good reason to carefully work out the details. But for the task at hands it seems that there is not so much to worry about...

Of course, τ itself is the map

$$\begin{aligned} \tau : &\mathrm{Hom}(E_1, E_2)_{|U} \to U \times \mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2}); \\ &\tau(\alpha) = (p, \tau_p(\alpha)), \qquad \forall p \in U,\ \alpha \in \mathrm{Hom}(E_{1,p}, E_{2,p}). \end{aligned}$$

We have $f(\pi_1^{-1}(U)) \subset \pi_2^{-1}(U)$ (since $\pi_2 \circ f = \pi_1$) and hence, since φ_1 and φ_2 are diffeomorphisms, in order to prove that $f_{|\pi_1^{-1}(U)}$ is C^∞ it suffices to prove that the map $\varphi_2 \circ f \circ \varphi_1^{-1} : U \times \mathbb{R}^{n_1} \to U \times \mathbb{R}^{n_2}$ is C^∞. However at each $p \in U$ we have

$$(\mathrm{pr}_2 \circ \varphi_2 \circ f \circ \varphi_1^{-1})_{|\{p\} \times \mathbb{R}^{n_1}} = \varphi_{2,p} \circ f_p \circ \varphi_{1,p}^{-1} = \tau_p(f_p) = \mathrm{pr}_2(\tau(s(p))),$$

and hence for all $(p, v) \in U \times \mathbb{R}^{n_1}$:

$$(\varphi_2 \circ f \circ \varphi_1^{-1})(p, v) = \Big(p, \big[\mathrm{pr}_2(\tau(s(p)))\big](v)\Big).$$

Here $\mathrm{pr}_2 \circ \tau \circ s$ is a C^∞ map from M to $\mathrm{Hom}(\mathbb{R}^{n_1}, \mathbb{R}^{n_2})$. Therefore, after passing to C^∞ charts on U, [25] we are reduced to the task stated in (99), (100)!]

We now continue with the solution. Let $\mathcal{H}$ be the set of bundle homomorphisms $E_1 \to E_2$. Then above we have constructed a map

$$\Gamma(\mathrm{Hom}(E_1, E_2)) \to \mathcal{H}, \quad \text{"}s \mapsto f\text{"}. \tag{101}$$

We next construct the inverse map. Thus let f be a bundle homomorphism $E_1 \to E_2$. Then by definition, for each $p \in M$, $f_p = f_{|E_{1,p}}$ is an $\mathbb{R}$-linear map from $E_{1,p}$ to $E_{2,p}$, i.e. $f_p \in \mathrm{Hom}(E_{1,p}, E_{2,p}) = \mathrm{Hom}(E_1, E_2)_p \subset \mathrm{Hom}(E_1, E_2)$. Let us define the map $s : M \to \mathrm{Hom}(E_1, E_2)$ by $s(p) := f_p$. Clearly $\pi \circ s = 1_M$, and one verifies that s is C^∞ using bundle charts in a manner very similar to what we did above. Hence $s \in \Gamma(\mathrm{Hom}(E_1, E_2))$, and so we have constructed a map

$$\mathcal{H} \to \Gamma(\mathrm{Hom}(E_1, E_2)), \quad \text{"}f \mapsto s\text{"}. \tag{102}$$

It is immediate from our definitions (in particular using "$s(p) = f_p$") that the two maps (101) and (102) are inverses to each other. Hence we the two maps are in fact *bijections.* $\square$

[25] Here one could expand and give many more details! :-)

Problem 41:

(a) If (U,φ) is any bundle chart for E such that $\varphi(\pi^{-1}(U)\cap E') = U\times\mathbb{R}^m$, then $E_p\cap E'$ is an m-dimensional subspace of E_p for every $p\in U$. [Proof: for any $p\in U$ the restriction of φ to $E_p=\pi^{-1}(\{p\})$ is a linear isomorphism from E_p onto $\{p\}\times\mathbb{R}^n$, and $\varphi(\pi^{-1}(U)\cap E')=U\times\mathbb{R}^m$ implies that $\varphi(E_p\cap E') = \{p\}\times\mathbb{R}^m$; hence $E_p\cap E'$ is indeed an m-dimensional subspace of E_p.]

By assumption the bundle charts for E with the special property above *cover* M; hence for every $p\in M$ the intersection $E_p\cap E'$ is a linear subspace of E_p. Set $\mu(p):=\dim(E_p\cap E')$; then μ is a function from M to $\{0,1,\ldots,n\}$. It is clear from the previous discussion that μ is locally constant. (Indeed, given $p\in M$, let (U,φ) be a bundle chart for E such that $p\in U$ and $\varphi(\pi^{-1}(U)\cap E')=U\times\mathbb{R}^m$ for some $m\leq n$; then we saw in the previous discussion that $\mu(q)=m$ for all $q\in U$.) Hence $\mu^{-1}(\{m\})$ is an *open* subset of M for each $m\in\{0,1,\ldots,n\}$. But the sets $\mu^{-1}(\{0\}),\ldots,\mu^{-1}(\{n\})$ form a partition of M; furthermore M is connected (since M is a manifold), and thus M cannot be represented as a union of two or more disjoint nonempty open subsets. It follows that all except one of the sets $\mu^{-1}(\{0\}),\ldots,\mu^{-1}(\{n\})$ are *empty*. In other words there is $m\in\{0,\ldots,n\}$ such that $\mu^{-1}(\{m\})=M$, i.e. $\dim(E_p\cap E')=m$ for *all* $p\in M$. Done! □

(b) Let $\mathcal{F}$ be the family of all bundle charts for E satisfying (1), i.e. $\varphi(E'\cap\pi^{-1}(U))=U\times\mathbb{R}^m$. (By part (a) we know that m is a *fixed* integer, $0\leq m\leq n$, independent of $(U,\varphi)\in\mathcal{F}$.) For each $(U,\varphi)\in\mathcal{F}$ we set

$$\widetilde{\varphi}:=\varphi_{|E'\cap\pi^{-1}(U)}.$$

We wish to prove that $(E',\pi_{|E'},M)$ together with the family

$$\{(U,\widetilde{\varphi})\ :\ (U,\varphi)\in\mathcal{F}\}$$

satisfy all the conditions required in Problem 36. For each $p\in M$ we set $E'_p=E'\cap E_p$; we noted in part (a) that E'_p is an m-dimensional subspace of E_p. Also for each $(U,\varphi)\in\mathcal{F}$, it holds by our assumptions that $\widetilde{\varphi}$ is a bijection from $E'\cap\pi^{-1}(U)$ onto $U\times\mathbb{R}^m$, and for each $p\in U$ we have $\widetilde{\varphi}_{|E'_p}=\varphi_{|E'_p}$ and this is a linear isomorphism of E'_p onto $\{p\}\times\mathbb{R}^m$. Also the sets U cover M as (U,φ) runs through $\mathcal{F}$, by assumption. Hence it only remains to prove that if both (U,φ), $(V,\psi)\in\mathcal{F}$, then the map $\widetilde{\psi}\circ\widetilde{\varphi}^{-1}$ from $(U\cap V)\times\mathbb{R}^m$ onto itself is C^∞. However that map is equal to the restriction of the map

$$\psi\circ\varphi^{-1}\ :\ (U\cap V)\times\mathbb{R}^n\to(U\cap V)\times\mathbb{R}^n$$

to the set $(U\cap V)\times\mathbb{R}^m$, and from this the desired smoothness is clear.

[Let us discuss the very last step in some detail: By Problem 8(c) it suffices to prove that both the maps $\text{pr}_1\circ\widetilde{\psi}\circ\widetilde{\varphi}^{-1}$ and $\text{pr}_2\circ\widetilde{\psi}\circ\widetilde{\varphi}^{-1}$; however the first of these equals $\text{pr}_1:(U\cap V)\times\mathbb{R}^m\to U\cap V$ which we know is C^∞;

hence it suffices to prove that the map

$$\mathrm{pr}_2 \circ \widetilde{\psi} \circ \widetilde{\varphi}^{-1} : (U \cap V) \times \mathbb{R}^m \to \mathbb{R}^m$$

is C^∞. That map is the restriction of the map

$$\mathrm{pr}_2 \circ \psi \circ \varphi^{-1} \ : \ (U \cap V) \times \mathbb{R}^n \to \mathbb{R}^n$$

to $(U \cap V) \times \mathbb{R}^m$. Hence after passing to charts on $(U \cap V) \times \mathbb{R}^m$, we see that it suffices to prove the following general fact: Given an open set $\Omega \subset \mathbb{R}^k$ (some $k \in \mathbb{Z}^+$) and a C^∞ map $f : \Omega \to \mathbb{R}^n$, *if* it happens that $f(x) \in \mathbb{R}^m$ for all $x \in \Omega$ then f is C^∞ also as a map $\Omega \to \mathbb{R}^m$. This is of course completely trivial from the definition of what it means for a map between $\mathbb{R}^d$-spaces to be C^∞.[26]]

We have proved above that all the conditions of Problem 36 are fulfilled and so by that problem, $(E', \pi_{|E'}, M)$ is a vector bundle of rank m, and $(U, \widetilde{\varphi})$ is a bundle chart for E' for every $(U, \varphi) \in \mathcal{F}$. □

(c) This is more or less immediate from our assumptions about existence of bundle charts satisfying (1), together with the following criterion for being a differentiable submanifold which we pointed out in the notes to lecture #2 (here formulated with notation adapted to our setting): *If E is any $d + n$ dimensional C^∞ manifold and E' is an arbitrary subset of E, then E' has a (uniquely determined) structure of a differentiable submanifold of E of dimension $d + m$ if and only if for every $x \in E'$ there is a C^∞ chart (V, ψ) of E such that $x \in V$, $\psi(x) = 0$, $\psi(V)$ is an open cube $(-\varepsilon, \varepsilon)^{d+n}$, and*

$$\psi(V \cap E') = (-\varepsilon, \varepsilon)^{d+m} \times \{0\}^{n-m}. \tag{103}$$

(Cf., e.g., [1, Sec. III.5, esp. Lemma 5.2].)

Details: Let $x \in E'$ be given. Set $p = \pi(x) \in M$. Then by assumption there is a bundle chart (U, φ) for E such that $p \in U$ and (1) holds, i.e. $\varphi(E' \cap \pi^{-1}(U)) = U \times \mathbb{R}^m$. Now choose also any C^∞ chart (W, τ) for M with $p \in W$. Of course we may assume $W \subset U$ (otherwise just replace W by $W \cap U$) and $\tau(p) = 0$ (otherwise just compose τ with a translation of $\mathbb{R}^d$). Then $\tau(W)$ is an open set in $\mathbb{R}^d$ containing 0; hence there is some $\varepsilon > 0$ such that $(-\varepsilon, \varepsilon)^d \subset \tau(W)$. Now we may replace W by the smaller open set $\tau^{-1}((-\varepsilon, \varepsilon)^d)$; after doing this we have $\tau(W) = (-\varepsilon, \varepsilon)^d$. Now the map

$$\psi := (\tau, 1_{\mathbb{R}^n}) \circ \varphi_{|\pi^{-1}(W)}$$

[26]The conclusion here has the following generalization: Let S, N be C^∞ manifolds, let $f : S \to N$ be a C^∞ map, and let $M \subset N$ be a differentiable submanifold of N. Assume that $f(S) \subset M$. Then f is C^∞ also as a map $S \to M$. Cf., e.g., [15, Cor. 5.30]. The proof of this fact basically reduces to what we have already done, if one uses charts as in [12, Lemma 1.3.1]. Note that if we merely assume that M is an *immersed* submanifold of N, then the corresponding statement is *false* in general! (Can you give an example?)

is a diffeomorphism of $\pi^{-1}(W)$ onto $(-\varepsilon,\varepsilon)^d\times\mathbb{R}^n\subset\mathbb{R}^{d+n}$ (since it is a composition of a diffeomorphism of $\pi^{-1}(W)$ onto $W\times\mathbb{R}^n$ and a diffeomorphism of $W\times\mathbb{R}^n$ onto $(-\varepsilon,\varepsilon)^d\times\mathbb{R}^n$), and so

$$(\pi^{-1}(W),\psi)$$

is a C^∞ chart for E'. It follows from $\varphi(E'\cap\pi^{-1}(U))=U\times\mathbb{R}^m$ (where "$\mathbb{R}^m$" really stands for $\mathbb{R}^m\times\{0\}^{n-m}$) that we have

$$\psi(E'\cap\pi^{-1}(W))=(-\varepsilon,\varepsilon)^d\times\mathbb{R}^m\times\{0\}^{n-m}.$$

Hence if we set $V:=\psi^{-1}((-\varepsilon,\varepsilon)^{d+n})$ (an open subset of $\psi^{-1}(W)$) then also

$$(V,\psi_{|V})$$

is a C^∞ chart for E', and this chart satisfies $x\in V$, $\psi(x)=0$, $\psi(V)=(-\varepsilon,\varepsilon)^{d+n}$, and

$$\psi(E'\cap V)=(-\varepsilon,\varepsilon)^{d+m}\times\{0\}^{n-m},$$

i.e. the condition (103). Done! □

Problem 42:

(a). Recall that as a set, f^*E is defined to be

$$f^*E := \{(p, v) \,:\, p \in M,\, v \in E_{f(p)}\} \subset M \times E, \tag{104}$$

and we define the projection map $\widetilde{\pi} : f^*E \to M$ to be simply $\widetilde{\pi} := \mathrm{pr}_1$, i.e. $\widetilde{\pi}(p, v) := p$ for all $(p, v) \in f^*E$. Also for any bundle chart (U, φ) for (E, π, N) we have specified that if $\widetilde{\varphi}$ is the map

$$\begin{aligned} &\widetilde{\varphi} : \widetilde{\pi}^{-1}(f^{-1}(U)) \to f^{-1}(U) \times \mathbb{R}^n \qquad (n := \operatorname{rank} E); \\ &\widetilde{\varphi}(p, v) := (p, \mathrm{pr}_2(\varphi(v))) \end{aligned} \tag{105}$$

then $(f^{-1}(U), \widetilde{\varphi})$ is a bundle chart for f^*E.

In order to prove that $(f^*E, \widetilde{\pi}, M)$ is a vector bundle, we now verify that $\widetilde{\pi} : f^*E \to M$ with the above proposed bundle charts satisfy all the conditions required in Problem 36. Firstly, $\widetilde{\pi}$ is obviously surjective, and for every $p \in M$, the set $(f^*E)_p := \widetilde{\pi}^{-1}(p)$ is seen to be

$$(f^*E)_p := \widetilde{\pi}^{-1}(p) = \{p\} \times E_{f(p)} \text{ ``}= E_{f(p)}\text{''}$$

(the last equality is our usual identification), and this set carries the structure of an n-dimensional real vector space since (E, π, N) is a vector bundle. Next let (U, φ) be any bundle chart for (E, π, N). Then $f^{-1}(U)$ of course is an open subset of M. Also

$$\widetilde{\pi}^{-1}(f^{-1}(U)) = \{(p, v) \,:\, p \in f^{-1}(U),\, v \in E_{f(p)}\},$$

and for any fixed $p \in f^{-1}(U)$, we know that $\mathrm{pr}_2 \circ \varphi_{|E_{f(p)}}$ is a linear isomorphism of $E_{f(p)}$ onto $\mathbb{R}^n$; hence from the definition of $\widetilde{\varphi}$, (105), it follows that the restriction of $\widetilde{\varphi}$ to the set $(f^*E)_p = \{p\} \times E_{f(p)}$ is a linear isomorphism onto $\{p\} \times \mathbb{R}^n$. In particular $\widetilde{\varphi}$ restricts to a *bijection* of $\{p\} \times E_{f(p)}$ onto $\{p\} \times \mathbb{R}^n$, and using this fact for every $p \in f^{-1}(U)$ it follows that $\widetilde{\varphi}$ is a bijection of $\widetilde{\pi}^{-1}(f^{-1}(U))$ onto $f^{-1}(U) \times \mathbb{R}^n$. Furthermore, $M = \bigcup f^{-1}(U)$ when the union is taken over all bundle charts (U, φ) for (E, π, N), since the family of such sets U cover N.

Now the only condition from Problem 36 which remains to be verified is that if (U, φ) and (V, ψ) are any two bundle charts for (E, π, N), then $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is a C^∞ map from $(f^{-1}(U) \cap f^{-1}(V)) \times \mathbb{R}^n$ to itself. To prove this, first note that $f^{-1}(U) \cap f^{-1}(V) = f^{-1}(U \cap V)$. Next, by parsing the maps one finds that

$$\begin{aligned} \widetilde{\psi} \circ \widetilde{\varphi}^{-1}(p, w) = \big(p, \mathrm{pr}_2(\psi(\varphi^{-1}(f(p), w)))\big), \\ \forall (p, w) \in f^{-1}(U \cap V) \times \mathbb{R}^n. \end{aligned} \tag{106}$$

[Details: Let $(p, w) \in f^{-1}(U \cap V) \times \mathbb{R}^n$. Then $f(p) \in U \cap V$ and so $\varphi^{-1}(f(p), w)$ is defined and lies in $E_{f(p)} \subset \pi^{-1}(U \cap V)$. Hence $(p, \varphi^{-1}(f(p), w)) \in$

$(f^*E)_p$, and using (105) we have $\widetilde{\varphi}(p, \varphi^{-1}(f(p), w)) = (p, w)$; hence

$$(p, \varphi^{-1}(f(p), w)) = \widetilde{\varphi}^{-1}(p, w);$$

and applying $\widetilde{\psi}$ to this relation and again using (105), we obtain (106)!]

Clearly (106) implies that $\widetilde{\psi} \circ \widetilde{\varphi}^{-1}$ is a C^∞ map from $f^{-1}(U \cap V) \times \mathbb{R}^n$ to itself (using the fact that the maps ψ, φ^{-1}, f are C^∞, and also using Problem 8(b),(c)).

Hence all the conditions from Problem 36 are satisfied, and so Problem 36 implies that $(f^*E, \widetilde{\pi}, M)$ is a vector bundle, and that for any bundle chart (U, φ) for (E, π, N), $(f^{-1}(U), \widetilde{\varphi})$ is a bundle chart for f^*E. $\square$

(b) Passing to local coordinates (viz., choosing appropriate charts and bundle charts) one reduces to the case when M is an open subset of $\mathbb{R}^d$, N is an open subset of $\mathbb{R}^{d'}$, and $E = N \times \mathbb{R}^n$. [27] Then the definition of f^*, (104), becomes:

$$f^*E = \{(p, f(p), v) \,:\, p \in M,\ v \in \mathbb{R}^n\} \subset M \times E = M \times N \times \mathbb{R}^n.$$

Now note that the map

$$\varphi : M \times N \times \mathbb{R}^n \to \mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R}^n,$$
$$\varphi(p, q, v) = (p, q - f(p), v)$$

is a diffeomorphism of $M \times N \times \mathbb{R}^n$ onto

$$\Omega = \{(p, q', v) \in M \times \mathbb{R}^{d'} \times \mathbb{R}^n \,:\, q' + f(p) \in N\},$$

which is an open subset of $\mathbb{R}^d \times \mathbb{R}^{d'} \times \mathbb{R}^n$ (inverse map: $\varphi^{-1}(p, q', v) = (p, q' + f(p), v)$). Hence $(M \times N \times \mathbb{R}^n, \varphi)$ is a C^∞ chart on $M \times N \times \mathbb{R}^n$, and we note that

$$\varphi(f^*E) = M \times \{0\} \times \mathbb{R}^n \subset \Omega.$$

The existence of such a chart immediately implies that for every $p \in f^*E$ there is a C^∞ chart (V, ψ) on $M \times N \times \mathbb{R}^n$ such that $p \in V$, $\psi(p) = 0$, $\psi(V) = (-\varepsilon, \varepsilon)^{d+d'+n}$ and $\psi(V \cap f^*E) = (-\varepsilon, \varepsilon)^{d+n} \times \{0\}^{d'}$. (Indeed, simply compose φ with a translation and a suitable permutation of the coordinates; then restrict the domain appropriately – cf. the solution to Problem 41(c)). Hence (by a result stated in the notes to Lecture #2; cf., e.g., [1, Sec. III.5]), f^*E is a differentiable submanifold of $M \times E = M \times N \times \mathbb{R}^n$.

(One should also verify that the C^∞ manifold structure of f^*E as a differentiable submanifold of $M \times E = M \times N \times \mathbb{R}^n$ agrees with the C^∞ manifold structure on f^*E defined in part (a) via Problem 36. This is "immediate"

[27] Here we are sweeping a lot of details under the carpet; however we have discussed similar things many times previously...

by comparing the C^∞ charts provided in each case. – but would take some effort to write out.) $\square$

Problem 43:

(a). This is very direct: We map any $(s_1, s_2) \in \Gamma(E_1) \oplus \Gamma(E_2)$ to the section $s \in \Gamma(E_1 \oplus E_2)$ defined by

$$s(p) := (s_{1,p}, s_{2,p}) \in E_{1,p} \oplus E_{2,p} = (E_1 \oplus E_2)_p.$$

One verifies that this map $(s_1, s_2) \mapsto s$ is an isomorphism of $C^\infty M$-modules. We leave out the details...

(b). This is a special case of (c). (Namely, take $E_2 = M \times \mathbb{R}$ in (c); then $\text{Hom}(E_1, E_2) = E_1^*$ and $\Gamma(E_2) = C^\infty(M)$, so that (c) gives the result that we want.)

(Cf. also [3, Prop. 6.2.11 and Prop. 7.5.4].)

(c). (Cf., e.g., [16, Prop. 1.53].)

Given $h \in \Gamma(\text{Hom}(E_1, E_2))$ and $s \in \Gamma E_1$, let us define the map

$$\Phi_{h,s} : M \to E_2, \qquad \Phi_{h,s}(p) := h(p)(s(p)). \tag{107}$$

Clearly $\pi_2 \circ \Phi_{h,s} = 1_{M_2}$; also $\Phi_{h,s}$ is a C^∞ map since h and s are C^∞ maps (passing to local coordinates this reduces to the basic fact pointed out around (99), (100)). Hence $\Phi_{h,s} \in \Gamma E_2$.

Next, given $h \in \Gamma(\text{Hom}(E_1, E_2))$, we consider the map $s \mapsto \Phi_{h,s}$. Actually let us change notation by setting

$$\Phi_h(s) := \Phi_{h,s} \qquad (s \in \Gamma E_1).$$

Our previous paragraph shows that Φ_h is a map

$$\Phi_h : \Gamma E_1 \to \Gamma E_2.$$

It is immediate from (107) that Φ_h is $C^\infty(M)$-linear. Hence

$$\Phi_h \in \text{Hom}(\Gamma E_1, \Gamma E_2).$$

We have thus defined a map

$$\Gamma(\text{Hom}(E_1, E_2)) \to \text{Hom}(\Gamma E_1, \Gamma E_2), \qquad h \mapsto \Phi_h. \tag{108}$$

It is again immediate from (107) that *this* map is $C^\infty(M)$-linear, i.e. a homomorphism of $C^\infty(M)$-modules. We are going to prove that the map (108) is a *bijection.* This will imply that it is an isomorphism of $C^\infty(M)$-modules, as desired!

The proof of injectivity is easy: It suffices to prove that the kernel of the map (108) is $\{0\}$. Thus let $h \in \Gamma(\text{Hom}(E_1, E_2))$ be given and assume

$\Phi_h = 0$. Then $\Phi_{h,s} = 0$ for all $s \in \Gamma E_1$, and so $h(p)(s(p)) = 0$ for all $s \in \Gamma E_1$ and $p \in M$. Hence by Problem 35(c), for every $p \in M$ we have $h(p)(v) = 0$, $\forall v \in E_{1,p}$, i.e. $h(p) = 0$ in $\mathrm{Hom}(E_{1,p}, E_{2,p}) = \mathrm{Hom}(E_1, E_2)_p$. Since this is true for every $p \in M$, we conclude that $h = 0$, as desired. This completes the proof that the map in (108) is injective.

It remains to prove surjectivity. Thus take an arbitrary element $\Phi \in \mathrm{Hom}(\Gamma E_1, \Gamma E_2)$, i.e. a $C^\infty(M)$-linear map $\Phi : \Gamma E_1 \to \Gamma E_2$. Let us start by proving that Φ is "local" in the following sense:

Lemma 3. *For any open set $U \subset M$ and any $s_1, s_2 \in \Gamma E_1$, if $s_{1|U} = s_{2|U}$ then $\Phi(s_1)_{|U} = \Phi(s_2)_{|U}$.*

Proof. Assume $s_{1|U} = s_{2|U}$. Now our task is to prove that $\Phi(s_1)(p) = \Phi(s_2)(p)$ for every $p \in U$. Thus fix a point $p \in U$. By Problem 7(c) there is a function $f \in C^\infty(M)$ which has compact support contained in U and which satisfy $f(p) = 1$. Using $s_{1|U} = s_{2|U}$ and $f(p) = 0$ for all $p \in M \setminus U$ it follows that $fs_1 = fs_2$ in ΓE_1; thus $\Phi(fs_1) = \Phi(fs_2)$. But Φ is $C^\infty(M)$-linear and thus $f\Phi(s_1) = f\Phi(s_2)$, and in particular $f(p)\Phi(s_1)(p) = f(p)\Phi(s_2)(p)$, and since $f(p) \neq 0$ this implies $\Phi(s_1)(p) = \Phi(s_2)(p)$. Done! □

Let $\mathcal{V}$ be the family of open sets $V \subset M$ such that there exist sections $b_1, \ldots, b_n \in \Gamma E_1$ and $c_1, \ldots, c_m \in \Gamma E_2$, such that $b_{1|V}, \ldots, b_{n|V}$ form a basis of sections of $E_{1|V}$ and $c_{1|V}, \ldots, c_{m|V}$ form a basis of sections of $E_{2|V}$. By Problem 35(b), $\mathcal{V}$ covers M. Now take any $V \in \mathcal{V}$, and choose sections $b_1, \ldots, b_n \in \Gamma E_1$ and $c_1, \ldots, c_m \in \Gamma E_2$ with the property just mentioned. For each $j \in \{1, \ldots, n\}$, $\Phi(b_j)_{|V} \in \Gamma(E_{2|V})$; hence (by Problem 34) there are unique $g_j^k \in C^\infty(V)$ ($k = 1, \ldots, m$) such that

$$\Phi(b_j)_{|V} = g_j^k c_{k|V}. \tag{109}$$

For each $q \in V$, let $h(q)$ be the linear map $E_{1,q} \to E_{2,q}$ which has matrix $(g_j^k(q))$ with respect to the bases $b_1(q), \ldots, b_n(q)$ and $c_1(q), \ldots, c_m(q)$; that is,

$$h(q)\big(\alpha^j b_j(q)\big) = \alpha^j g_j^k(q) c_k(q), \qquad \forall \alpha = (\alpha^j)_{j=1,\ldots,n} \in \mathbb{R}^n. \tag{110}$$

Note that $h(q) \in \mathrm{Hom}(E_1, E_2)_q$, and since all $g_j^k \in C^\infty(V)$, we have

$$h \in \Gamma(\mathrm{Hom}(E_1, E_2)_{|V}). \tag{111}$$

Lemma 4. *For every $s \in \Gamma E_1$ and every $q \in V$, $\Phi(s)(q) = h(q)(s(q))$.*

Proof. Let $s \in \Gamma E_1$ and $q \in V$ be given, and take $f^1, \ldots, f^n \in C^\infty(V)$ so that $s_{|V} = f^j b_{j|V}$. By Problem 7(e) there exist an open set $U \subset V$ with $q \in U$ and functions $\widetilde{f}^1, \ldots, \widetilde{f}^n \in C^\infty(M)$ such that $\widetilde{f}^j_{|U} \equiv f^j_{|U}$. Hence

$s_{|U} = (\widetilde{f}^j b_j)_{|U}$, and so by Lemma 3, $\Phi(s)_{|U} = \Phi(\widetilde{f}^j b_j)_{|U}$, and in particular

$$\Phi(s)(q) = \Phi(\widetilde{f}^j b_j)(q) = \widetilde{f}^j(q)\Phi(b_j)(q) = f^j(q)g_j^k(q)c_k(q) = h(q)\big(f^j(q)b_j(q)\big) \\ = h(q)(s(q)).$$

(In the second equality we used the assumption that Φ is $C^\infty(M)$-linear; in the third equality we used (109); in the fourth equality we used (110), and in the last equality we used $s_{|V} = f^j b_{j|V}$.) □

Lemma 5. *$h(q)$ depends only on Φ and q, and not on V or $b_1, \dots, b_n$ or $c_1, \dots, c_m$.*

Proof. This is clear from Lemma 4 and Problem 35(c). □

It follows from Lemma 5 that $h(q) \in \mathrm{Hom}(E_1, E_2)_q$ can be unambiguously defined for any point $q \in M$ which lies in some $V \in \mathcal{V}$. But as we have pointed out, $\mathcal{V}$ covers M; hence we have in fact defined a map

$$h : M \to \mathrm{Hom}(E_1, E_2)$$

which satisfies $\pi \circ h = 1_M$ and

$$\Phi(s)(q) = h(q)(s(q)) \tag{112}$$

for all $s \in \Gamma E_1$ and $q \in M$ (cf. Lemma 4). But we also have $h_{|V} \in \Gamma(\mathrm{Hom}(E_1, E_2)_{|V})$ for every $V \in \mathcal{V}$; hence h is C^∞, and so $h \in \Gamma(\mathrm{Hom}(E_1, E_2))$. Now by (112), $\Phi = \Phi_h$. This completes the proof that $h \mapsto \Phi_h$ is surjective.
□

(d). (Cf. [3, Thm. 7.5.5] and stackexchange.)

Incomplete solution: Given any $f \in \Gamma(E_1)$ and $g \in \Gamma(E_2)$, let us write $f \otimes g$ for the map

$$\begin{aligned} &f \otimes g \, : \, M \to E_1 \otimes E_2, \\ &(f \otimes g)(p) := f(p) \otimes g(p) \in E_{1,p} \otimes E_{2,p} \qquad (p \in M). \end{aligned} \tag{113}$$

Then clearly $\pi \circ (f \otimes g) = 1_M$, where π is the projection map $E_1 \times E_2 \to M$. Let us verify that the map $f \otimes g$ is C^∞. Assume that (U, φ_j) is a bundle chart for E_j, for $j = 1, 2$. Then by the definition of $E_1 \otimes E_2$ – cf. Problem 39 – if we define $\tau : \pi^{-1}(U) \to U \times \mathbb{R}^{mn}$ by $\tau(p, v) = (p, \tau_p(v))$ where $\tau_p = \varphi_{1,p} \otimes \varphi_{2,p}$ (and we have fixed an identification $\mathbb{R}^m \otimes \mathbb{R}^n = \mathbb{R}^{mn}$), then (U, τ) is a bundle chart for $E_1 \otimes E_2$, i.e. τ is a diffeomorphism from $\pi^{-1}(U)$ onto $U \times \mathbb{R}^{mn}$. Hence it suffices to verify that the map $\tau \circ (f \otimes g) \, : \, U \to U \times \mathbb{R}^{mn}$ is $\mathbb{C}^\infty$, for any (U, φ_j) $(j = 1, 2)$ as above. As usual it suffices to verify that $\text{pr}_1 \circ \tau \circ (f \otimes g)$ and $\text{pr}_2 \circ \tau \circ (f \otimes g)$ are C^∞; the first of these is the identity map on U which is trivially C^∞; and the second map is seen to equal

$$p \mapsto \varphi_{1,p}(f(p)) \otimes \varphi_{2,p}(g(p)) \, : \, U \to \mathbb{R}^{mn}. \tag{114}$$

Here we know that the maps $p \mapsto \varphi_{1,p}(f(p))$ and $p \mapsto \varphi_{2,p}(g(p))$ are C^∞. But also the map $\langle v, w \rangle \mapsto v \otimes w \, : \, \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{mn}$ is C^∞; hence the map (114) is C^∞, as desired. Hence we have proved:

$$f \otimes g \in \Gamma(E_1 \otimes E_2).$$

We have thus constructed a map

$$\Gamma(E_1) \times \Gamma(E_2) \to \Gamma(E_1 \otimes E_2), \qquad (f, g) \mapsto f \otimes g.$$

Note that this map is $C^\infty(M)$-bilinear; hence there is a unique $C^\infty(M)$-linear map

$$J : \Gamma(E_1) \otimes \Gamma(E_2) \to \Gamma(E_1 \otimes E_2)$$

such that

$$J(f \otimes g) = f \otimes g, \qquad \forall f \in \Gamma(E_1), \; g \in \Gamma(E_2).$$

We claim that J is an isomorphism of $C^\infty(M)$-modules.

Proof that J is *surjective:* By Problem 38 [28] there exists a finite open cover $U_1, \ldots, U_r$ of M such that $E_{1|U_\ell}$ and $E_{2|U_\ell}$ are trivial for $\ell = 1, \ldots, r$. Then for each fixed $\ell \in \{1, \ldots, r\}$ there is a basis of sections $b_1, \ldots, b_m \in \Gamma E_{1|U_\ell}$ and a basis of sections $b'_1, \ldots, b'_n \in \Gamma E_{2|U_\ell}$ (here we are writing $m = \text{rank}\, E_1$ and $n = \text{rank}\, E_2$). Now for each $p \in U_\ell$, the vectors $b_j(p) \otimes b'_k(p)$ $(j \in$

[28] Apply Problem 38 to both E_1 and E_2; then consider the common refinement of the two open covers, i.e. the family of pair-wise intersections of the open sets.

$\{1, \ldots, m\}$, $k \in \{1, \ldots, n\}\}$), form a basis of the $\mathbb{R}$-linear space $E_{1,p} \otimes E_{2,p}$. Hence

$$\{b_j \otimes b'_k \, : \, j \in \{1, \ldots, m\}, \ k \in \{1, \ldots, n\}\}$$

is a basis of sections in $\Gamma(E_1 \otimes E_2)_{|U_\ell}$. (Here the notation "$b_j \otimes b'_k$" is the one introduced above in (113), but applied for the vector bundles $E_{1|U_\ell}$ and $E_{2|U_\ell}$.) This means that for every section $s \in \Gamma(E_1 \otimes E_2)_{|U_\ell}$ there exists a unique choice of functions $g^{jk} \in C^\infty(U_\ell)$ such that

$$s = \sum_{j=1}^{m} \sum_{k=1}^{n} g^{jk} \cdot b_j \otimes b'_k = \sum_{j=1}^{m} \sum_{k=1}^{n} (g^{jk} b_j) \otimes b'_k.$$

Now let a global section $s \in \Gamma(E_1 \otimes E_2)$ be given. From the above discussion (and passing to a slightly different notation) we conclude that for each $\ell \in \{1, \ldots, r\}$ there exist sections $\sigma_j^{(\ell)} \in \Gamma E_{1|U_\ell}$, $\tau_j^{(\ell)} \in \Gamma E_{2|U_\ell}$, $j = 1, \ldots, mn$, such that

$$s_{|U_\ell} = \sum_{j=1}^{mn} \sigma_j^{(\ell)} \otimes \tau_j^{(\ell)}.$$

By Problem 11(a),(b) (partition of unity), there exist C^∞ functions $\varphi_1, \ldots, \varphi_r : M \to [0,1]$ satisfying $\operatorname{supp} \varphi_\ell \subset U_\ell$ for each ℓ and

$$\sum_{\ell=1}^{r} \varphi_\ell(p)^2 = 1, \qquad \forall p \in M.$$

Define the function $\widetilde{\sigma}_j^{(\ell)} : M \to E_1$ by $\widetilde{\sigma}_j^{(\ell)}(p) = \varphi_\ell(p) \sigma_j^{(\ell)}(p)$ for $p \in U_\ell$ and $\widetilde{\sigma}_j^{(\ell)} = 0 \in E_{1,p}$ for $p \in M \setminus U_\ell$. Then $\widetilde{\sigma}_j^{(\ell)}$ is C^∞ by the argument in the solution to Problem 7(a) (indeed the restriction of $\widetilde{\sigma}_j^{(\ell)}$ to the two open sets U_ℓ and $M \setminus \operatorname{supp}(\varphi_\ell)$ is C^∞, and these two open sets cover M). Also $\pi_1 \circ \widetilde{\sigma}_j^{(\ell)} = 1_M$. Hence

$$\widetilde{\sigma}_j^{(\ell)} \in \Gamma E_1.$$

Similarly define

$$\widetilde{\tau}_j^{(\ell)} \in \Gamma E_2$$

by $\widetilde{\tau}_j^{(\ell)}(p) = \varphi_\ell(p) \tau_j^{(\ell)}(p)$ for $p \in U_\ell$ and zero elsewhere. Now

$$\sum_{j=1}^{mn} \sum_{\ell=1}^{r} \widetilde{\sigma}_j^{(\ell)} \otimes \widetilde{\tau}_j^{(\ell)} \in \Gamma E_1 \otimes \Gamma E_2,$$

and for every $p \in M$ we have

$$\begin{aligned}
J\Big(\sum_{j=1}^{mn}\sum_{\ell=1}^{r}\widetilde{\sigma}_j^{(\ell)}\otimes\widetilde{\tau}_j^{(\ell)}\Big)(p) &= \sum_{j=1}^{mn}\sum_{\ell=1}^{r}\widetilde{\sigma}_j^{(\ell)}(p)\otimes\widetilde{\tau}_j^{(\ell)}(p)\\
&= \sum_{j=1}^{mn}\sum_{\substack{\ell=1\\(p\in U_\ell)}}^{r}(\varphi_\ell(p)\sigma_j^{(\ell)}(p))\otimes(\varphi_\ell(p)\tau_j^{(\ell)}(p))\\
&= \sum_{\substack{\ell=1\\(p\in U_\ell)}}^{r}\varphi_\ell(p)^2\sum_{j=1}^{mn}\sigma_j^{(\ell)}(p)\otimes\tau_j^{(\ell)}(p)\\
&= \sum_{\substack{\ell=1\\(p\in U_\ell)}}^{r}\varphi_\ell(p)^2 s(p)\\
&= s(p).
\end{aligned}$$

Hence

$$J\Big(\sum_{j=1}^{mn}\sum_{\ell=1}^{r}\widetilde{\sigma}_j^{(\ell)}\otimes\widetilde{\tau}_j^{(\ell)}\Big) = s,$$

and we have proved that J is surjective.

Proof that J is *injective:* This seems somewhat more complicated to carry out in the direct approach fashion used above, and we skip it for now... However see [3, Thm. 7.5.5] for an elegant (but slightly less direct) proof. □

Problem 44:

(a). (Cf. Poor, [16, Prop. 1.60].)

We start by proving that there is a natural *bijection* between Γf^*E and $\Gamma_f E$. Remeber from Problem 42 that f^*E is a differentiable submanifold of $M \times E$, and we know from Problem 8 that $\text{pr}_2 : M \times E \to E$ is a C^∞ map. Now if $s \in \Gamma f^*E$ then $\text{pr}_2 \circ s$ is a C^∞ map from M to E. Also $\pi \circ \text{pr}_2 \circ s = f$, since $\text{pr}_2(s(p)) \in E_{f(p)}$ for all $p \in M$. Hence $\text{pr}_2 \circ s \in \Gamma_f E$. Thus we have constructed a map

$$\Gamma f^*E \to \Gamma_f E; \quad s \mapsto \text{pr}_2 \circ s. \tag{115}$$

We next construct the inverse map. Given $\sigma \in \Gamma_f E$, we define the map

$$\widehat{\sigma} : M \to f^*E; \qquad \widehat{\sigma}(p) := (p, \sigma(p)).$$

Note that for any $p \in M$ we have $\sigma(p) \in E_{f(p)}$ and hence

$$\widehat{\sigma}(p) \in \{p\} \times E_{f(p)} = (f^*E)_p.$$

This relation implies in particular $\widehat{\sigma}(p) \in f^*E$, i.e. $\widehat{\sigma}$ is indeed a well-defined map from M to f^*E, and it also implies that $\widetilde{\pi}(\widehat{\sigma}(p)) = p$; thus $\widetilde{\pi} \circ \widehat{\sigma} = 1_M$. In order to prove that $\widehat{\sigma}$ is C^∞, it suffices to prove that $\widetilde{\varphi} \circ \widehat{\sigma}_{|f^{-1}(U)}$ is C^∞ for every bundle chart (U, φ) for (E, π, N). [29] Now for every $p \in f^{-1}(U)$,

$$\widetilde{\varphi}(\widehat{\sigma}(p)) = \widetilde{\varphi}(p, \sigma(p)) = (p, \text{pr}_2(\varphi(\sigma(p)))),$$

and the last expression is clearly a C^∞ function of $p \in f^{-1}(U)$ (using Problem 8(d) and the fact that any composition of C^∞ maps is a C^∞ map). Hence $\widehat{\sigma} \in \Gamma f^*E$. Thus we have constructed a map

$$\Gamma_f E \to \Gamma f^*E; \quad \sigma \mapsto \widehat{\sigma}. \tag{116}$$

Next we prove that the two maps (115) and (116) are inverses to each other. For every $\sigma \in \Gamma_f E$ we have, for every $p \in M$,

$$\text{pr}_2 \circ \widehat{\sigma}(p) = \text{pr}_2(p, \sigma(p)) = \sigma(p).$$

Hence $\text{pr}_2 \circ \widehat{\sigma} = \sigma$. Next, for every $s \in \Gamma f^*E$ we have, for every $p \in M$,

$$\widehat{\text{pr}_2 \circ s}(p) = (p, \text{pr}_2(s(p))) = (\widetilde{\pi}(s(p)), \text{pr}_2(s(p))) = (\text{pr}_1(s(p)), \text{pr}_2(s(p))) \\ = s(p).$$

Hence $\widehat{\text{pr}_2 \circ s} = s$. Done!

Hence we have proved that the map (115) is a bijection of Γf^*E onto $\Gamma_f E$, with inverse given by (116). Hence there is a unique $C^\infty M$-module structure on $\Gamma_f E$ which such that (115) is an isomorphism of $C^\infty M$-modules!

[29] Here we use the notation from the solution of Problem 42(a); thus $\widetilde{\varphi}$ is the C^∞ diffeomorphism of $\widetilde{\pi}^{-1}(f^{-1}(U))$ onto $f^{-1}(U) \times \mathbb{R}^n$ given by $\widetilde{\varphi}(p, v) := (p, \text{pr}_2(\varphi(v)))$; then $(f^{-1}(U), \widetilde{\varphi})$ is a bundle chart for $(f^*E, \widetilde{\pi}, M)$.

It should here also be pointed out that the $C^\infty M$-module operations in $\Gamma_f E$ are completely natural, namely they are just pointwise addition and pointwise multiplication by scalar(s). Indeed, let $\alpha \in C^\infty M$ and $\sigma_1, \sigma_2 \in \Gamma_f E$. Then $\sigma_1 + \sigma_2 \in \Gamma_f E$ is just the pointwise sum, i.e.

$$(\sigma_1 + \sigma_2)(p) = \sigma_1(p) + \sigma_2(p) \in E_{f(p)}, \qquad \forall p \in M;$$

and $\alpha\sigma_1 \in \Gamma_f E$ is the pointwise product, i.e.

$$(\alpha\sigma_1)(p) = \alpha(p)\sigma_1(p) \in E_{f(p)}, \qquad \forall p \in M.$$

To prove the formula for $\sigma_1 + \sigma_2$, note that since we require that (116) is a $C^\infty M$-module isomorphism, we should have $\widehat{\sigma_1 + \sigma_2} = \widehat{\sigma}_1 + \widehat{\sigma}_2$ in $\Gamma f^* E$, and by definition of the $C^\infty M$-module structure of $\Gamma f^* E$, $\widehat{\sigma}_1 + \widehat{\sigma}_2$ is just pointwise sum, i.e. $(\widehat{\sigma}_1 + \widehat{\sigma}_2)(p) = \widehat{\sigma}_1(p) + \widehat{\sigma}_2(p)$ for all $p \in M$. Hence

$$\begin{aligned}(p, (\sigma_1 + \sigma_2)(p)) = \widehat{\sigma_1 + \sigma_2}(p) = \widehat{\sigma}_1(p) + \widehat{\sigma}_2(p) &= (p, \sigma_1(p)) + (p, \sigma_2(p)) \\ &= (p, \sigma_1(p) + \sigma_2(p)),\end{aligned}$$

and therefore $(\sigma_1 + \sigma_2)(p) = \sigma_1(p) + \sigma_2(p)$. Done! The proof of the formula for $\alpha\sigma_1$ is completely similar. □

(b) (Recall that "basis of sections" is defined in Problem 33(c).)

Our task is to prove that for every $p \in U$,

(117) $\qquad s_1(f(p)), \ldots, s_n(f(p))$ form a basis for $(f^*E)_p$.

However $(f^*E)_p = E_{f(p)}$ and $f(p) \in V$, and therefore (117) follows from our assumption that $s_1, \ldots, s_n$ is a basis of sections in $\Gamma E_{|V}$. □

(c) By Problem 38 there is a finite open cover $\{U_1, \ldots, U_r\}$ of N such that $E_{|U_j}$ is trivial for each j. For each $j \in \{1, \ldots, r\}$, let $s_{j,1}, \ldots, s_{j,n}$ be a basis of sections in $\Gamma E_{|U_j}$ (here $n = \operatorname{rank} E$). Also let $\varphi_1, \ldots, \varphi_r$ be a subordinate partition of unity as in Problem 11(a), i.e. each φ_j is a C^∞ function $N \to [0, 1]$ with $\operatorname{supp} \varphi_j \subset U_j$ (but $\operatorname{supp} \varphi_j$ is not necessarily compact) and $\sum_{j=1}^r \varphi_j(y) = 1$ for all $y \in N$. By Problem 11(b), we may furthermore assume that each function $\rho_j := \sqrt{\varphi_j}$ is C^∞. Note that these functions satisfy $\operatorname{supp} \rho_j = \operatorname{supp} \varphi_j \subset U_j$ and

$$\sum_{j=1}^r \rho_j(y)^2 = 1, \qquad \forall y \in N.$$

For each $j \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, n\}$ we define a function $\widetilde{s}_{j,k} : N \to E$ by

$$\widetilde{s}_{j,k}(y) = \begin{cases} \rho_j(y) s_{j,k}(y) & \text{if } y \in U_j \\ 0 \text{ (in } E_y) & \text{if } y \notin U_j. \end{cases}$$

Then $\operatorname{supp}(\widetilde{s}_{j,k}) \subset \operatorname{supp}(\rho_j) \subset U_j$ and hence by mimicking the solution to Problem 7(a) one shows that $\widetilde{s}_{j,k}$ is C^∞. Also $\pi \circ \widetilde{s}_{j,k} = 1_N$; hence $\widetilde{s}_{j,k} \in \Gamma E$.

Now let $s \in \Gamma_f E$ be given. We then define $\alpha_{j,k} : M \to \mathbb{R}$ for $j \in \{1, \ldots, r\}$ and $k \in \{1, \ldots, n\}$ by the requirement

$$\begin{cases} s(x) = \sum_{k=1}^{n} \alpha_{j,k}(x) \cdot s_{j,k}(f(x)) & \text{if } x \in f^{-1}(U_j); \\ \alpha_{j,1}(x) = \cdots = \alpha_{j,n}(x) = 0 & \text{if } x \notin f^{-1}(U_j). \end{cases} \tag{118}$$

By part (b) together with Problem 34, this makes the functions $\alpha_{j,k}$ uniquely determined, and $\alpha_{j,k|f^{-1}(U_j)} \in C^\infty(f^{-1}(U_j))$ for all j, k. Next define the function $\widetilde{\alpha}_{j,k} : M \to \mathbb{R}$ by

$$\widetilde{\alpha}_{j,k}(x) := \rho_j(f(x)) \cdot \alpha_{j,k}(x).$$

Then $\widetilde{\alpha}_{j,k|f^{-1}(U_j)} \in C^\infty(f^{-1}(U_j))$, and also $\text{supp}(\widetilde{\alpha}_{j,k}) \subset \text{supp}(\rho_j \circ f) \subset f^{-1}(\text{supp}(\rho_j)) \subset f^{-1}(U_j)$, [30] and hence by Problem 7(a), $\widetilde{\alpha}_{j,k} \in C^\infty(M)$.

Now for each $x \in M$,

$$\sum_{j=1}^{r} \sum_{k=1}^{n} \widetilde{\alpha}_{j,k}(x) \widetilde{s}_{j,k}(f(x)) = \sum_{\substack{j=1 \\ (f(x) \in U_j)}}^{r} \sum_{k=1}^{n} \rho_j(f(x))^2 \alpha_{j,k}(x) s_{j,k}(f(x))$$

$$\{\text{use (118)}\} \qquad = \sum_{\substack{j=1 \\ (f(x) \in U_j)}}^{r} \rho_j(f(x))^2 s(x) = s(x).$$

(The sum over j is taken over all $j \in \{1, \ldots, r\}$ for which $f(x) \in U_j$.) The last equality holds since $\sum_{j=1}^{r} \rho_j(f(x))^2 = 1$ and $\rho_j(f(x)) = 0$ for all j with $f(x) \notin U_j$. Hence we have expressed s as a finite sum of the desired form. □

[30] Here the inclusion $\text{supp}(\rho_j \circ f) \subset f^{-1}(\text{supp}(\rho_j))$ holds since $f^{-1}(\text{supp}(\rho_j))$ is a closed subset of M which contains every $x \in M$ satisfying $\rho_j(f(x)) \neq 0$.

Problem 45:

(a). The solution to Problem 40 is generalized to the present situation without any new difficulties arising (apart from a little extra amount of book-keeping):

Let us write $n_j = \operatorname{rank} E_j$ $(j = 1, 2)$ and let π be the projection map $\pi : \operatorname{Hom}(E_1, f^*E_2) \to M$.

Let $s \in \Gamma(\operatorname{Hom}(E_1, f^*E_2))$. Then for each $p \in M$,

$$s(p) \in \operatorname{Hom}(E_1, f^*E_2)_p = \operatorname{Hom}(E_{1,p}, E_{2,f(p)}),$$

and so s gives rise to a map

$$h : E_1 \to E_2, \qquad h(x) := s(\pi_1(x))(x) \quad (x \in E_1).$$

By construction this map h satisfies $\pi_2 \circ h = f \circ \pi_1$, and furthermore for each $p \in M$,

$$h_p := h_{|E_{1,p}} = s(p) \in \operatorname{Hom}(E_{1,p}, E_{2,f(p)}),$$

i.e. h_p is a linear map from $E_{1,p}$ to $E_{2,f(p)}$. Hence if we can only prove that h is C^∞ then h is a bundle homomorphism $E_1 \to E_2$ along f.

To prove that h is C^∞ is a local problem, and by passing to appropriate charts and bundle charts it is seen to follow from the basic fact pointed out around (99), (100) in the solution of Problem 40. (We leave out the details...)

Hence, writing $\mathcal{H}$ for the set of bundle homomorphisms $E_1 \to E_2$ along f, then above we have constructed a map

$$\Gamma(\operatorname{Hom}(E_1, f^*E_2)) \to \mathcal{H}, \qquad \text{“}s \mapsto h\text{”}. \tag{119}$$

We next construct the inverse map. Thus let h be a bundle homomorphism $E_1 \to E_2$ along f. Then by definition, for each $p \in M$, $h_p = h_{|E_{1,p}}$ is an $\mathbb{R}$-linear map from $E_{1,p}$ to $E_{2,f(p)}$, i.e.

$$h_p \in \operatorname{Hom}(E_{1,p}, E_{2,f(p)}) = \operatorname{Hom}(E_1, f^*E_2)_p.$$

Let us define the map $s : M \to \operatorname{Hom}(E_1, f^*E_2)$ by $s(p) := h_p$. Clearly $\pi \circ s = 1_M$, and one verifies that s is C^∞ by passing to local coordinates (we again leave out the details). Hence $s \in \Gamma(\operatorname{Hom}(E_1, f^*E_2))$, and so we have constructed a map

$$\mathcal{H} \to \Gamma(\operatorname{Hom}(E_1, f^*E_2)), \quad \text{“}h \mapsto s\text{”}. \tag{120}$$

It is immediate from our definitions (in particular using “$s(p) = h_p$”) that the two maps (119) and (120) are inverses to each other. Hence we the two maps are in fact *bijections*. $\square$

(b). Consider the special case $N = M$ and $f = 1_M$. Then $h : E_1 \to E_2$ is a "bundle homomorphism along f" iff h is a bundle homomorphism. Also $f^*E_2 = E_2$. Hence in this special case, part (a) of the present problem says the same as Problem 40 (and one verifies that the bijection is really the same as there).

Next consider the special case where E_1 is the trivial vector bundle of rank 1 over M, i.e. $E_1 = M \times \mathbb{R}$ (but $f : M \to N$ is again a general C^∞ map between C^∞ manifolds; also (E_2, π_2, N) is an arbitrary vector bundle). Then there is an "obvious" bijection between the family of bundle homomorphism $h : E_1 \to E_2$ along f and the family $\Gamma_f E_2$ of sections s of E_2 along f: This bijection is given by $s(p) := h(p, 1)$ $(\forall p \in M)$; inverse: $h(p, r) = r \cdot s(p)$ $(\forall (p, r) \in E_1 = M \times \mathbb{R})$. Furthermore we have an "obvious" identification $\mathrm{Hom}(E_1, f^*E_2) = f^*E_2$, via the identifications

$$\mathrm{Hom}(E_1, f^*E_2)_p = \mathrm{Hom}(E_{1,p}, (f^*E_2)_p) = \mathrm{Hom}(\mathbb{R}, (f^*E_2)_p) = (f^*E_2)_p$$

$(\forall p \in M)$. In the light of these identifications, part (a) of the present problem now says that there is a natural bijection between Γf^*E_2 and the set $\Gamma_f E_2$. This is exactly the statement of Problem 44(a) (and one verifies that the bijection is really the same as there). □

Problem 46:

Let (U, φ) be a bundle chart for E with $c(t_0) \in U$. Take $\varepsilon > 0$ so small that $c(t) \in U$ for all $t \in (t_0 - \varepsilon, t_0 + \varepsilon)$. Then via (U, φ), the section $s_{|(t_0-\varepsilon,t_0+\varepsilon)}$ is identified with a C^∞ map from $(t_0 - \varepsilon, t_0 + \varepsilon)$ to $\mathbb{R}^n$, where $n := \operatorname{rank} E$. Let s^j be the j:th coordinate of this function; thus s^j is a C^∞ map from $(t_0 - \varepsilon, t_0 + \varepsilon)$ to $\mathbb{R}$, for $j = 1, \ldots, n$. By Problem 12(b), for each j there exists some $\varepsilon_j \in (0, \varepsilon)$ and a C^∞ function $g^j : U \to \mathbb{R}$ such that $g^j(c(t)) = s^j(t)$ for all $t \in (t_0 - \varepsilon_j, t_0 + \varepsilon_j)$. Let $g : U \to \mathbb{R}^n$ be the function whose jth coordinate is g^j; this is a C^∞ function from U to $\mathbb{R}^n$, and via our bundle chart (U, φ), g defines a section $\widetilde{s} \in \Gamma E_{|U}$ satisfying $\widetilde{s}(c(t)) = s(t)$ for all $t \in (t_0 - \varepsilon', t_0 + \varepsilon')$, where $\varepsilon' := \min(\varepsilon_1, \ldots, \varepsilon_n)$. Now by Problem 35(a), there exists some $s_1 \in \Gamma E$ and an open set $V \subset U$ containing $c(t_0)$ satisfying $s_{1|V} = \widetilde{s}_{|V}$. Now take $\varepsilon'' \in (0, \varepsilon']$ so that $c(t) \in V$ for all $t \in (t_0 - \varepsilon'', t_0 + \varepsilon'')$. Then we have $s_1(c(t)) = s(t)$ for all $t \in (t_0 - \varepsilon'', t_0 + \varepsilon'')$. Done! $\square$

[Some pedantic details: In more precise notation, we have in the previous discussion:

$$s^j := \widetilde{\mathrm{pr}}_j \circ \mathrm{pr}_2 \circ \varphi \circ s : (t_0 - \varepsilon, t_0 + \varepsilon) \to \mathbb{R},$$

where pr_2 is the projection from $U \times \mathbb{R}^n$ onto the second factor $\mathbb{R}^n$, and $\mathrm{pr}_j : \mathbb{R}^n \to \mathbb{R}$ is projection onto the jth coordinate. Also $\widetilde{s}$ is defined by

$$\widetilde{s}(p) := \varphi^{-1}(p, g(p)), \qquad \forall p \in U,$$

which by inspection is indeed a C^∞ map from U to E with $\pi \circ \widetilde{s} = 1_U$; thus $\widetilde{s} \in \Gamma E_{|U}$. Our choice of g, $g^1, \ldots, g^n$ and ε' implies that for every $t \in (t_0 - \varepsilon', t_0 + \varepsilon')$ we have

$$g^j(c(t)) = s^j(t) = \widetilde{\mathrm{pr}}_j \circ \mathrm{pr}_2 \circ \varphi \circ s(t);$$

hence

$$g(c(t)) = \mathrm{pr}_2 \circ \varphi \circ s(t),$$

and since also $\mathrm{pr}_1 \circ \varphi \circ s(t) = \pi \circ s(t) = c(t)$, it follows that:

$$\varphi(s(t)) = (c(t), g(c(t))),$$

and thus

$$s(t) = \varphi^{-1}(c(t), g(c(t))) = \widetilde{s}(c(t)), \qquad \forall t \in (t_0 - \varepsilon', t_0 + \varepsilon'),$$

just as we claimed in the above discussion.]

Problem 47:

(a). The map

$$f \mapsto X(Y(f)) - Y(X(f)), \qquad C^\infty(M) \to C^\infty(M), \tag{121}$$

is clearly $\mathbb{R}$-linear, since X and Y are (or give) $\mathbb{R}$-linear maps on $C^\infty(M)$ (cf. Problem 15(b)). Furthermore for any $f, g \in C^\infty(M)$, using the fact that X and Y are derivations we have

$$\begin{aligned} X(Y(fg)) &= X\big((Yf)\cdot g + f\cdot(Yg)\big) \\ &= (X(Yf))\cdot g + (Yf)\cdot(Xg) + (Xf)\cdot(Yg) + f\cdot(X(Yg)). \end{aligned}$$

Similarly,

$$Y(X(fg)) = (Y(Xf))\cdot g + (Xf)\cdot(Yg) + (Yf)\cdot(Xg) + f\cdot(Y(Xg)),$$

and subtracting the two we get

$$X(Y(fg)) - Y(X(fg)) = \Big(X(Y(f)) - Y(X(f))\Big)\cdot g + f\cdot\Big(X(Y(g)) - Y(X(g))\Big).$$

Hence the map in (121) is a derivation of $C^\infty(M)$. Hence by Problem 15(b) there is a unique vector field Z on M such that

$$Z(f) = X(Y(f)) - Y(X(f)), \qquad \forall f \in C^\infty(M).$$

Done. □

(b). ...

(c). For any $f \in C^\infty(M)$ we have, by definition of the Lie product,

$$\begin{aligned} [[X,Y],Z](f) &= [X,Y](Zf) - Z([X,Y]f) \\ &= X(Y(Z(f))) - Y(X(Z(f))) - Z(X(Y(f))) + Z(Y(X(f))). \end{aligned}$$

Adding this to the corresponding formulas for $[[Y,Z],X](f)$ and $[[Z,X],Y](f)$ we obtain

$$\Big([[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y]\Big)(f) = 0.$$

This is true for all $f \in C^\infty(M)$; hence we obtain (via Problem 15(b)):

$$[[X,Y],Z] + [[Y,Z],X] + [[Z,X],Y] = 0.$$

□

(d). ...

Problem 48:

(a). The task is to prove that if (V, y) is any other C^∞ chart on M, with respect to which

$$\omega_{|V} = \sum_I \widetilde{\omega}_I \, dy^I$$

(with $\widetilde{\omega}_I \in C^\infty(U)$), then we have

$$\sum_I d\omega_I \wedge dx^I = \sum_I d\widetilde{\omega}_I \wedge dy^I \qquad \text{in } U \cap V. \tag{122}$$

We start by noticing that the fact that

$$\sum_I \widetilde{\omega}_I \, dy^I = \sum_I \omega_I \, dx^I \qquad \text{in } U \cap V \tag{123}$$

(since both these are $= \omega_{|U \cap V}$ in $U \cap V$) means that

$$\sum_I \widetilde{\omega}_I \, dy^I = f^* \Big(\sum_I \omega_I \, dx^I \Big) \qquad \text{in } y(U \cap V) \subset \mathbb{R}^d. \tag{124}$$

where f is the coordinate transformation

$$f = x \circ y^{-1} : y(U \cap V) \to x(U \cap V).$$

[Pedantic explanation: In (124) we are stating an equality between two r-forms living on an open subset of $\mathbb{R}^d$, and so we are no longer viewing dx^I or dy^I as r-forms on $U \cap V$, but rather as r-forms on $x(U) \subset \mathbb{R}^d$ and on $y(V) \subset \mathbb{R}^d$, respectively. This is the reason why we can both have (123) and (124) although on first look they seem to contradict each other! Namely, in (123) we are viewing dx^I and dy^I as r-forms on (subsets of) M, but in (124) we are viewing them more concretely as r-forms on (subsets of) $\mathbb{R}^d$. Note that also ω_I and $\widetilde{\omega}_I$ stand for different things in (124) versus (123): in (123) ω_I and $\widetilde{\omega}_I$ are functions on U and V respectively, whereas in (124) they are functions on $x(U)$ and $y(V)$, respectively (and a more pedantically correct notation for these would be $\omega_I \circ x^{-1}$ and $\widetilde{\omega}_I \circ y^{-1}$). The situation is exactly the same regarding the relationship between (122) and the computation below. Note that we saw a similar example of such[31] abuse of notation already when we introduced tangent spaces: We have $\frac{\partial}{\partial x^j} = \frac{\partial y^k}{\partial x^j} \frac{\partial}{\partial y^k}$ in $T_p M$ for any $p \in U \cap V$, but the corresponding relation certainly does *not* hold (in general) when we view $\frac{\partial}{\partial x^j}$ as vectors in $T_{x(p)}\mathbb{R}^d = \mathbb{R}^d$ and $\frac{\partial}{\partial y^k}$ as vectors in $T_{y(p)}\mathbb{R}^d = \mathbb{R}^d$.]

[31] very convenient!

Now in $y(U \cap V)$ we have:

$$\begin{aligned}\sum_I d\widetilde{\omega}_I \wedge dy^I &= d\Big(\sum_I \widetilde{\omega}_I\, dy^I\Big)\\ &= d\Big(f^*\Big(\sum_I \omega_I\, dx^I\Big)\Big)\\ &= f^*\Big(d\Big(\sum_I \omega_I\, dx^I\Big)\Big)\\ &= f^*\Big(\sum_I d\omega_I \wedge dx^I\Big).\end{aligned}$$

[Details: The first equality holds by the definition of d; the second by (124); the third by Jost, [12, Lemma 2.1.3]; and finally the fourth equality again holds by the definition of d.]

The equality proved in the above computation says exactly that (122) holds! This completes the proof that the map $d : \Omega^r(M) \to \Omega^{r+1}(M)$ is well-defined. It is now immediate to verify that this map is $\mathbb{R}$-linear. $\square$

(b). Strictly speaking, we need to prove this formula first in the special case when M and N are open subsets of $\mathbb{R}^d$ and $\mathbb{R}^{d'}$, since we make use of this fact in the proof that the general exterior derivative $d : \Omega^r(M) \to \Omega^{r+1}(M)$ is well-defined; cf. part (a) above!

Thus assume that M is an open subset of $\mathbb{R}^d$ and N is an open subset of $\mathbb{R}^{d'}$ (and $f : M \to N$ is a C^∞ map). Let us write $x = (x^1, \ldots, x^{d'})$ for a variable point in N and $y = (y^1, \ldots, y^d)$ for a variable point in M. Let $\omega \in \Omega^r(N)$, and take functions $\omega_I \in C^\infty(N)$ so that $\omega = \sum_I \omega_I \, dx^I$. Then $f^*(\omega) \in \Omega^r(M)$ is given by (recall that $I = (i_1, \ldots, i_r)$ runs through all r-tuples with $1 \leq i_1 < \cdots < i_r \leq d'$):

$$\begin{aligned} f^*(\omega) &= \sum_I f^*(\omega_I) \, f^*(dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \\ &= \sum_I (\omega_I \circ f) \, f^*(dx^{i_1}) \wedge \cdots \wedge f^*(dx^{i_r}) \\ &= \sum_I (\omega_I \circ f) \, d(x^{i_1} \circ f) \wedge \cdots \wedge d(x^{i_r} \circ f) \\ &= \sum_I (\omega_I \circ f) \, df^{i_1} \wedge \cdots \wedge df^{i_r}. \end{aligned}$$

(In this computation we used some basic properties of f^* which we pointed out in Lecture #8.) Hence (making use of [12, Lemma 2.1.2] for r-forms on $\mathbb{R}^d$):

$$\begin{aligned} d(f^*(\omega)) &= \sum_I d\Big((\omega_I \circ f) \, df^{i_1} \wedge \cdots \wedge df^{i_r} \Big) \\ &= \sum_I \Big(d(\omega_I \circ f) \wedge df^{i_1} \wedge \cdots \wedge df^{i_r} \\ &\qquad + (\omega_I \circ f)(d(df^{i_1})) \wedge df^{i_2} \wedge \cdots \wedge df^{i_r} \\ &\qquad - (\omega_I \circ f)(df^{i_1}) \wedge (d(df^{i_2})) \wedge df^{i_3} \wedge \cdots \wedge df^{i_r} \\ &\qquad + \cdots - (-1)^r (\omega_I \circ f)(df^{i_1}) \wedge (df^{i_2}) \wedge \cdots \wedge d(df^{i_r}) \Big). \end{aligned} \tag{125}$$

But note that for any $g \in C^\infty(M)$ we have

$$d(dg) = d\Big(\frac{\partial g}{\partial y^j} dy^j \Big) = d\Big(\frac{\partial g}{\partial y^j} \Big) \wedge dy^j = \frac{\partial^2 g}{\partial y^k \partial y^j} dy^k \wedge dy^j = 0. \tag{126}$$

(The last equality holds since we are adding over $k, j \in \{1, \ldots, d\}$, and since $dy^k \wedge dy^j = -dy^j \wedge dy^k$.) Hence all inner terms in (125) except the first *vanish,* i.e. we obtain:

$$d(f^*(\omega)) = \sum_I d(\omega_I \circ f) \wedge df^{i_1} \wedge \cdots \wedge df^{i_r}. \tag{127}$$

(Of course, (126) is a special case of the general relation $d \circ d = 0$ [12, Theorem 2.1.5], and once we know this the proof of (127) is much shorter.)

On the other hand,

$$f^*(d\omega) = f^*\left(\sum_I d\omega_I \wedge dx^I\right) = \sum_I f^*(d\omega_I) \wedge f^*(dx^I)$$

$$= \sum_I d(\omega_I \circ f) \wedge df^{i_1} \wedge \cdots \wedge df^{i_r}. \tag{128}$$

Comparing with (127), we conclude that indeed $d(f^*(\omega)) = f^*(d\omega)$. □

Later, when we have defined the exterior derivative for general manifolds M (cf. part (a) of this problem), the above proof carries over, with very small changes, to the case of a C^∞ map $f : M \to N$ between C^∞ manifolds. Indeed, one fixes a chart (U, x) on N and assumes $\omega_{|U} = \sum_I \omega_I dx^I$ with $\omega_I \in C^\infty(U)$. Then the computation up to and including (125) is still valid, *in the open subset* $f^{-1}(U)$ *of* M. (Note that each f^j is a C^∞ function $f^{-1}(U) \to \mathbb{R}$, defined by $f^j := x^j \circ f$.) Also for any open set $W \subset M$ and any $g \in C^\infty(W)$, the computation in (126) shows that $d(dg) = 0$; namely if we work locally wrt any chart (V, y) on W. Hence we obtain (127), as an equality of $(r+1)$-forms restricted to the set $f^{-1}(U)$. Similarly the computation (128) is valid in $f^{-1}(U)$, and so we conclude that

$$d(f^*(\omega))_{|f^{-1}(U)} = f^*(d\omega)_{|f^{-1}(U)}.$$

But this is true for (U, x) being an arbitrary C^∞ chart on N; hence we actually have $d(f^*(\omega)) = f^*(d\omega)$ on all M. □

(c). Let (U, x) be a C^∞ chart on M. We first prove the stated formula on U, and in the special case when

$$X_j := \frac{\partial}{\partial x^{\ell_j}} \in \Gamma(TU), \qquad (j = 0, \ldots, r), \tag{129}$$

for some $\ell_0, \ldots, \ell_r \in \{1, \ldots, d\}$ $(d = \dim M)$. In this case $[X_j, X_k] = 0$ for all $j, k \in \{0, \ldots, r\}$, and hence the formula that we wish to prove states that for any $\omega \in \Omega^r(U)$,

$$[d\omega](X_0, \ldots, X_r) = \sum_{j=0}^{r} (-1)^j X_j\big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_r)\big). \tag{130}$$

Let us first note that both sides of (130) are *alternating* in $X_0, \ldots, X_r$, i.e. if we replace $X_0, \ldots, X_r$ by $X_{\sigma(0)}, \ldots, X_{\sigma(r)}$ for some $\sigma \in \mathfrak{S}_{r+1}$ (the group of permutations of $\{0, 1, \ldots, r\}$) then the effect is that the expressions on both sides of (130) get multiplied by $\operatorname{sgn} \sigma$. [Detailed proof: For the left hand side this holds since $d\omega$ is alternating by definition. Now consider the right hand side. It suffices to study what happens when X_i and X_{i+1} are switched for some $i \in \{0, 1, \ldots, r-1\}$, since $\mathfrak{S}_{r+1}$ is generated by such transpositions. Then for each $j \notin \{i, i+1\}$ the corresponding term in the sum is *negated*, since $\omega(X_0, \ldots, \hat{X}_j, \ldots, X_r)$ gets negated by the transposition. Furthermore the contribution from $j \in \{i, i+1\}$ to the sum after the transposition is

$$(-1)^i X_{i+1}\big(\omega(X_0, \ldots, \hat{X}_{i+1}, \ldots, X_r)\big) + (-1)^{i+1} X_i\big(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_r)\big),$$

and this is again equal to -1 times the contribution from $j \in \{i, i+1\}$ in the original sum. Hence we conclude that the whole expression in the right hand side of (130) gets *negated* when switching $X_i \leftrightarrow X_{i+1}$, and this completes the proof of the claim.]

It follows that it suffices to prove (130) in the special case

$$\ell_0 < \ell_1 < \cdots < \ell_r. \tag{131}$$

(Note in particular that if any two of the ℓ_j's are equal then the alternating property proved above implies that both sides of (130) equal zero and so the equality holds.)

Now take $\omega_I \in C^\infty(U)$ so that $\omega = \sum_I \omega_I dx^I$ (notation as before). Then $d\omega = \sum_I d\omega_I \wedge dx^I$, and recalling $I = (i_1, \ldots, i_r)$ and the definition of wedge product, we get:

$$\begin{aligned}[d\omega](X_0, \ldots, X_r) &= \sum_I (d\omega_I \wedge dx^I)(X_0, \ldots, X_r) \\ &= \sum_I \sum_{\sigma \in \mathfrak{S}_{r+1}} (\operatorname{sgn} \sigma) \cdot d\omega_I(X_{\sigma(0)}) \cdot \prod_{j=1}^{r} dx^{i_j}(X_{\sigma(j)}).\end{aligned} \tag{132}$$

Recalling (129), we see that the last product equals one if $I = (i_1, \ldots, i_r) = (\ell_{\sigma(1)}, \ldots, \ell_{\sigma(r)})$, otherwise zero. Recalling now (131) and the fact that I runs through all r-tuples with $1 \le i_1 < i_2 < \cdots < i_r \le d$, it follows that $\sigma \in \mathfrak{S}_{r+1}$ contributes to the last sum iff $\sigma(1) < \sigma(2) < \cdots < \sigma(r)$. There exist exactly $r+1$ such permutations σ, namely one for each choice of $a := \sigma(0) \in \{0, 1, \ldots, r+1\}$; explicitly the unique admissible permutation $\sigma = \sigma_a$ with $\sigma_a(0) = a$ is given by $\sigma_a(j) = j - 1$ for $1 \leq j \leq a$ and $\sigma_a(j) = j$ for $a < j \leq r$. Note also that this permutation σ can be obtained as a product of a transpositions $i \leftrightarrow i+1$; hence $\text{sgn}(\sigma_a) = (-1)^a$. Given $\sigma = \sigma_a$, we get contribution from $I = (\ell_{\sigma_a(1)}, \ldots, \ell_{\sigma_a(r)})$ and no other I, and using $\omega = \sum_I \omega_I dx^I$ we get for $I = (\ell_{\sigma(1)}, \ldots, \ell_{\sigma(r)})$:

$$\omega_I = \omega(X_{\sigma_a(1)}, \ldots, X_{\sigma_a(r)}) = \omega(X_0, \ldots, \hat{X}_a, \ldots, X_r) \in C^\infty(U)$$

and thus

$$d\omega_I(X_{\sigma_a(0)}) = X_a\big(\omega(X_0, \ldots, \hat{X}_a, \ldots, X_r)\big).$$

Using these facts in (132) we get:

$$[d\omega](X_0, \ldots, X_r) = \sum_{a=0}^{r} (-1)^a X_a\big(\omega(X_0, \ldots, \hat{X}_a, \ldots, X_r)\big),$$

i.e. (130) is proved! Recall that this was under the assumption that $X_j = \frac{\partial}{\partial x^{\ell_j}}$ for $j = 0, \ldots, r$; cf. (129).

Next, let us call the right hand side of the general formula which we wish to prove "$F(X_0, \ldots, X_r)$". That is:

$$\begin{aligned} F(X_0, \ldots, X_r) := & \sum_{j=0}^{r} (-1)^j X_j\big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_r)\big) \\ & + \sum_{0 \leq j < k \leq r} (-1)^{j+k} \omega([X_j, X_k], X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r). \end{aligned} \tag{133}$$

Thus F is a map $F : \Gamma(TM) \times \cdots \times \Gamma(TM) \to C^\infty(M)$. Let us prove that F is $C^\infty(M)$-multilinear! It is clear by inspection that F is $\mathbb{R}$-multilinear. Let us also note that F is *alternating.* (Proof: The argument we gave just below (130) applies to show that the first sum is alternating. For the second sum a similar argument works; we leave out the details.) Hence to show the $C^\infty(M)$-multilinearity, it now suffices to prove that for any $f \in C^\infty(M)$,

$$F(fX_0, X_1, \ldots, X_r) = f \cdot F(X_0, X_1, \ldots, X_r).$$

But when we replace X_0 by fX_0, the $j = 0$ term in the first sum of (133) clearly gets multiplied by f, whereas each $j > 0$ term becomes $(-1)^j$ times

$$\begin{aligned} X_j\big(\omega(fX_0, \ldots, \hat{X}_j, \ldots, X_r)\big) &= X_j\big(f \cdot \omega(X_0, \ldots, \hat{X}_j, \ldots, X_r)\big) \\ &= (X_j f) \cdot \omega(X_0, \ldots, \hat{X}_j, \ldots, X_r) + f \cdot X_j\big(\omega(X_0, \ldots, \hat{X}_j, \ldots, X_r)\big). \end{aligned}$$

In the second sum, each term with $0 < j < k \leq r$ gets multiplied by f, whereas each term with $0 = j < k \leq r$ becomes $(-1)^{0+k}$ times

$$\begin{aligned}
&\omega([fX_0, X_k], X_1, \ldots, \hat{X}_k, \ldots, X_r) \\
&= \omega\big(f[X_0, X_k] - (X_k f)X_0, X_1, \ldots, \hat{X}_k, \ldots, X_r\big) \\
&= f \cdot \omega([X_0, X_k], X_1, \ldots, \hat{X}_k, \ldots, X_r) - (X_k f) \cdot \omega(X_0, \ldots, \hat{X}_k, \ldots, X_r).
\end{aligned}$$

(Cf. Problem 47(d).) Hence in total we get:

$$\begin{aligned}
F(fX_0, X_1, \ldots, X_r) =& f \cdot F(X_0, X_1, \ldots, X_r) \\
&+ \sum_{j=1}^{r} (-1)^j (X_j f) \cdot \omega(X_0, \ldots, \hat{X}_j, \ldots, X_r) \\
&+ \sum_{k=1}^{r} (-1)^k \big(-(X_k f)\big) \cdot \omega(X_0, \ldots, \hat{X}_k, \ldots, X_r) \\
&= f \cdot F(X_0, X_1, \ldots, X_r).
\end{aligned}$$

Hence the $C^\infty(M)$-multilinearity is proved.

Now the proof of the formula in the general case is easily completed: Again let (U, x) be an arbitrary C^∞ chart on M; it suffices to prove that the desired formula holds *in* U. Hence from now on we may just as well assume $M = U$. We keep the notation "F" from (133) (but now with $M = U$). We have proved above that F is $C^\infty(U)$-multilinear, and also that

$$[d\omega](X_0, \ldots, X_r) = F(X_0, \ldots, X_r) \tag{134}$$

when each X_j is of the form $X_j = \frac{\partial}{\partial x^{\ell_j}}$ (indeed, cf. (130) and again recall that for such $X_0, \ldots, X_r$, all Lie brackets $[X_j, X_k]$ vanish). But every $X \in \Gamma(TU)$ can be expressed as $X = f_\ell \frac{\partial}{\partial x^\ell}$ for some (unique) $f_1, \ldots, f_d \in C^\infty(U)$; hence it follows that (134) holds for arbitrary $X_1, \ldots, X_r \in \Gamma(TU)$. This is the desired formula! $\square$

Problem 49:

(a). By the definition of tensor product (of $C^\infty(M)$-modules), every element in $\Gamma(E_1)\otimes\Gamma(\bigwedge^r M)$ can be written as a finite sum of pure tensors $\mu_1\otimes\omega_1$ (where $\mu_1 \in \Gamma E_1$, $\omega_1 \in \Gamma(\bigwedge^r M) = \Omega^r(M)$). Hence by Problem 43(d), the same is true for any section in $\Gamma(E_1 \otimes \bigwedge^r M) = \Omega^r(E_1)$. The analogous fact of course also holds for $\Gamma(E_2 \otimes \bigwedge^s M) = \Omega^s(E_2)$. Hence the stated formula,

$$(\mu_1 \otimes \omega_1) \wedge (\mu_2 \otimes \omega_2) = (\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2), \quad \forall \mu_1 \in \Gamma(E_1),\ \omega_1 \in \Omega^r(M),\ \mu_2 \in \Gamma(E_2),\ \omega_2 \in \Omega^s(M), \tag{135}$$

together with the requirement that $\wedge$ should be a $C^\infty(M)$-bilinear map $\Omega^r(E_1) \times \Omega^s(E_2) \to \Omega^{r+s}(E_1 \otimes E_2)$, certainly makes $s_1 \wedge s_2$ *uniquely determined* for any $s_1 \in \Omega^r(E_1)$, $s_2 \in \Omega^s(E_2)$. Hence it remains to prove that such a $C^\infty(M)$-bilinear map *exists.*

For the existence proof, we start by considering the map

$$F : \Gamma(E_1) \times \Omega^r(M) \times \Gamma(E_2) \times \Omega^s(M) \to \Omega^{r+s}(E_1 \otimes E_2),$$
$$F(\mu_1, \omega_1, \mu_2, \omega_2) = (\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2).$$

This map is immediately verified to be $C^\infty(M)$-multilinear. Hence by the definition of tensor product, there exists a unique $C^\infty(M)$-linear map

$$\widetilde{F} : \Gamma(E_1) \otimes \Omega^r(M) \otimes \Gamma(E_2) \otimes \Omega^s(M) \to \Omega^{r+s}(E_1 \otimes E_2)$$

satisfying

$$\widetilde{F}(\mu_1 \otimes \omega_1 \otimes \mu_2 \otimes \omega_2) = (\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2), \quad \forall \mu_1 \in \Gamma(E_1),\ \omega_1 \in \Omega^r(M),\ \mu_2 \in \Gamma(E_2),\ \omega_2 \in \Omega^s(M). \tag{136}$$

Using Problem 43(d) and $\Omega^r(M) = \Gamma(\bigwedge^r M)$, $\widetilde{F}$ becomes identified with a $C^\infty(M)$-linear map

$$\widetilde{F} : \Omega^r(E_1) \otimes \Omega^s(E_2) \to \Omega^{r+s}(E_1 \otimes E_2).$$

Composing $\widetilde{F}$ with the canonical map $(s_1, s_2) \mapsto s_1 \otimes s_2$ from $\Omega^r(E_1) \times \Omega^s(E_2)$ to $\Omega^r(E_1) \otimes \Omega^s(E_2)$ (which is $C^\infty(M)$-bilinear by the definition of tensor product), we obtain a $C^\infty(M)$-bilinear map

$$\Omega^r(E_1) \times \Omega^s(E_2) \to \Omega^{r+s}(E_1 \otimes E_2)$$

which maps $\big(\mu_1 \otimes \omega_1, \mu_2 \otimes \omega_2\big)$ to $(\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2)$ for all $\mu_1, \omega_1, \mu_2, \omega_2$ as in (136). This map satisfies all requirements imposed on "$\wedge$", i.e. we have proved the (unique) existence of such a map "$\wedge$"! □

As an addendum, let us note that the wedge product $s_1 \wedge s_2 \in \Omega^{r+s}(E_1 \otimes E_2)$ of any two sections $s_1 \in \Omega^r(E_1)$ and $s_2 \in \Omega^s(E_2)$ can be computed "fiber by fiber":

$$(s_1 \wedge s_2)(p) = s_1(p) \wedge s_2(p), \qquad \forall p \in M,$$

where in the right hand side, the "$\wedge$" [32] denotes the unique $\mathbb{R}$-bilinear map $(E_{1,p} \otimes \bigwedge^r(T_p^*(M))) \times (E_{2,p} \otimes \bigwedge^s(T_p^*(M))) \to (E_{1,p} \otimes E_{2,p}) \otimes \bigwedge^{r+s}(T_p^*(M))$ satisfying

$$(v_1 \otimes \varphi_1) \wedge (v_2 \otimes \varphi_2) = (v_1 \otimes v_2) \otimes (\varphi_1 \wedge \varphi_2),$$
$$\forall v_1 \in E_{1,p},\ \varphi_1 \in \textstyle\bigwedge^r(T_p^*(M)),\ v_2 \in E_{2,p},\ \varphi_2 \in \bigwedge^s(T_p^*(M)). \tag{137}$$

Indeed, this is clear by parsing through the identifications in the above proof (in particular see equation (113) in the solution to Problem 43(d)). Note also that the fact that there indeed exists a unique $\mathbb{R}$-bilinear map "$\wedge$" satisfying (137) is proved by an argument completely similar to what we did above. □

(b). This is quite immediate from the corresponding formulas for the "standard" wedge product on $\Omega(M)$. Indeed, for the associativity relation, by $C^\infty(M)$-multilinearity and the argument at the beginning of our solution to part (a), it suffices to prove the identity s_1, s_2, s_3 of the form $s_j = \mu_j \otimes \omega_j$ $(j = 1, 2, 3)$, where $\mu_j \in \Gamma(E_j)$ and $\omega_1 \in \Omega^r(M)$, $\omega_2 \in \Omega^s(M)$, $\omega_3 \in \Omega^t(M)$. But in that case we have

$$\begin{aligned} s_1 \wedge (s_2 \wedge s_3) &= (\mu_1 \otimes \omega_1) \wedge \Big((\mu_2 \otimes \omega_2) \wedge (\mu_3 \otimes \omega_3)\Big) \\ &= (\mu_1 \otimes \omega_1) \wedge \Big((\mu_2 \otimes \mu_3) \otimes (\omega_2 \wedge \omega_3)\Big) \\ &= (\mu_1 \otimes \mu_2 \otimes \mu_3) \otimes (\omega_1 \wedge (\omega_2 \wedge \omega_3)) \\ &= (\mu_1 \otimes \mu_2 \otimes \mu_3) \otimes ((\omega_1 \wedge \omega_2) \wedge \omega_3) \\ &= \cdots \\ &= (s_1 \wedge s_2) \wedge s_3. \end{aligned}$$

In the fourth equality we used the fact that the wedge product on $\Omega(M)$ is associative.

Similarly for any s_1, s_2 as above we have

$$\begin{aligned} (-1)^{rs} \cdot J(s_2 \wedge s_1) &= (-1)^{rs} \cdot J((\mu_2 \otimes \mu_1) \otimes (\omega_2 \wedge \omega_1)) \\ &= (-1)^{rs} \cdot (\mu_1 \otimes \mu_2) \otimes (\omega_2 \wedge \omega_1) \\ &= (\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2) \\ &= s_1 \wedge s_2. \end{aligned}$$

In the third equality we used the fact that $\omega_1 \wedge \omega_2 = (-1)^{rs} \omega_2 \wedge \omega_1$. □

(c). This is immediate from the second relation in part (b); indeed note that $m' = m \circ J : \Omega^{r+s}(E_2 \otimes E_1) \to \Omega^{r+s}(\widetilde{E})$.

[32] Note that we are now using "$\wedge$" in quite a few different ways; however it will always be clear from the type of the two arguments which "$\wedge$" is used in each instance.

(d). Similarly, this is fairly immediate from the first relation in part (b); we leave out a detailed discussion. (One has to change order between certain operations and this becomes somewhat tedious to spell out.) Instead we give a direct proof by *mimicking* the proof of the first relation in part (b): For any s_1, s_2, s_3 of the form $s_j = \mu_j \otimes \omega_j$ $(j = 1, 2, 3)$, where $\mu_j \in \Gamma(E_j)$ and $\omega_1 \in \Omega^r(M)$, $\omega_2 \in \Omega^s(M)$, $\omega_3 \in \Omega^t(M)$, we have (note carefully that now certain "$\wedge$" have a different meaning than in part (b)...):

$$\begin{aligned}
s_1 \wedge (s_2 \wedge s_3) &= (\mu_1 \otimes \omega_1) \wedge \Big((\mu_2 \otimes \omega_2) \wedge (\mu_3 \otimes \omega_3) \Big) \\
&= (\mu_1 \otimes \omega_1) \wedge \Big((\mu_2 \cdot \mu_3) \otimes (\omega_2 \wedge \omega_3) \Big) \\
&= (\mu_1 \cdot (\mu_2 \cdot \mu_3)) \otimes (\omega_1 \wedge (\omega_2 \wedge \omega_3)) \\
&= ((\mu_1 \cdot \mu_2) \cdot \mu_3) \otimes ((\omega_1 \wedge \omega_2) \wedge \omega_3) \\
&= \cdots \\
&= (s_1 \wedge s_2) \wedge s_3.
\end{aligned}$$

Done! $\square$

Problem 50: We have

$$(\mu \circ \eta)_{|U} = \Big(\beta^{j*} \otimes \gamma_k \otimes \mu_j^k\Big) \circ \Big(\alpha^{i*} \otimes \beta_\ell \otimes \eta_i^\ell\Big).$$

and by the definitions in Problem 49(a),(c), this is

$$= \Big(\big(\beta^{j*} \otimes \gamma_k\big) \circ \big(\alpha^{i*} \otimes \beta_\ell\big)\Big) \otimes \big(\mu_j^k \wedge \eta_i^\ell\big).$$

But using the definitions of composition of homomorphisms and of the identifications $\text{Hom}(E_2, E_3) = E_2^* \otimes E_3$ and $\text{Hom}(E_1, E_2) = E_1^* \otimes E_2$, we find that

$$\big(\beta^{j*} \otimes \gamma_k\big) \circ \big(\alpha^{i*} \otimes \beta_\ell\big) = \delta_{j,\ell} \alpha^{i*} \otimes \gamma_k.$$

(Here $\delta_{j,\ell}$ is the standard Kronecker symbol; $\delta_{j,\ell} = 1$ if $j = \ell$ otherwise $= 0$. Note that in terms of matrices the last formula is simply a formula for the product of a matrix with "1" in position k, j and all other entries zero, and a matrix with "1" in position ℓ, i and all other entries zero.) Using the last formula in the previous one, we obtain:

$$\begin{aligned}(\mu \circ \eta)_{|U} &= \big(\delta_{j,\ell} \alpha^{i*} \otimes \gamma_k\big) \otimes \big(\mu_j^k \wedge \eta_i^\ell\big) \\ &= \big(\alpha^{i*} \otimes \gamma_k\big) \otimes \big(\mu_j^k \wedge \eta_i^j\big),\end{aligned}$$

qed. □

Problem 51:

(a). To start, by the definition of tensor product, there is a natural bijection between the $C^\infty(M)$-multilinear maps $\Gamma(TM)^{(r)} \to \Gamma E$ and the $C^\infty(M)$-linear maps from

$$\underbrace{\Gamma(TM) \otimes \cdots \otimes \Gamma(TM)}_{r \text{ times}} = \Gamma(\underbrace{TM \otimes \cdots \otimes TM}_{r \text{ times}}) = \Gamma(T_0^r M)$$

to ΓE (here we used Problem 43(d)). But the space of such maps is

$$\mathrm{Hom}(\Gamma(T_0^r M), \Gamma E) = \Gamma \mathrm{Hom}(T_0^r M, E) = \Gamma(E \otimes (T_0^r M)^*) = \Gamma(E \otimes T_r^0 M).$$

(For the first equality see Problem 43(c); for the next equality cf. p. 9 in Lecture #7.) Working through the identifications used above, we see that viewing a given $s \in \Gamma(E \otimes T_r^0 M)$ as a $C^\infty(M)$-multilinear map $\Gamma(TM)^{(r)} \to \Gamma E$, means that for any vector fields $X_1, \ldots, X_r \in \Gamma(TM)$, the section

$$s(X_1, \ldots, X_r) \in \Gamma E$$

is explicitly given by

(138)
$$\big(s(X_1, \ldots, X_r)\big)(p) = C_p\big(s(p) \otimes (X_1(p) \otimes \cdots \otimes X_r(p))\big) \qquad (\forall p \in M),$$

where C_p is the unique $\mathbb{R}$-linear map ("contraction at p")

$$C_p : \big(E_p \otimes T_r^0(M)_p\big) \otimes T_0^r(M)_p \to E_p$$

which maps

(139) $$C_p(w \otimes \eta \otimes \alpha) = \eta(\alpha) \cdot w, \qquad \forall w \in E_p,\ \eta \in T_r^0(M)_p,\ \alpha \in T_0^r(M)_p.$$

Next, note that $\bigwedge^r M$ by definition is a subset of $T_r^0(M)$, and one verifies immediately that it is in fact a subbundle of $T_r^0(M)$. [33] Hence $E \otimes \bigwedge^r M$ is a subbundle of $E \otimes T_r^0(M)$. [34]

In order to complete the solution, we have to prove that for any $s \in \Gamma(E \otimes T_r^0(M))$, we have $s \in \Gamma(E \otimes \bigwedge^r M)$ iff s as a $C^\infty(M)$-multilinear map $\Gamma(TM)^{(r)} \to \Gamma E$ is *alternating.* Now $s \in \Gamma(E \otimes \bigwedge^r M)$ is equivalent with $s(p) \in E_p \otimes \bigwedge^r(T_p^* M)$, $\forall p \in M$, and on the other hand $s : \Gamma(TM)^{(r)} \to \Gamma E$ is alternating iff

$$s(X_{\sigma(1)}, \ldots, X_{\sigma(r)})(p) = (\operatorname{sgn} \sigma) \cdot s(X_1, \ldots, X_r)(p),$$
$$\forall \sigma \in \mathfrak{S}_r,\ X_1, \ldots, X_r \in \Gamma(TM),\ p \in M.$$

[33] Indeed, using standard bundle charts, this boils down to the local fact that $U \times \bigwedge^r((\mathbb{R}^n)^*)$ is a subbundle of $U \times (\mathbb{R}^n \otimes \cdots \otimes \mathbb{R}^n)^*$ (for U open $\subset M$).

[34] General fact (complement to Problems 39(a) and 41): If E_1, E_2, F are vector bundles over M and E_1 is a subbundle of E_2, then $E_1 \otimes F$ is a subbundle of $E_2 \otimes F$.

Hence, by also using (138) and Problem 35(c) (applied to the vector bundle TM) we see that it suffices to prove the following: *For every $p \in M$ and every $z \in E_p \otimes T_r^0(M)_p$, we have $z \in E_p \otimes \bigwedge^r(T_p^*M)$ iff*

$$C_p(z \otimes (v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)})) = (\operatorname{sgn}\sigma) \cdot C_p(z \otimes (v_1 \otimes \cdots \otimes v_r)), \qquad \forall \sigma \in \mathfrak{S}_r,\ v_1, \ldots, v_r \in T_pM. \tag{140}$$

Proof of the last claim: Let $b_1, \ldots, b_n$ be a basis of E_p. Then every $z \in E_p \otimes T_r^0(M)_p$ can be expressed as $z = \sum_{j=1}^n b_j \otimes \eta_j$ with uniquely determined $\eta_1, \ldots, \eta_n \in T_r^0(M)_p$, and we have $z \in E_p \otimes \bigwedge^r(T_p^*M)$ iff $\eta_j \in \bigwedge^r(T_p^*M)$ for all j. It now follows from the definition of C_p, (139), that

$$C_p(z \otimes \alpha) = \sum_{j=1}^n \eta_j(\alpha) \cdot b_j, \qquad \forall \alpha \in T_0^r(M)_p.$$

Hence since $b_1, \ldots, b_n$ is a basis of E_p, we see that (140) holds iff

$$\eta_j(v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(r)}) = \eta_j(v_1 \otimes \cdots \otimes v_r), \qquad \forall j \in \{1, \ldots, n\},\ \sigma \in \mathfrak{S}_r,\ v_1, \ldots, v_r \in T_pM.$$

In other words, (140) is equivalent with $\eta_j \in \bigwedge^r(T_p^*M)$ for all j, and as we have already pointed out this is equivalent with $z \in E_p \otimes \bigwedge^r(T_p^*M)$. Done!

□

Addendum: Note that for any $s \in \Omega^r(E)$ of the form $s = \mu \otimes \omega$ with $\mu \in \Gamma(E)$ and $\omega \in \Omega^r(M)$ (and more generally for any $s \in \Gamma(E \otimes T_r^0 M)$ of the form $s = \mu \otimes \omega$ with $\mu \in \Gamma(E)$ and $\omega \in \Gamma(T_r^0 M)$), the corresponding multilinear map satisfies

$$(\mu \otimes \omega)(X_1, \ldots, X_r) = \omega(X_1, \ldots, X_r) \cdot \mu \qquad (\text{in } \Gamma E), \qquad \forall X_1, \ldots, X_r \in \Gamma(TM). \tag{141}$$

Indeed, this is clear from (138) and (139). □

(b). The expressions on both sides in the given formula clearly depends $C^\infty(M)$-linearly on s_1 and $C^\infty(M)$-linearly on s_2. Hence, by the argument at the beginning of the solution to Problem 49(a), it suffices to prove the stated formula when $s_j = \mu_j \otimes \omega_j$ $(j = 1, 2)$, with $\mu_j \in \Gamma(E_j)$ and $\omega_1 \in \Omega^r(M)$, $\omega_2 \in \Omega^s(M)$. In this case we get by (141):

$$\frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma)\, s_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \otimes s_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)})$$

$$= \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma) \Big(\omega_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \cdot \mu_1\Big) \otimes \Big(\omega_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}) \cdot \mu_2\Big),$$

$$= \frac{1}{r!s!} \bigg(\sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma)\, \omega_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \cdot \omega_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}) \bigg) \cdot \mu_1 \otimes \mu_2.$$

By the definition of exterior product on $\Omega(M)$, this is

$$= (\omega_1 \wedge \omega_2)(X_1, \ldots, X_{r+s}) \cdot (\mu_1 \otimes \mu_2).$$

But we have $s_1 \wedge s_2 = (\mu_1 \otimes \mu_2) \otimes (\omega_1 \wedge \omega_2)$, and so by one more application of (141), the above is

$$= (s_1 \wedge s_2)(X_1, \ldots, X_{r+s}).$$

Done! □

Finally, the analogous formula for $s_1 \cdot s_2$ is:

$$(s_1 \cdot s_2)(X_1, \ldots, X_{r+s})$$
$$= \frac{1}{r!s!} \sum_{\sigma \in \mathfrak{S}_{r+s}} \operatorname{sgn}(\sigma)\, s_1(X_{\sigma(1)}, \ldots, X_{\sigma(r)}) \cdot s_2(X_{\sigma(r+1)}, \ldots, X_{\sigma(r+s)}),$$
$$\forall X_1, \ldots, X_{r+s} \in \Gamma(TM),$$

Problem 52:

(a). This was proved in the lecture.

(b). We first prove that the connection $D_{|U}$, if it exists, is unique. Let $s \in \Gamma(E_{|U})$ be given. We claim that if V is any open subset of U and $s' \in \Gamma(E)$ is such that $s'_{|V} = s_{|V}$, then

$$(D_{|U}s)_{|V} = (Ds')_{|V}. \tag{142}$$

By Problem 35(a), U can be covered by open sets V for which such a section $s' \in \Gamma(E)$ exists; hence (142) implies that the whole section $D_{|U}s$ is uniquely determined.

[Proof of (142): It follows from the requirement on $D_{|U}$ that

$$D_{|U}(s'_{|U}) = (Ds')_{|U} \tag{143}$$

Furthermore, applying part (a) to $D_{|U}$, and using $(s'_{|U})_{|V} = s'_{|V} = s_{|V}$, we have

$$(D_{|U}(s'_{|U}))_{|V} = (D_{|U}s)_{|V}. \tag{144}$$

Combining (143) and (144) we get (142).]

Next let us verify that there indeed exists a well-defined section "$D_{|U}s$" in $\Gamma(E_{|U} \otimes T^*U)$ such that (142) holds whenever V is open in U and $s' \in \Gamma(E)$ is such that $s'_{|V} = s_{|V}$. For this, it suffices to verify that for any two open subset $V_1, V_2 \subset U$ and any $s'_1, s'_2 \in \Gamma(E)$, if $s'_{j|V_j} = s_{|V_j}$ for $j = 1, 2$ then $(Ds'_1)_{|V_1 \cap V_2} = (Ds'_2)_{|V_1 \cap V_2}$. However this is clear from $(s'_1)_{|V_1 \cap V_2} = s_{|V_1 \cap V_2} = (s'_2)_{|V_1 \cap V_2}$, and part (a) of the lemma (i.e. the fact that D is local).

Hence we have shown that our requirements on $D_{|U}$ imply that $D_{|U}$ is a uniquely defined map from $\Gamma(E_{|U})$ to $\Gamma(E_{|U} \otimes T^*U)$. Note that it is immediate from our construction that this map has the desired property, i.e. that $(Ds)_{|U} = D_{|U}(s_{|U})$ for all $s \in \Gamma(E)$. (Indeed, let $s \in \Gamma(E)$ be given. Then for the section $s_{|U} \in \Gamma(E_{|U})$, (142) applies with $s' = s$ and $V = U$, and then says that $(D_{|U}(s_{|U}))_{|U} = (Ds)_{|U}$, as desired.)

It remains to prove that $D_{|U}$ is a connection on $E_{|U}$. Clearly $D_{|U}$ is $\mathbb{R}$-linear, and so it remains to prove that

$$D_{|U}(fs) = s \otimes df + fD_{|U}s, \qquad \forall f \in C^\infty(U),\ s \in \Gamma(E_{|U}). \tag{145}$$

For this let $f \in C^\infty(U)$ and $s \in \Gamma(E_{|U})$ be given. It suffices to prove that for any given point $p \in U$, there is an open subset $V \subset U$ with $p \in V$ such that

$$(D_{|U}(fs))_{|V} = (s \otimes df + fD_{|U}s)_{|V}.$$

However, given $p \in U$, we know by Problem 35(a) (applied to the two vector bundles E and $M \times \mathbb{R}$) that there exist $f' \in C^\infty(M)$ and $s' \in \Gamma(E)$ such that $f'_{|V} = f_{|V}$ and $s'_{|V} = s_{|V}$ for some open set $V \subset U$ containing p. Then also $(f's')_{|V} = (fs)_{|V}$, and now

$$(D_{|U}(fs))_{|V} = (D(f's'))_{|V} = (s' \otimes df' + f'Ds')_{|V} = s'_{|V} \otimes d(f'_{|V}) + f'_{|V}(Ds')_{|V}$$
$$= s_{|V} \otimes d(f_{|V}) + f_{|V}(D_{|U}s)_{|V} = (s \otimes df + fD_{|U}s)_{|V},$$

as desired! Hence we have proved (145), and so $D_{|U}$ *is a connection on* $E_{|U}$. □

(c) The requirement on D is that

(146) $$(Ds)_{|U_\alpha} = D_\alpha(s_{|U_\alpha}), \qquad \forall s \in \Gamma(E),\ \alpha \in A.$$

Let $s \in \Gamma(E)$ be given. Since $\cup_{\alpha\in A} U_\alpha = M$, the condition (146) determines Ds uniquely (if it exists at all). To prove that there really exists a section $Ds \in \Gamma(E \otimes T^*M)$ which satisfies (146), it suffices to verify that for any two $\alpha, \beta \in A$ with $V := U_\alpha \cap U_\beta \neq \emptyset$, we have $(D_\alpha(s_{|U_\alpha}))_V = (D_\beta(s_{|U_\beta}))_V$. However this is immediate from our assumption that $(D_\alpha)_{|V} = (D_\beta)_{|V}$.

Hence we have shown that our requirement on D implies that D is a uniquely defined map from $\Gamma(E)$ to $\Gamma(E \otimes T^*M)$. Clearly this map is $\mathbb{R}$-linear. In order to prove that D is a connection, it remains to verify that $D(fs) = s \otimes df + fDs$ for all $f \in C^\infty(M)$, $s \in \Gamma(E)$. Thus let $f \in C^\infty(M)$ and $s \in \Gamma(E)$ be given. Since $\cup_{\alpha\in A} U_\alpha = M$, it suffices to prove that $(D(fs))_{|U_\alpha} = (s \otimes df + fDs)_{|U_\alpha}$ for every $\alpha \in A$. Thus let $\alpha \in A$ be given. Then

$$(D(fs))_{|U_\alpha} = D_\alpha((fs)_{|U_\alpha}) = s_{|U_\alpha} \otimes df_{|U_\alpha} + f_{|U_\alpha} D_\alpha(s_{|U_\alpha})$$
$$= (s \otimes df + fDs)_{|U_\alpha},$$

as desired! □

Remark 1. The following (apriori) *stronger* version of part (b) is actually technically slightly more *direct* to prove:

Let $(U_\alpha)_{\alpha\in A}$ be an open covering of M, and for each $\alpha \in A$ let D_α be a connection on $E_{|U_\alpha}$. Assume that $(D_\alpha(s_{|U_\alpha}))_{|U_\alpha\cap U_\beta} = (D_\beta(s_{|U_\beta}))_{|U_\alpha\cap U_\beta}$ for all $s \in \Gamma E$ and all $\alpha, \beta \in A$. Then there exists a unique connection D on E such that $(Ds)_{|U_\alpha} = D_\alpha(s_{|U_\alpha})$ for all $s \in \Gamma E$ and $\alpha \in A$.

Proof. Given any $s \in \Gamma E$, the section $Ds \in \Gamma(T^*M \otimes E)$ is clearly uniquely determined (if it exists) by the given requirement, since $M = \cup_{\alpha\in A} U_\alpha$. On the other hand the given compatibility assumption easily implies that Ds is indeed as well-defined (C^∞!) section of $T^*M \otimes E$! Hence we obtain a well-defined map from $\Gamma(E)$ to $\Gamma(T^*M \otimes E)$. The rest is as above! □

Problem 53: Let $p \in M$ be the base point of v, so that $v \in T_pM$. Take an open neighborhood $U \subset M$ of p such that both $TM_{|U} = TU$ and $E_{|U}$ are trivial, and choose bases of sections $X_1, \ldots, X_d \in \Gamma(TU)$ and $\sigma_1, \ldots, \sigma_n \in \Gamma(E_{|U})$. Let $\Gamma_{ij}^k \in C^\infty(U)$ be the corresponding Christoffel symbols, so that $D_{X_i}\sigma_j = \Gamma_{ij}^k \sigma_k$ for all $i \in \{1, \ldots, d\}$ and $j \in \{1, \ldots, n\}$. Take $a^1, \ldots, a^n \in C^\infty(U)$ and $b^1, \ldots, b^n \in C^\infty(U)$ so that

$$s_{1|U} = a^j \sigma_j \qquad \text{and} \qquad s_{2|U} = b^j \sigma_j$$

(cf. Problem 34). Also take $\gamma^1, \ldots, \gamma^n \in \mathbb{R}$ so that

$$v = \gamma^i \cdot X_i(p) \in T_pM.$$

Then as shown in Lecture #9, we have

$$D_v(s_1) = v(a^k) \cdot \sigma_k(p) + \gamma^j \cdot a^k(p) \cdot \Gamma_{jk}^\ell(p) \cdot \sigma_\ell(p). \tag{147}$$

and

$$D_v(s_2) = v(b^k) \cdot \sigma_k(p) + \gamma^j \cdot b^k(p) \cdot \Gamma_{jk}^\ell(p) \cdot \sigma_\ell(p). \tag{148}$$

[Details: In Lecture #9 we noted a formula saying that if $X = \gamma^i X_i \in \Gamma(TU)$ then $D_X(s_1) = X(a^k) \cdot \sigma_k + \gamma^j a^k \Gamma_{jk}^\ell \sigma_\ell$ in $\Gamma(E_{|U})$, and evaluating this section at p we get (147). But of course we don't need to refer to Lecture #9; the proof of (147) is immediate using Leibniz' rule: We have

$$D_v(s_1) = D_v(a^k \sigma_k) = v(a^k) \cdot \sigma_k(p) + a^k(p) \cdot D_v(\sigma_k),$$

and here $D_v(\sigma_k) = \gamma^i D_{X_i(p)}(\sigma_k) = \gamma^i \big(D_{X_i}(\sigma_k)\big)(p) = \gamma^i \Gamma_{ik}^\ell(p) \sigma_\ell(p)$, and combining these two we get (147). The proof of (148) is the same.]

Now we are assuming $v = \dot{c}(0)$; hence (by "fact" on p. 9 in Lecture #2; cf. Problem 13(f)):

$$v(a^j) = (a^j \circ c)'(0) \quad \text{and} \quad v(b^j) = (b^j \circ c)'(0).$$

We are also assuming that for every $t \in (-\varepsilon, \varepsilon)$ we have $s_1(c(t)) = s_2(c(t))$, i.e.

$$a^j(c(t)) \cdot \sigma_j(c(t)) = b^j(c(t)) \cdot \sigma_j(c(t)) \qquad \text{in } E_{c(t)},$$

and thus

$$a^j(c(t)) = b^j(c(t)), \qquad \forall j \in \{1, \ldots, n\},\ t \in (-\varepsilon, \varepsilon).$$

Combining the above facts, we conclude that

$$v(a^j) = v(b^j), \qquad \forall j \in \{1, \ldots, n\}.$$

Also of course $a^j(p) = a^j(c(0)) = b^j(c(0)) = b^j(p)$. Hence we see by inspection in (147) and (148) that $D_v(s_1) = D_v(s_2)$. $\square$

Problem 54: The map d is clearly $\mathbb{R}$-linear. Furthermore, for any $f \in C^\infty U$,

$$\begin{aligned} d(fa^k s_k) = s_k \otimes d(fa^k) &= s_k \otimes (a^k \cdot df + f \cdot da^k) \\ &= (a^k s_k) \otimes df + f \cdot s_k \otimes da^k \\ &= (a^k s_k) \otimes df + f \cdot d(a^k s_k). \end{aligned}$$

Hence d is indeed a connection. □

Problem 55: The first statement is an immediate verification; indeed the restricted map is, by assumption, a map $D : \Gamma E' \to \Gamma(E' \otimes T^*M)$, and it satisfies $D(fs) = f \cdot Ds + s \otimes df$ for all $f \in C^\infty(M)$, $s \in \Gamma E'$, since this holds more generally when $s \in \Gamma E$.

Here's a simple example showing that the condition is not always satisfied: Let E be the trivial vector bundle $E = M \times \mathbb{R}^2$ over $M = \mathbb{R}$, equipped with the corresponding 'trivial' connection D (i.e. the connection which Jost would call "d" with respect to the bundle chart $\varphi = 1_E : E \to M \times \mathbb{R}^2$). Set

$$E' = \{(x, v) \in E \,:\, v \in \mathbb{R}(\cos x, \sin x)\}.$$

This is easily verified to be a vector subbundle of E. Now consider the section $s \in \Gamma E'$, $s(x) := (\cos x, \sin x)$. Then $Ds(x) = (-\sin x, \cos x)$ in $\Gamma(E \otimes T^*M) = \Gamma E$. (Note: For our $M = \mathbb{R}$ we have $T^*M = M \times \mathbb{R}$ and thus $E \otimes T^*M = E$ under obvious identifications.) Hence Ds is *not* in $\Gamma(E' \otimes T^*M) = \Gamma(E')$; indeed $Ds(x) \notin E'$ for *every* $x \in M$. □

Problem 57: The requirement on f^*D is:

$$(f^*D)(s \circ f) = D_{df(\cdot)}(s) \in \Gamma(\mathrm{Hom}(TM, f^*E)), \qquad \forall s \in \Gamma E \tag{149}$$

Let us first verify that this formula makes sense! For any $s \in \Gamma E$ we have $s \circ f \in \Gamma_f E = \Gamma f^*E$ (cf. Problem 44); hence if f^*D is a connection on f^*E then we should indeed have

$$(f^*D)(s \circ f) \in \Gamma(f^*E \otimes T^*M) = \Gamma(\mathrm{Hom}(TM, f^*E)).$$

Next let us prove that $D_{df(\cdot)}(s)$, i.e. the map $v \mapsto D_{df(v)}(s)$, is indeed a bundle homomorphism $TM \to E$ along f, so that it can be viewed as an element of $\Gamma(\mathrm{Hom}(TM, f^*E))$ by Problem 45. First of all, df is a C^∞ function $TM \to TN$ (cf. Problem 17(a)); also $D(s) \in \Gamma(\mathrm{Hom}(TN, E))$ and so $D(s)$ can be viewed as a bundle homomorphism $TN \to E$ (cf. Problem 40). Now $D_{df(\cdot)}(s)$ is the composition of these two C^∞ maps, hence itself C^∞. Also $D_{df(\cdot)}(s)$ clearly restricts to an $\mathbb{R}$-linear map $T_pM \to E_{f(p)}$ for all $p \in M$. Hence $D_{df(\cdot)}(s)$ is indeed a bundle homomorphism $TM \to E$ along f. Done!

By Problem 44(c), every section in Γf^*E be written as a finite $C^\infty(M)$-linear combination of sections of the form $s \circ f$ with $s \in \Gamma E$. Hence the connection f^*D, if it exists at all, is *uniquely determined* by the relation (149).

It remains to prove that there exists such a connection f^*D. Let us first prove the existence of f^*D in the special case when (E, π, N) is a *trivial* vector bundle. Then fix a basis of sections $s_1, \ldots, s_n \in \Gamma E$ (cf. Problem 33). Then for any $\sigma \in \Gamma f^*E$ there exist unique 'coefficient functions' $\alpha_1, \ldots, \alpha_n \in C^\infty(M)$ such that

$$\sigma = \sum_{j=1}^{n} \alpha_j \cdot (s_j \circ f)$$

(cf. Problem 44(b) and Problem 34), and we now define the map

$$\widetilde{D} : \Gamma f^*E \to \Gamma(f^*E \otimes T^*M)$$

by setting

$$\widetilde{D}(\sigma) := \sum_{j=1}^{n} \Big(\alpha_j \cdot D_{df(\cdot)}(s_j) + (s_j \circ f) \otimes d\alpha_j \Big). \tag{150}$$

(Here $D_{df(\cdot)}(s_j) \in \Gamma(\text{Hom}(TM, f^*E)) = \Gamma(f^*E \otimes T^*M)$ as in (149).) This map $\widetilde{D}$ is clearly well-defined and $\mathbb{R}$-linear. Furthermore, for any $g \in C^\infty(M)$ and $\sigma = \sum_{j=1}^{n} \alpha_j \cdot (s_j \circ f)$ as above, we have

$$g \cdot \sigma = \sum_{j=1}^{n} (g \cdot \alpha_j) \cdot (s_j \circ f),$$

and hence by definition:

$$\begin{aligned}
\widetilde{D}(g \cdot \sigma) &= \sum_{j=1}^{n} \Big(g\alpha_j \cdot D_{df(\cdot)}(s_j) + (s_j \circ f) \otimes d(g\alpha_j) \Big) \\
&\Big\{ \text{Use } d(g\alpha_j) = g \cdot d\alpha_j + \alpha_j \cdot dg. \Big\} \\
&= g \cdot \sum_{j=1}^{n} \Big(\alpha_j \cdot D_{df(\cdot)}(s_j) + (s_j \circ f) \otimes d\alpha_j \Big) + \sum_{j=1}^{n} \alpha_j (s_j \circ f) \otimes dg \\
&= g \cdot \widetilde{D}(\sigma) + \sigma \otimes dg,
\end{aligned}$$

This proves that $\widetilde{D}$ is a connection! Finally we prove that $\widetilde{D}$ satisfies (149). Thus let $s \in \Gamma E$ be given. Then there exist unique $\beta_1, \ldots, \beta_n \in C^\infty(N)$ such that $s = \sum_{j=1}^{n} \beta_j s_j$ (cf. Problem 34); hence $s \circ f = \sum_{j=1}^{n} (\beta_j \circ f) \cdot (s_j \circ f)$; and so by our definition of $\widetilde{D}$,

$$\widetilde{D}(s \circ f) = \sum_{j=1}^{n} \Big((\beta_j \circ f) \cdot D_{df(\cdot)}(s_j) + (s_j \circ f) \otimes d(\beta_j \circ f) \Big). \tag{151}$$

On the other hand for any $p \in M$ and $v \in T_pM$ we have

$$D_{df(v)}(s) = D_{df(v)}\Big(\sum_{j=1}^{n} \beta_j s_j\Big)$$
$$= \sum_{j=1}^{n}\Big(\beta_j(f(p)) \cdot D_{df(v)}(s_j) + [df_p(v)](\beta_j) \cdot s_j(f(p))\Big).$$

Here

$$[df_p(v)](\beta_j) = (d\beta_j)_{f(p)}(df_p(v)) = [d(\beta_j \circ f)](v) \in \mathbb{R}$$

(by the chain rule), and hence by comparing the last two formulas we see that $\widetilde{D}(s \circ f) = D_{df(\cdot)}(s)$. Hence we have proved that the connection $\widetilde{D}$ satisfies the relation required for f^*D, (149). This proves that f^*D exists, namely $f^*D = \widetilde{D}$!

We have thus proved that the pullback of any connection on a *trivial* vector bundle exists; and it is unique since we have noted that the pullback of *any* connection is unique if it exists.

Finally we will prove that the pullback connection f^*D exists for an *arbitrary* vector bundle (E, π, N). Fix an open covering $(V_\alpha)_{\alpha \in A}$ of N such that $E_{|V_\alpha}$ is trivial for each $\alpha \in A$. Let $U_\alpha = f^{-1}(V_\alpha)$; then $(U_\alpha)_{\alpha\in A}$ is an open covering of M. Now $f_{|U_\alpha}$ is a C^∞ map of manifolds $U_\alpha \to V_\alpha$, and $D_{|V_\alpha}$ is a connection on the trivial vector bundle $E_{|V_\alpha}$ (cf. Problem 52); hence by what we have proved above, there exists a uniquely defined pullback connection $\widetilde{D}_\alpha := f^*_{|U_\alpha}(D_{|V_\alpha})$ on $f^*_{|U_\alpha}(E_{|V_\alpha}) = (f^*E)_{|U_\alpha}$. [35] Let us prove that these connections $\widetilde{D}_\alpha$ ($\alpha \in A$) are compatible in the appropriate sense. Thus let $\alpha, \beta \in A$ and set $U' := U_\alpha \cap U_\beta$; assume $U' \neq \emptyset$. Note that $U' = f^{-1}(V')$ where $V' := V_\alpha \cap V_\beta$. We claim that

$$\widetilde{D}_{\alpha|U'}(s' \circ f_{|U'}) = D_{df_{|U'}(\cdot)}(s'), \qquad \forall s' \in \Gamma E_{|V'}. \tag{152}$$

(Here in the right hand side, "D" really stands for "$D_{|U'}$"; we will use this type of mild abuse of notation several times in the discussion below.) To prove (152), let $s' \in \Gamma E_{|V'}$ be given, and take $p \in U'$. By Problem 35(a), there is a section $s \in \Gamma E_{|V_\alpha}$ such that $s_{|V''} = s'_{|V''}$ for some open set $V'' \subset V'$ containing $f(p)$. Then $s \circ f_{|U_\alpha}$ and $s' \circ f_{|U'}$ have the same restrictions to $U'' := f^{-1}(V'')$, i.e. $(s \circ f_{|U_\alpha})_{|U''} = (s' \circ f_{|U'})_{|U''}$. Note also that $p \in U''$.

[35] Prove this identification, "$f^*_{|U_\alpha}(E_{|V_\alpha}) = (f^*E)_{|U_\alpha}$", as a complement to Problem 42.

Now we get:

$$
\begin{aligned}
&\widetilde{D}_{\alpha|U'}(s' \circ f_{|U'})_{|U''} \\
&\quad =(\widetilde{D}_\alpha(s \circ f_{|U_\alpha}))_{|U''} \quad \left\{ \begin{array}{l} \text{since } (s \circ f_{|U_\alpha})_{|U''} = (s' \circ f_{|U'})_{|U''}\text{; cf. the} \\ \text{solution to Problem 52(b).} \end{array} \right\} \\
&\quad =(D_{df_{|U_\alpha}(\cdot)}(s))_{|U''} \quad \left\{ \text{by the defining condition for } \widetilde{D}_\alpha. \right\} \\
&\quad =(D_{df_{|U'}(\cdot)}(s'))_{|U''} \quad \left\{ \begin{array}{l} \text{indeed for every } v \in T(U'') \text{ we have } df(v) \in \\ T(V'') \text{ and thus } D_{df(v)}(s) = D_{df(v)}(s')\text{, since} \\ D(s)_{|V''} = D(s')_{|V''} \text{ (cf. the solution to Prob-} \\ \text{lem 52(b)).} \end{array} \right\}.
\end{aligned}
$$

Since every $p \in U'$ has such a neighborhood U'', it follows that (152) holds!

But (152) says exactly that $\widetilde{D}_{\alpha|U'}$ is the $f_{|U'}$-pullback of $D_{|V'}$ (which we know is unique if it exists). Changing the roles of α and β, it also follows that $\widetilde{D}_{\beta|U'}$ is the $f_{|U'}$-pullback of $D_{|V'}$. Hence by the uniqueness of "the $f_{|U'}$-pullback of $D_{|V'}$", we conclude:

$$
\widetilde{D}_{\alpha|U'} = \widetilde{D}_{\beta|U'}. \tag{153}
$$

The fact that (153) holds for all $\alpha, \beta \in A$ with $U_\alpha \cap U_\beta \neq \emptyset$ implies by Problem 52(c) that there exists a unique connection $\widetilde{D}$ on f^*E satisfying $\widetilde{D}_{|U_\alpha} = \widetilde{D}_\alpha$ for all $\alpha \in A$. One easily proves that

$$
\widetilde{D}(s \circ f) = D_{df(\cdot)}(s), \qquad \forall s \in \Gamma E. \tag{154}
$$

(Indeed, let $s \in \Gamma E$ be given. Then for every $\alpha \in A$ we have $(\widetilde{D}(s \circ f))_{|U_\alpha} = \widetilde{D}_\alpha((s \circ f)_{|U_\alpha}) = \widetilde{D}_\alpha(s_{|V_\alpha} \circ f_{|U_\alpha}) = D_{df_{|U_\alpha}(\cdot)}(s_{|V_\alpha}) = D_{df(\cdot)}(s)_{|U_\alpha}$, where each step is justified by arguments similar to those in the proof of (152). The fact that $(\widetilde{D}(s \circ f))_{|U_\alpha} = D_{df(\cdot)}(s)_{|U_\alpha}$ for each $\alpha \in A$ implies that $\widetilde{D}(s \circ f) = D_{df(\cdot)}(s)$. Done!)

The relation (154) says exactly that the connection $\widetilde{D}$ satisfies the requirements on the pullback bundle f^*D; hence we have proved that f^*D exists! □

(b). Let us first prove that the connection f^*D defined in part (a) indeed satisfies the stated condition. Thus let $s \in \Gamma f^*E = \Gamma_f E$ and let $c : (-\varepsilon, \varepsilon) \to M$ be a C^∞ curve; also let $s_1 \in \Gamma E$, and assume that $s_1(f(c(t))) = s(c(t))$ for all $t \in (-\varepsilon, \varepsilon)$. The assumption means that the two sections $s_1 \circ f$ and s in Γf^*E are equal along the curve c. Hence by Problem 53,

$$
(f^*D)_{\dot{c}(0)}(s) = (f^*D)_{\dot{c}(0)}(s_1 \circ f).
$$

But also, by the defining property of f^*D,

$$(f^*D)_{\dot{c}(0)}(s_1 \circ f) = D_{df(\dot{c}(0))}(s_1).$$

Hence $(f^*D)_{\dot{c}(0)}(s) = D_{df(\dot{c}(0))}(s_1)$, and so we have proved that f^*D satisfies the desired condition.

Next let us prove that f^*D is the *only* connection on f^*E which satisfies the stated condition. Thus let ∇ be *any* connection on f^*E such that for any $s \in \Gamma f^*E = \Gamma_f E$, any $s_1 \in \Gamma E$, and any curve $c : (-\varepsilon, \varepsilon) \to M$, if $s_1(f(c(t))) = s(c(t))$ $(\forall t \in (-\varepsilon, \varepsilon))$, then $\nabla_{\dot{c}(0)}(s) = D_{df(\dot{c}(0))}(s_1)$.

Consider an arbitrary section $s_1 \in \Gamma E$ and an arbitrary $v \in TM$. Then there is a C^∞ curve $c : (-\varepsilon, \varepsilon) \to M$ such that $v = \dot{c}(0)$. Note that $s_1 \circ f \in \Gamma f^*E$ and *obviously* $s_1(f(c(t))) = (s_1 \circ f)(c(t))$ for all $t \in (-\varepsilon, \varepsilon)$. Hence by our assumption, $\nabla_{\dot{c}(0)}(s_1 \circ f) = D_{df(\dot{c}(0))}(s_1)$, i.e.

$$\nabla_v(s_1 \circ f) = D_{df(v)}(s_1).$$

The fact that this holds for all $s_1 \in \Gamma E$ and all $v \in TM$ means exactly that ∇ satisfies the defining condition for "f^*D"; hence by the uniqueness proved in part (a) we must have $\nabla = f^*D$. $\square$

(c). Let $V \subset N$ be an open set containing q such that there exists a basis of sections $s_1, \ldots, s_n \in \Gamma E_{|V}$. Set $U = f^{-1}(V) \subset M$. Then, given $s \in \Gamma f^* E$, there exist unique $\alpha_1, \ldots, \alpha_n \in C^\infty(U)$ such that

$$s_{|U} = \sum_{j=1}^{n} \alpha_j \cdot (s_j \circ f_{|U})$$

(cf. Problem 44(b) and Problem 34). Now by the definition of f^*D (cf. part a),

$$(f^*D)(s)_{|U} = \sum_{j=1}^{n} \Big((s_j \circ f_{|U}) \otimes d\alpha_j + \alpha_j \cdot D_{df(\cdot)}(s_j) \Big) \quad \text{in } \Gamma((f^*E \otimes T^*M)_{|U}).$$

In particular,

$$(f^*D)_{\dot{c}(0)}(s) = \sum_{j=1}^{n} \Big((\alpha_j \circ c)'(0) \cdot s_j(q) + \alpha_j(c(0)) \cdot D_{df(\dot{c}(0))}(s_j) \Big).$$

Here $df(\dot{c}(0)) = 0$ since $f \circ c$ is constant; thus we are left with:

$$(f^*D)_{\dot{c}(0)}(s) = \sum_{j=1}^{n} (\alpha_j \circ c)'(0) \cdot s_j(q). \tag{155}$$

On the other hand we have $s(c(t)) = \alpha_j(c(t)) \cdot s_j(q)$ for all $t \in (-\varepsilon, \varepsilon)$, and hence the right hand side of (155) equals the tangent vector $(\frac{d}{dt}(s \circ c)(t))_{|t=0}$ in $T_{s(c(0))}(E_q) = E_q$. Done! □

Problem 58: The general structure of the following proof is very similar to the solution to Problem 57(a).

The requirement on D is

$$D(\mu \otimes \nu) = (D_1\mu) \otimes \nu + \mu \otimes (D_2\nu), \qquad \forall \mu \in \Gamma E_1,\ \nu \in \Gamma E_2. \tag{156}$$

By the definition of tensor product (of $C^\infty(M)$-modules), every element in $\Gamma(E_1) \otimes \Gamma(E_2)$ can be written as a finite sum of pure tensors $\mu_1 \otimes \mu_2$ ($\mu_1 \in \Gamma E_1$, $\mu_2 \in \Gamma E_2$); hence by Problem 43(d), the same is true for any section in $\Gamma(E_1 \otimes E_2)$. (Note that a main portion of the solution to Problem 43(d) was spent on proving exactly this fact.) Hence the formula (156), together with the requirement that D should be $\mathbb{R}$-linear (or merely additive) certainly makes the connection D *uniquely defined,* if it exists at all.

Thus it remains to prove that there exists such a connection D. Let us first prove the existence of D in the special case when both E_1 and E_2 are *trivial* vector bundles. Then fix a basis of sections $\mu_1, \ldots, \mu_n \in \Gamma E_1$ and a basis of sections $\nu_1, \ldots, \nu_m \in \Gamma E_2$ (cf. Problem 33; we set $n = \operatorname{rank} E_1$ and $m = \operatorname{rank} E_2$). Then $\{\mu_i \otimes \nu_j\}$ (with $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, m\}$) is a basis of sections in $\Gamma(E_1 \otimes E_2)$, and so for any $s \in \Gamma(E_1 \otimes E_2)$ there exist unique 'coefficient functions' $\alpha_{i,j} \in C^\infty(M)$ such that

$$s = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j}\, \mu_i \otimes \nu_j, \tag{157}$$

and we now define the map

$$D : \Gamma(E_1 \otimes E_2) \to \Gamma(E_1 \otimes E_2 \otimes T^*M)$$

by setting[36]

$$D(s) := \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(\mu_i \otimes \nu_j \otimes d\alpha_{i,j} + \alpha_{i,j}(D_1\mu_i) \otimes \nu_j + \alpha_{i,j}\mu_i \otimes (D_2\nu_j) \Big).$$

(Here we use the natural isomorphism between $\Gamma(E_1 \otimes T^*M \otimes E_2)$ and $\Gamma(E_1 \otimes E_2 \otimes T^*M)$ to identify $(D_1\mu_i) \otimes \nu_j \in \Gamma(E_1 \otimes T^*M \otimes E_2)$ with an element in $\Gamma(E_1 \otimes E_2 \otimes T^*M)$.) This map D is clearly well-defined and $\mathbb{R}$-linear. Furthermore, for any $f \in C^\infty(M)$ and s as in (157), we have

$$fs = \sum_{i=1}^{n} \sum_{j=1}^{m} f\alpha_{i,j}\, \mu_i \otimes \nu_j,$$

[36] Note that this definition of D apriori depends on the choice of the bases of sections $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$; however we will soon prove that D is a connection satisfying (156), and *then* it follows that our D is in fact independent of the choices of $\mu_1, \ldots, \mu_n$ and $\nu_1, \ldots, \nu_n$, since we noted from start that there exists *at most one* connection satisfying (156)!

and hence by our definition,

$$D(fs) = \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(\mu_i \otimes \nu_j \otimes d(f\alpha_{i,j}) + f\alpha_{i,j}(D_1\mu_i) \otimes \nu_j + f\alpha_{i,j}\mu_i \otimes (D_2\nu_j) \Big)$$

$$\big\{\text{use } d(f\alpha_{i,j}) = \alpha_{i,j}df + fd\alpha_{i,j} \text{ and our formula for } D(s)\big\}$$

$$= f \cdot D(s) + \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_{i,j}\mu_i \otimes \nu_j \otimes df$$

$$= f \cdot D(s) + s \otimes df.$$

This proves that D is a connection! We next prove that D satisfies (156). Thus let $\mu \in \Gamma E_1$ and $\nu \in \Gamma E_2$ be given. Then there exist unique $\alpha_1, \ldots, \alpha_n \in C^\infty(M)$ such that $\mu = \sum_{i=1}^{n} \alpha_i \mu_i$, and there exist unique $\beta_1, \ldots, \beta_m \in C^\infty(M)$ such that $\nu = \sum_{j=1}^{m} \beta_j \nu_j$. Then

$$\mu \otimes \nu = \sum_{i=1}^{n} \sum_{j=1}^{m} \alpha_i \beta_j \mu_i \otimes \nu_j,$$

and hence by our definition,

$$D(\mu \otimes \nu) = \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(\mu_i \otimes \nu_j \otimes d(\alpha_i\beta_j) + \alpha_i\beta_j(D_1\mu_i) \otimes \nu_j + \alpha_i\beta_j\mu_i \otimes (D_2\nu_j) \Big)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} \Big(\mu_i \otimes (\beta_j\nu_j) \otimes d\alpha_i + (\alpha_i D_1 \mu_i) \otimes (\beta_j \nu_j)$$

$$+ (\alpha_i\mu_i) \otimes \nu_j \otimes d\beta_j + \alpha_i\mu_i \otimes (\beta_j D_2 \nu_j) \Big)$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (D_1(\alpha_i\mu_i)) \otimes (\beta_j\nu_j) + \sum_{i=1}^{n} \sum_{j=1}^{m} (\alpha_i\mu_i) \otimes (D_2(\beta_j\nu_j))$$

$$= (D_1\mu) \otimes \nu + \mu \otimes (D_2\nu).$$

Hence we have proved that D satisfies (156). This completes the proof in the special case when both E_1 and E_2 are trivial vector bundles.

Finally we will prove that the connection D on $\Gamma(E_1 \otimes E_2)$ exists when E_1, E_2 are *arbitrary* vector bundles over M. Fix an open covering $(U_\alpha)_{\alpha \in A}$ of M such that both $E_{1|U_\alpha}$ and $E_{2|U_\alpha}$ are trivial for all $\alpha \in A$. Then by what we have proved above, for each $\alpha \in A$ there exists a unique connection D_α on $(E_1 \otimes E_2)_{|U_\alpha} = E_{1|U_\alpha} \otimes E_{2|U_\alpha}$ satisfying

(158) $$D_\alpha(\mu \otimes \nu) = (D_1\mu) \otimes \nu + \mu \otimes (D_2\nu), \qquad \forall \mu \in \Gamma E_{1|U_\alpha},\ \nu \in \Gamma E_{2|U_\alpha}.$$

(Here "D_1" really stands for $D_{1|U_\alpha}$ and similarly for D_2; cf. Problem 52.) Now one proves that these connections D_α are compatible in the sense that $(D_\alpha)_{|U_\alpha \cap U_\beta} = (D_\beta)_{|U_\alpha \cap U_\beta}$ for all $\alpha, \beta \in A$. (We leave out the details for this; but cf. the solution to Problem 57 where we give a detailed proof of the same kind of compatibility in a different situation.) Hence by Problem 52(c), there exists a unique connection D on $\Gamma(E_1 \otimes E_2)$ satisfying $D_{|U_\alpha} = D_\alpha$ for all $\alpha \in A$. One easily proves that this connection D satisfies (156). This completes the proof. $\square$

Problem 59: (a). By the now familiar argument (cf., e.g., the beginning of the solution to Problem 58), any section in $\Gamma(E_1 \otimes E_2)$ can be written as a finite sum of pure tensor sections $s_1 \otimes s_2$ with $s_1 \in \Gamma E_1$ and $s_2 \in \Gamma E_2$. The analogous fact holds for $\Gamma(E_1^* \otimes E_3)$. Hence, by $\mathbb{R}$-linearity, it suffices to prove the desired formula when $\alpha = s_1 \otimes s_2$ and $\beta = u \otimes s_3$ for some $s_1 \in \Gamma E_1$, $s_2 \in \Gamma E_2$, $u \in \Gamma E_1^*$, $s_3 \in \Gamma E_3$. In this case,

$$(\alpha, \beta) = (s_1 \otimes s_2, u \otimes s_3) = (s_1, u) \cdot s_2 \otimes s_3,$$

and hence

$$\begin{aligned} D(\alpha,\beta) &= s_2 \otimes s_3 \otimes d(s_1,u) + (s_1,u) \cdot D(s_2 \otimes s_3) \\ &= s_2 \otimes s_3 \otimes \Big((Ds_1,u) + (s_1,Du)\Big) + (s_1,u) \cdot \Big((Ds_2) \otimes s_3 + s_2 \otimes (Ds_3)\Big) \\ &= \Big((Ds_1) \otimes s_2 + s_1 \otimes (Ds_2)\,,\ u \otimes s_3\Big) \\ &\qquad\qquad + \Big(s_1 \otimes s_2\,,\ (Du) \otimes s_3 + u \otimes (Ds_3)\Big) \\ &= (D\alpha,\beta) + (\alpha, D\beta). \end{aligned}$$

Done! □

(b). We have standard identifications $\Gamma(\mathrm{Hom}(E_2,E_3)) = \Gamma(E_2^* \otimes E_3)$ and $\Gamma(\mathrm{Hom}(E_1,E_2)) = \Gamma(E_1^* \otimes E_2)$, and under these identifications, the composition $\alpha \circ \beta$ is the section in $\Gamma(E_1^* \otimes E_3)$ obtained by contracting the E_2^*-part of α against the E_2-part of β. (Cf. the solution to Problem 50 where this is discussed in matrix notation.) Hence the stated formula is equivalent to the formula proved in part a. (Of course the formula in part a remains true regardless of the exact ordering of the factors in the tensor products.) □

(c). We have the standard identification $\Gamma(\mathrm{Hom}(E_1,E_2)) = \Gamma(E_1^* \otimes E_2)$, and under this identification the composition $\alpha \circ \beta$ is the section in $\Gamma(E_2)$ obtained by contracting the E_1^*-part of α against β. Hence the stated formula is a special case of the formula in part a (namely: take $E_3 = M \times \mathbb{R}$ and replace E_1 by E_1^* in the formula in part a). □

Problem 60:

(a). The general structure of the following proof is again similar to the solutions of Problems 57(a) and 58.

The requirement on $D : \Omega^p(E) \to \Omega^{p+1}(E)$ is $\mathbb{R}$-linearity and

$$D(\mu \otimes \omega) = (D\mu) \wedge \omega + \mu \otimes d\omega, \qquad \forall \mu \in \Gamma E,\ \omega \in \Omega^p(M). \tag{159}$$

We call such a map an *exterior covariant derivative* with respect to the connection D. It follows from $\Omega^p(E) = \Gamma(E \otimes \bigwedge^p M) = \Gamma(E) \otimes_{C^\infty M} \Omega^p(M)$ (cf. Problem 43(d)) that every $s \in \Omega^p(E)$ can be written as a *finite* sum of pure tensors $s \otimes \omega$ ($s \in \Gamma(E)$, $\omega \in \Omega^p(M)$). Hence, since D is required to be $\mathbb{R}$-linear (in particular additive), the condition (159) certainly makes $D : \Omega^p(E) \to \Omega^{p+1}(E)$ *uniquely determined*, if it exists at all.

Thus it remains to prove that there *exists* such an exterior covariant derivative. We start by proving three lemmata:

Lemma 6. *Let $(U_\alpha)_{\alpha \in A}$ be an open covering of M, and for each $\alpha \in A$ let $D_\alpha : \Omega^p(E_{|U_\alpha}) \to \Omega^{p+1}(E_{|U_\alpha})$ be an exterior covariant derivative wrt the connection $D_{|U_\alpha}$ on $E_{|U_\alpha}$. Assume that $(D_\alpha(s_{|U_\alpha}))_{|U_\alpha \cap U_\beta} = (D_\beta(s_{|U_\beta}))_{|U_\alpha \cap U_\beta}$ for all $s \in \Omega^p(E)$ and all $\alpha, \beta \in A$. Then there exists a unique $\mathbb{R}$-linear map $D : \Omega^p(E) \to \Omega^{p+1}(E)$ satisfying $(Ds)_{|U_\alpha} = D_\alpha(s_{|U_\alpha})$ for all $s \in \Omega^p(E)$ and $\alpha \in A$, and this map is an exterior covariant derivative wrt D.*

This lemma is proved by the same type of arguments as in the solution to Problem 52 (cf. in particular Remark 1).

Lemma 7. *Let $U_2 \subset U_1$ be open subsets of M, and for $j = 1, 2$ let $D_j : \Omega^p(E_{|U_j}) \to \Omega^{p+1}(E_{|U_j})$ be an exterior covariant derivative wrt the connection $D_{|U_j}$. Then*

$$(D_1 s)_{|U_2} = D_2(s_{|U_2}), \qquad \forall s \in \Omega^p(E_{|U_1}). \tag{160}$$

Proof. Since every $s \in \Omega^p(E_{|U_1})$ can be written as a finite sum of pure tensors $\mu \otimes \omega$, where $\mu \in \Gamma(E_{|U_1})$ and $\omega \in \Omega^p(U_1)$, it suffices to prove (160) when s is such a pure tensor; $s = \mu \otimes \omega$. But then

$$\begin{aligned}(D_1 s)_{|U_2} = (D_1(\mu \otimes \omega))_{|U_2} &= ((D_{|U_1}\mu) \wedge \omega + \mu \otimes d\omega)_{|U_2} \\ &= (D_{|U_1}\mu)_{|U_2} \wedge \omega_{|U_2} + \mu_{|U_2} \otimes d(\omega_{|U_2}) \\ &= (D_{|U_2}\mu_{|U_2}) \wedge \omega_{|U_2} + \mu_{|U_2} \otimes d(\omega_{|U_2}) \\ &= D_2(\mu_{|U_2} \otimes \omega_{|U_2}) = D_2(s_{|U_2}).\end{aligned}$$

(In the above computation we used the fact that $(D_{|U_1})_{|U_2} = D_{|U_2}$ – this fact is immediate from the solution of Problem 52. □

Lemma 8. *Let U be any open subset of M such that both the vector bundles TU and $E_{|U}$ are trivializable. Then there exists a unique exterior covariant derivative $\widetilde{D} : \Omega^p(E_{|U}) \to \Omega^{p+1}(E_{|U})$ wrt $D_{|U}$.*

Proof. The assumptions imply that there exists a basis of sections $\omega_1, \ldots, \omega_r$ for $\Omega^p(U)$ ($r = \binom{d}{p}$) and a basis of sections $\mu_1, \ldots, \mu_n$ for $\Gamma(E_{|U})$. Then $(\mu_j \otimes \omega_k)$ form a basis of sections of $\Omega^p(E_{|U})$ and hence every $s \in \Omega^p(E_{|U})$ can be uniquely expressed as $s = a^{jk}\mu_j \otimes \omega_k$ with $a^{jk} \in C^\infty U$. We now define the map $\widetilde{D} : \Omega^p(E_{|U}) \to \Omega^{p+1}(E_{|U})$ by

$$\widetilde{D}(a^{jk}\mu_j \otimes \omega_k) := (D(a^{jk}\mu_j)) \wedge \omega_k + a^{jk}\mu_j \otimes d\omega_k. \tag{161}$$

(In the right hand side, "D" of course stands for $D_{|U}$.) This map $\widetilde{D}$ is clearly $\mathbb{R}$-linear. Let us verify that $\widetilde{D}$ is an exterior covariant derivative wrt $D_{|U}$. Thus let $\mu \in \Gamma(E_{|U})$ and $\omega \in \Omega^p(U)$ be given. Then there exist unique $b^1, \ldots, b^n, c^1, \ldots, c^r \in C^\infty(U)$ such that $\mu = b^j\mu_j$ and $\omega = c^k\omega_k$, and thus $\mu \otimes \omega = b^jc^k\mu_j \otimes \omega_k$. Hence by our definition,

$$\widetilde{D}(\mu \otimes \omega) = (D(b^jc^k\mu_j)) \wedge \omega_k + b^jc^k\mu_j \otimes d\omega_k,$$

and this can be manipulated as follows:

$$\begin{aligned}
&= \big(c^kD(b^j\mu_j) + b^j\mu_j \otimes dc^k\big) \wedge \omega_k + (b_j\mu_j) \otimes (c^k \cdot d\omega_k) \\
&= (D(b^j\mu_j)) \wedge (c^k\omega_k) + (b^j\mu_j) \otimes (dc^k \wedge \omega_k + c^k \cdot d\omega_k) \\
&= (D\mu) \wedge \omega + \mu \otimes d(c^k\omega_k) \\
&= (D\mu) \wedge \omega + \mu \otimes d\omega.
\end{aligned}$$

Hence $\widetilde{D}$ is indeed an exterior covariant derivative wrt $D_{|U}$.

The uniqueness follows by the argument immediately below (159). (In particular this shows that $\widetilde{D}$ is independent of the choice of bases of sections $\omega_1, \ldots, \omega_r$ and $\mu_1, \ldots, \mu_n$.) □

We now complete the proof of existence: Let $\{U_\alpha\}$ be a family of open subsets satisfying the assumption of Lemma 8, covering M. Let $D_\alpha : \Omega^p(E_{|U_\alpha}) \to \Omega^{p+1}(E_{|U_\alpha})$ be the exterior covariant derivative provided by Lemma 8. For any $\alpha, \beta \in A$ the set $V := U_\alpha \cap U_\beta$ also satisfies the assumption of Lemma 8 (assume $V \neq \emptyset$ for nontriviality), and so there exists a unique exterior covariant derivative $\widetilde{D} : \Omega^p(E_{|V}) \to \Omega^{p+1}(E_{|V})$ wrt $D_{|V}$. Then Lemma 7 (applied twice) implies that for every $s \in \Omega^p(E)$ we have

$$(D_\alpha(s_{|U_\alpha}))_{|V} = \widetilde{D}(s_{|V}) = (D_\alpha(s_{|U_\beta}))_{|V}.$$

Hence all assumptions of Lemma 6 are fulfilled and now that lemma proves the existence of an exterior covariant derivative $\Omega^p(E) \to \Omega^{p+1}(E)$ wrt D. Done! □ □

(b). In order to simplify the notation let us replace M by U; thus from now on we can write "E" in place of "$E_{|U}$".

Note that both sides of the stated formula are $\mathbb{R}$-linear in μ; hence by the argument below (159) it suffices to prove the formula for μ of the form $\mu = s \otimes \omega$, with $s \in \Gamma(E)$ and $\omega \in \Omega^p(M)$. For such μ we have by the definition in part (a):

$$\begin{aligned} D\mu = D(s \otimes \omega) = (Ds) \wedge \omega + \mu \otimes d\omega &= (ds + As) \wedge \omega + s \otimes d\omega \\ &= \big(ds \wedge \omega + s \otimes d\omega\big) + As \wedge \omega \\ &= d(s \otimes \omega) + As \wedge \omega. \end{aligned}$$

(In the last equality we again used the definition in part (a), this time for the naive connection d.) Note also that the "$\wedge$" in "$As \wedge \omega$" can be viewed as the combined vector-wedge-product (cf. Problem 49(c)) $\Omega^1(E) \times \Omega^p(M) \to \Omega^{p+1}(E)$ coming from the standard "scalar product map" $\Gamma(E) \times C^\infty(M) \to \Gamma(E)$. As in the problem formulation, let us also write "$\wedge$" for the vector-wedge-product

$$\Omega^1(\operatorname{End} E) \times \Omega^r(E) \to \Omega^{r+1}(E) \tag{162}$$

coming from the standard contraction ("evaluation") $\Gamma(\operatorname{End} E) \times \Gamma E \to \Gamma E$. (Thus "$As$" appearing above is the same as $A \wedge s$, namely the image of A and s under the map in (162) with $r = 0$.) Noticing that the given multiplication rules $\Gamma(\operatorname{End} E) \times \Gamma E \to \Gamma E$ and $\Gamma E \times C^\infty(M) \to \Gamma E$ satisfy the obvious associativity relation[37], it follows by Problem 49(d) that

$$As \wedge \omega = (A \wedge s) \wedge \omega = A \wedge (s \wedge \omega).$$

Using this in the previous computation gives

$$D\mu = d(s \otimes \omega) + A \wedge (s \wedge \omega) = d\mu + A \wedge \mu.$$

Done! □

[37] This merely captures the fact that the map $\Gamma(\operatorname{End} E) \times \Gamma E \to \Gamma E$ is $C^\infty(M)$-linear in its second argument (in fact it is also $C^\infty(M)$-linear in its first argument).

(c). Again by $\mathbb{R}$-(bi-)linearity it suffices to prove the stated formula for μ_1, μ_2 of the form $\mu_1 = s_1 \otimes \omega_1$ and $\mu_2 = s_2 \otimes \omega_2$, with $\mu_j \in \Gamma E_j$, $\omega_1 \in \Omega^r(M)$ and $\omega_2 \in \Omega^s(M)$. In this case we have

$$\begin{aligned} D(\mu_1 \wedge \mu_2) &= D((s_1 \cdot s_2) \otimes (\omega_1 \wedge \omega_2)) \\ &= (D(s_1 \cdot s_2)) \wedge (\omega_1 \wedge \omega_2) + (s_1 \cdot s_2) \otimes d(\omega_1 \wedge \omega_2), \end{aligned}$$

where we used the definition of $\wedge$ (Problem 49(c)) and then the definition of D (part (a) of this problem). Using now the assumption that the given connections respect our "$\cdot$", we get

$$\begin{aligned} &= ((Ds_1) \wedge s_2) \wedge (\omega_1 \wedge \omega_2) + (s_1 \wedge Ds_2) \wedge (\omega_1 \wedge \omega_2) \\ &\qquad + (s_1 \cdot s_2) \otimes (d\omega_1 \wedge \omega_2) + (-1)^r (s_1 \cdot s_2) \otimes (\omega_1 \wedge d\omega_2). \end{aligned} \tag{163}$$

Now note that the multiplication rule from E_1, E_2 to $\widetilde{E}$ satisfy the associativity relation $(s_1 \cdot s_2) \cdot f = s_1 \cdot (s_2 \cdot f)$ for all $s_1 \in \Gamma E_1$, $s_2 \in \Gamma E_2$, $f \in C^\infty(M) = \Gamma(M \times \mathbb{R})$ (namely since the multiplication rule is $C^\infty(M)$-linear in s_2). By Problem 49(d), this implies that

$$(\varphi_1 \wedge \varphi_2) \wedge \varphi_3 = \varphi_1 \wedge (\varphi_2 \wedge \varphi_3), \qquad \forall \varphi_1 \in \Omega^{r_1}(E_1),\ \varphi_2 \in \Omega^{r_2}(E_2),\ \varphi_3 \in \Omega^{r_3}(M).$$

Similarly we also have

$$(\varphi_1 \wedge \varphi_2) \wedge \varphi_3 = \varphi_1 \wedge (\varphi_2 \wedge \varphi_3), \qquad \forall \varphi_1 \in \Omega^{r_1}(E_2),\ \varphi_2 \in \Omega^{r_2}(M),\ \varphi_3 \in \Omega^{r_3}(M),$$

and other similar associativity relations. Furthermore by Problem 49(c),

$$\varphi_1 \wedge \varphi_2 = (-1)^{r_1 r_2} \varphi_2 \wedge \varphi_1, \qquad \forall \varphi_1 \in \Omega^{r_1}(E_2),\ \varphi_2 \in \Omega^{r_2}(M).$$

Using these facts (and $Ds_2 \in \Omega^1(E_2)$, and, again, the definition of $\wedge$), the expression in (163) is seen to be

$$\begin{aligned} &= (Ds_1) \wedge \omega_1 \wedge s_2 \wedge \omega_2 + (-1)^r s_1 \wedge \omega_1 \wedge (Ds_2) \wedge \omega_2 \\ &\qquad + (s_1 \otimes d\omega_1) \wedge (s_2 \otimes \omega_2) + (-1)^r (s_1 \otimes \omega_1) \wedge (s_2 \otimes d\omega_2) \\ &= \Big((Ds_1) \wedge \omega_1 + s_1 \otimes d\omega_1\Big) \wedge \mu_2 + (-1)^r \mu_1 \wedge \Big((Ds_2) \wedge \omega_2 + s_2 \otimes d\omega_2\Big) \\ &= (D\mu_1) \wedge \mu_2 + (-1)^r \mu_1 \wedge (D\mu_2). \end{aligned}$$

Done! □

(d). The statement that the given connections respect the multiplication rule can be expressed as:

(164)
$$D\big(m(s_1 \otimes s_2)\big) = m(Ds_1 \otimes s_2) + m(s_1 \otimes Ds_2) \qquad (\forall s_1 \in \Gamma E_1,\ s_2 \in \Gamma E_2),$$

where in the right hand side, "$m(\alpha)$" for $\alpha \in \Omega^1(E_1 \otimes E_2)$ is the output of the vector-wedge-product

$$\Gamma \mathrm{Hom}(E_1 \otimes E_2, \widetilde{E}) \times \Omega^1(E_1 \otimes E_2) \to \Omega^1(\widetilde{E})$$

which extends the standard evaluation map

$$\Gamma \mathrm{Hom}(E_1 \otimes E_2, \widetilde{E}) \times \Gamma(E_1 \otimes E_2) \to \Gamma(\widetilde{E}).$$

However by Problem 59(c) and Problem 58 we have:

$$D\big(m(s_1 \otimes s_2)\big) = (Dm)(s_1 \otimes s_2) + m(Ds_1 \otimes s_2) + m(s_1 \otimes Ds_2),$$

for any $s_1 \in \Gamma E_1,\ s_2 \in \Gamma E_2$. Hence (164) is equivalent with:

(165) $$(Dm)(s_1 \otimes s_2) = 0 \qquad (\forall s_1 \in \Gamma E_1,\ s_2 \in \Gamma E_2).$$

But every section in $\Gamma(E_1 \otimes E_2)$ can be written as a finite sum of sections of the form $s_1 \otimes s_2$; hence (165) is equivalent with $(Dm)(s) = 0$ for all $s \in \Gamma(E_1 \otimes E_2)$. This is equivalent with $Dm = 0$ in $\Omega^1(\mathrm{Hom}(E_1 \otimes E_2, \widetilde{E}))$ (via Problem 35(c)). □

Problem 61:

It suffices to prove the stated formula when $s = \mu \otimes \omega$ ($\mu \in \Gamma(E)$, $\omega \in \Omega^r(M)$), since an arbitrary section in $\Omega^r(E)$ can be expressed as a finite sum of such "pure tensor" sections. Now when $s = \mu \otimes \omega$, we find that the right hand side of the stated formula equals

$$\sum_{j=0}^{r}(-1)^j D_{X_j}\big(\omega(X_0,\ldots,\hat{X}_j,\ldots,X_r)\cdot\mu\big)$$
$$+\sum_{0\leq j<k\leq r}(-1)^{j+k}\omega([X_j,X_k],X_0,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_r)\cdot\mu.$$
$$=\sum_{j=0}^{r}(-1)^j X_j\big(\omega(X_0,\ldots,\hat{X}_j,\ldots,X_r)\big)\cdot\mu$$
$$+\sum_{j=0}^{r}(-1)^j\omega(X_0,\ldots,\hat{X}_j,\ldots,X_r)\cdot D_{X_j}\mu$$
$$+\sum_{0\leq j<k\leq r}(-1)^{j+k}\omega([X_j,X_k],X_0,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_r)\cdot\mu.$$

On the other hand we have by definition $Ds = (D\mu)\wedge\omega + \mu\otimes d\omega$ (cf. Problem 60(a)), and thus

$$[Ds](X_0,\ldots,X_r) = \big((D\mu)\wedge\omega\big)(X_0,\ldots,X_r) + [d\omega](X_0,\ldots,X_r)\cdot\mu.$$

For the first term we now use the definition of wedge product[38], and for the second term we apply Problem 48(c); this gives:

$$[Ds](X_0,\ldots,X_r) = \sum_{j=0}^{r}(-1)^j\cdot\omega(X_0,\ldots,\hat{X}_j,\ldots,X_r)\cdot(D\mu)(X_j)$$
$$+\sum_{j=0}^{r}(-1)^j X_j\big(\omega(X_0,\ldots,\hat{X}_j,\ldots,X_r)\big)\cdot\mu$$
$$+\sum_{0\leq j<k\leq r}(-1)^{j+k}\omega([X_j,X_k],X_0,\ldots,\hat{X}_j,\ldots,\hat{X}_k,\ldots,X_r)\cdot\mu.$$

Here "$(D\mu)(X_j)$" stands for the contraction of the form part of $D\mu \in \Omega^1(E)$ against X_j; thus $(D\mu)(X_j) = D_{X_j}\mu$. Hence the last expression equals our previous expression for the right hand side of the stated formula. Hence the stated formula is proved! □

[38] together with a computation reducing the "$\mathcal{A}$-sum" over $\mathfrak{S}_{r+1}$ to a sum over only $r+1$ distinct permutations; we leave this step to the reader.

Problem 62:

By Problem 59, for each $j \in \{0, \ldots, r\}$ we have

$$
\begin{aligned}
&\Big([\begin{smallmatrix}\nabla \\ D\end{smallmatrix}]_{X_j} \widetilde{s} \Big)(X_0, \ldots, \hat{X}_j, \ldots, X_r) \\
&= D_{X_j}\Big(s(X_0, \ldots, \hat{X}_j, \ldots, X_r) \Big) - \sum_{k=0}^{j-1} s\Big(X_0, \ldots, \nabla_{X_j} X_k, \ldots, \hat{X}_j, \ldots, X_r \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \sum_{k=j+1}^{r} s\Big(X_0, \ldots, \hat{X}_j, \ldots, \nabla_{X_j} X_k, \ldots, X_r \Big) \\
&= D_{X_j}\Big(s(X_0, \ldots, \hat{X}_j, \ldots, X_r) \Big) - \sum_{k=0}^{j-1} (-1)^k s\Big(\nabla_{X_j} X_k, X_0, \ldots, \hat{X}_k, \ldots, \hat{X}_j, \ldots, X_r \Big) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - \sum_{k=j+1}^{r} (-1)^{k-1} s\Big(\nabla_{X_j} X_k, X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r \Big),
\end{aligned}
$$

where in the last step we used the fact that the form part of s is alternating. Using the above it follows that

$$
\begin{aligned}
&\sum_{j=0}^{r} (-1)^j \Big([\begin{smallmatrix}\nabla \\ D\end{smallmatrix}]_{X_j} \widetilde{s} \Big)(X_0, \ldots, \hat{X}_j, \ldots, X_r) \\
&= \sum_{j=0}^{r} (-1)^j D_{X_j}\Big(s(X_0, \ldots, \hat{X}_j, \ldots, X_r) \Big) \\
&- \sum_{0 \le k < j \le r} (-1)^{j+k} s\Big(\nabla_{X_j} X_k, X_0, \ldots, \hat{X}_k, \ldots, \hat{X}_j, \ldots, X_r \Big) \\
&+ \sum_{0 \le j < k \le r} (-1)^{j+k} s\Big(\nabla_{X_j} X_k, X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r \Big).
\end{aligned}
$$

In the middle sum we change names between j and k; this gives:

$$
\begin{aligned}
&= \sum_{j=0}^{r} (-1)^j D_{X_j}\Big(s(X_0, \ldots, \hat{X}_j, \ldots, X_r) \Big) \\
&+ \sum_{0 \le j < k \le r} (-1)^{j+k} s\Big(\nabla_{X_j} X_k - \nabla_{X_k} X_j, X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r \Big). \\
&= \sum_{j=0}^{r} (-1)^j D_{X_j}\Big(s(X_0, \ldots, \hat{X}_j, \ldots, X_r) \Big) \\
&+ \sum_{0 \le j < k \le r} (-1)^{j+k} s\Big([X_j, X_k], X_0, \ldots, \hat{X}_j, \ldots, \hat{X}_k, \ldots, X_r \Big),
\end{aligned}
$$

where in the last step we used the assumption that ∇ is torsion free. By Problem 61, the above equals $[Ds](X_0, \ldots, X_r)$. Hence we have proved the desired formula. $\square$

Problem 63:

(a). Let (U, φ) be any bundle chart for E, and let $s_1, \ldots, s_n \in \Gamma E_{|U}$ be the corresponding basis of sections. Then we get a corresponding bundle chart $(U, \widetilde{\varphi})$ *for* $\mathrm{End}\, E$ by mapping any $B \in \mathrm{End}\, E_p$ $(p \in U)$ to the matrix for B with respect to the basis $s_1(p), \ldots, s_n(p)$ of E_p.

(Then $\widetilde{\varphi}$ is a C^∞ diffeomorphism from $\mathrm{End}\, E_{|U}$ onto $U \times M_n(\mathbb{R})$; pedantically for this to be a bundle chart we also need to fix an identification of $M_n(\mathbb{R})$ with $\mathbb{R}^{n^2}$. Furthermore: The bundle chart described here is the same as the one which we give in the solution to Problem 39, after identifying $\mathrm{Hom}(\mathbb{R}^n, \mathbb{R}^n)$ with $M_n(\mathbb{R})$ in the obvious way.)

Now if (U, φ) is a *metric* bundle chart then the image of $\mathrm{Ad}E_{|U}$ under $\widetilde{\varphi}$ is exactly $U \times \mathfrak{o}(n)$ where $\mathfrak{o}(n)$ is the set of *skew-symmetric* matrices in $M_n(\mathbb{R})$. Hence since $\mathfrak{o}(n)$ is a linear subspace of $M_n(\mathbb{R})$ [39], and since (E, π, M) can be covered with metric bundle charts [12, Thm. 2.1.3], it follows that $\mathrm{Ad}E$ is a vector subbundle of $\mathrm{End}\, E$. □

Remark: Recall that we write $\mathfrak{gl}(E)$ for the vector bundle $\mathrm{End}\, E$ equipped with its standard Lie algebra bundle structure. Similarly we write $\mathfrak{gl}_n(\mathbb{R})$ for $M_n(\mathbb{R})$ equipped with its standard Lie algebra structure; and $\mathfrak{o}(n)$ is in fact a *Lie subalgebra* of $\mathfrak{gl}_n(\mathbb{R})$. Also for each $p \in U$, $\widetilde{\varphi}_p$ is in fact a Lie algebra isomorphism $\mathfrak{gl}(E_p) \xrightarrow{\sim} \mathfrak{gl}_n(\mathbb{R})$ which maps $\mathrm{Ad}E_p$ onto $\mathfrak{o}(n)$. Hence $\mathrm{Ad}E$ is a *Lie algebra subbundle* of $\mathfrak{gl}(E)$.

[39] and so we can fix a linear isomorphism $M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ under which $\mathfrak{o}(n)$ becomes identified with $\mathbb{R}^k = \{(*, \cdots, *, 0, \cdots, 0)\} \subset \mathbb{R}^{n^2}$ for some k. (In fact $k = n(n-1)/2$.)

(b). Assume $s \in \Gamma(\mathrm{Ad}E)$. Let (U,φ) be any metric bundle chart for E, and let $\mu_1,\ldots,\mu_n$ be the corresponding basis of sections in $\Gamma E_{|U}$. With respect to (U,φ) we write $D_{|U} = d + A$ with $A = (A_j^k) \in \Omega^1(\operatorname{End} E)$; thus each A_j^k is in $\Omega^1(U)$ and $D(\mu_j) = A(\mu_j) = \mu_k \otimes A_j^k$ for all $j \in \{1,\ldots,n\}$; cf. #9, p. 6. Then $A_j^k = -A_k^j$ for all $j,k \in \{1,\ldots,n\}$, by Lemma 2 in #11.

Let $\mu^{1*},\ldots,\mu^{n*}$ be the basis of sections in $\Gamma E^*_{|U}$ which is dual to $\mu_1,\ldots,\mu_n$; then $\{\mu^{j*} \otimes \mu_k \ : \ j,k \in \{1,\ldots,n\}\}$ is a basis of sections in $\Gamma \operatorname{End} E$. Take $a_j^k \in C^\infty(U)$ for $j,k \in \{1,\ldots,n\}$ so that $s_{|U} = a_j^k \, \mu^{j*} \otimes \mu_k$. This means that for any $p \in U$, $(a_j^k(p))$ is the matrix for $s(p) \in \operatorname{End} E_p$ with respect to the basis for E_p which comes from (U,φ); hence by the definition of $\mathrm{Ad}E$ (Cf. #11, Def. 2) we have $a_j^k = -a_k^j$ throughout U, for all $j,k \in \{1,\ldots,n\}$.[40]

Now we have

$$(Ds)_{|U} = ds + [A,s].$$

(This was seen in the proof of the second Bianchi identity; cf. #11, p. 6.) Here since A and s have the matrices (A_j^k) and (a_j^k), respectively, we find that $[A,s]$ has the matrix $(A_j^k a_i^j - a_j^k A_i^j)_{i,k}$.[41] Hence:

$$(Ds)_{|U} = ds + [A,s] = \mu^{i*} \otimes \mu_k \otimes \big(da_i^k + a_i^j A_j^k - a_j^k A_i^j\big).$$

Using now the fact that $a_j^k = -a_k^j$ and $A_j^k = -A_k^j$ throughout U $(\forall j,k)$ it follows that $da_i^k = -da_k^i$ and $a_i^j A_j^k - a_j^k A_i^j = -(a_k^j A_j^i - a_j^i A_k^j)$ throughout U $(\forall i,k)$. Hence $(Ds)_{|U}$ is represented by a skew-symmetric matrix wrt the basis coming from (U,φ), and therefore $(Ds)_{|U} \in \Gamma(\mathrm{Ad}E_{|U})$. Since this holds for any metric bundle chart (U,φ) for E, it follows that $s \in \Gamma(\mathrm{Ad}E)$. Done! $\square$

See also alternative solution on the next page.

[40] Here's a more explicit version of exactly the same argument: By the definition of $\mathrm{Ad}E$ we have $\langle s(\mu_i),\mu_\ell\rangle = -\langle \mu_i, s(\mu_\ell)\rangle$ throughout U, for all $i,\ell \in \{1,\ldots,n\}$. But $\langle s(\mu_i),\mu_\ell\rangle = \langle a_i^k \mu_k, \mu_\ell\rangle = a_i^\ell$ and similarly $\langle \mu_i, s(\mu_\ell)\rangle = a_\ell^i$. Hence $a_i^\ell = -a_\ell^i$ throughout U.

[41] Details: We have $[A,s] = A\circ s - s \circ A$ since $A \in \Omega^1(\operatorname{End} E_{|U})$ and $s \in \Omega^0(\operatorname{End} E)$; cf. #11, p. 7. Now

$$\begin{aligned}[A,s] &= A\circ s - s\circ A \\ &= (\mu^{j*}\otimes \mu_k \otimes A_j^k)\circ(a_i^\ell\,\mu^{i*}\otimes\mu_\ell) - (a_j^k\,\mu^{j*}\otimes\mu_k)\circ(\mu^{i*}\otimes\mu_\ell\otimes A_i^\ell) \\ &= \mu^{i*}\otimes\mu_k\otimes(a_i^j A_j^k - a_j^k A_i^j).\end{aligned}$$

Alternative (not using local coordinates): For any $s \in \Gamma(\operatorname{End} E)$ and $X, Y \in \Gamma E$ we have

$$(Ds)(X) = D(s(X)) - s(DX)$$

by Problem 59(c), and hence

$$\langle (Ds)(X), Y\rangle = \langle D(s(X)), Y\rangle - \langle s(DX), Y\rangle. \tag{166}$$

(Here of course $\langle \cdot, \cdot \rangle$ stands for the vector-wedge-product $\Omega^1(E) \times \Gamma(E) \to \Omega^1(M)$ which comes from the given bundle metric $\Gamma(E) \times \Gamma(E) \to \Gamma(M)$.) Using also the fact that D is metric, we can write the above relation as:

$$\langle (Ds)(X), Y\rangle = d\langle s(X), Y\rangle - \langle s(X), DY\rangle - \langle s(DX), Y\rangle. \tag{167}$$

Switching X and Y we also have:

$$\langle X, (Ds)(Y)\rangle = d\langle X, s(Y)\rangle - \langle DX, s(Y)\rangle - \langle X, s(DY)\rangle. \tag{168}$$

Now assume $s \in \Gamma(\mathrm{Ad}E)$. Then $\langle s(Z_1), Z_2\rangle = -\langle Z_1, s(Z_2)\rangle$ for any two sections $Z_1, Z_2 \in \Gamma E$. This implies that more generally

$$\langle s(\mu_1), \mu_2\rangle = -\langle \mu_1, s(\mu_2)\rangle, \qquad \forall \mu_1 \in \Omega^p(E), \mu_2 \in \Omega^q(E). \tag{169}$$

[Indeed, if $\mu_1 = Z_1 \otimes \omega_1$ and $\mu_2 = Z_2 \otimes \omega_2$ with $Z_1, Z_2 \in \Gamma E$, $\omega_1 \in \Omega^p(M)$, $\omega_2 \in \Omega^q(M)$, then

$$\begin{aligned}\langle s(\mu_1), \mu_2\rangle = \langle s(Z_1) \otimes \omega_1, Z_2 \otimes \omega_2\rangle &= \langle s(Z_1), Z_2\rangle \cdot \omega_1 \wedge \omega_2 \\ &= -\langle Z_1, s(Z_2)\rangle \cdot \omega_1 \wedge \omega_2 = -\langle \mu_1, s(\mu_2)\rangle,\end{aligned}$$

i.e. (169) holds. The general case follows by $\mathbb{R}$-(bi)-linearity.] Applying (169) it follows that the right hand side of (167) equals the negative of the right hand side of (168). Hence:

$$\langle (Ds)(X), Y\rangle = -\langle X, (Ds)(Y)\rangle \qquad \text{in } \Omega^1(M). \tag{170}$$

The fact that this holds for all $X, Y \in \Gamma E$ implies that

$$Ds \in \Omega^1(\mathrm{Ad}E). \tag{171}$$

Done! □

[Detailed proof that (170) implies (171): Let (U, x) be any C^∞ chart for M; then $dx^1, \ldots, dx^d$ is a basis of sections in ΓT^*U. Hence there exist unique $\beta_1, \ldots, \beta_d \in \Gamma \operatorname{End} E_{|U}$ such that $Ds_{|U} = \beta_j \otimes dx^j$, and now the above relation says that

$$\langle \beta_j(X), Y\rangle \cdot dx^j = -\langle X, \beta_j(Y)\rangle \cdot dx^j \qquad \text{in } \Omega^1(U),$$

and therefore

$$\langle \beta_j(X), Y\rangle = -\langle X, \beta_j(Y)\rangle \qquad \text{in } C^\infty(U), \qquad \forall j.$$

Using Problem 35(c) and the definition of $\mathrm{Ad}E$, this implies that $\beta_j(p) \in \mathrm{Ad}E_p$, $\forall p \in U$, i.e. $\beta_j \in \Gamma(\mathrm{Ad}E_{|U})$, for all j. Therefore $Ds_{|U} \in \Omega^1(\mathrm{Ad}E_{|U})$. Since M can be covered by C^∞ charts, it follows that $Ds \in \Omega^1(\mathrm{Ad}E)$.]

Problem 64:

(a). First assume that the statement in $\bigwedge^r(V)$ holds. Note that $v_1 \wedge \cdots \wedge v_r \neq 0$ implies that $v_1, \ldots, v_r$ are linearly independent. (Proof: exercise!) Hence $r \leq n := \dim V$ and we can choose $v_{r+1}, \ldots, v_n \in V$ so that $v_1, \ldots, v_n$ is a basis for V. Then we know (cf., e.g., "Prop. 3" in Sec. 7.2 in the lecture notes) that $(v_I)_{I \in \mathcal{I}}$ is a basis for $\bigwedge^r(V)$, where $\mathcal{I}$ is the family of all r-tuples $I = (i_1, \ldots, i_r) \in \{1, \ldots, n\}^r$ with $i_1 < \cdots < i_r$, and

$$v_I := v_{i_1} \wedge \cdots \wedge v_{i_r}.$$

(Thus $\dim \bigwedge^r(V) = \#\mathcal{I} = \binom{n}{r}$.) Also since $v_1, \ldots, v_n$ is a basis for V, there exist unique constants $c_j^k \in \mathbb{R}$ ($j \in \{1, \ldots, r\}$, $k \in \{1, \ldots, n\}$) such that $w_j = c_j^k v_k$ for $j = 1, \ldots, r$. Then

$$\begin{aligned} w_1 \wedge \cdots \wedge w_r &= (c_1^{k_1} v_{k_1}) \wedge (c_2^{k_2} v_{k_2}) \wedge \cdots \wedge (c_r^{k_r} v_{k_r}) \\ &= c_1^{k_1} c_2^{k_2} \cdots c_r^{k_r} \cdot v_{k_1} \wedge v_{k_2} \wedge \cdots \wedge v_{k_r}. \end{aligned}$$

Note that the last expression is a sum over all $(k_1, \ldots, k_r) \in \{1, \ldots, n\}^r$, and for each such $(k_1, \ldots, k_r)$ there exist a unique $I \in \mathcal{I}$ and a unique permutation $\sigma \in \mathfrak{S}_r$ such that

$$(k_1, \ldots, k_r) = (i_{\sigma(1)}, \ldots, i_{\sigma(r)}).$$

Hence:

$$\begin{aligned} w_1 \wedge \cdots \wedge w_r &= \sum_{I \in \mathcal{I}} \sum_{\sigma \in \mathfrak{S}_r} c_1^{i_{\sigma(1)}} c_2^{i_{\sigma(2)}} \cdots c_r^{i_{\sigma(r)}} \cdot v_{i_{\sigma(1)}} \wedge v_{i_{\sigma(2)}} \wedge \cdots \wedge v_{i_{\sigma(r)}} \\ &= \sum_{I \in \mathcal{I}} \Big(\sum_{\sigma \in \mathfrak{S}_r} (\operatorname{sgn} \sigma) c_1^{i_{\sigma(1)}} c_2^{i_{\sigma(2)}} \cdots c_r^{i_{\sigma(r)}} \Big) \cdot v_I \\ &= \sum_{I \in \mathcal{I}} \det(c_\ell^{i_j}) \cdot v_I. \end{aligned}$$

(In the second equality we made repeated use of the rule $u_1 \wedge u_2 = -u_2 \wedge u_1$, $\forall u_1, u_2 \in V$. In the last line $(c_\ell^{i_j})$ is an $r \times r$-matrix; $\ell, j \in \{1, \ldots, r\}$.) Now from our assumption $v_1 \wedge \cdots \wedge v_r = c \cdot w_1 \wedge \cdots \wedge w_r$ and the fact that $(v_I)_{I \in \mathcal{I}}$ is a basis for $\bigwedge^r(V)$, it follows that $c \neq 0$, $\det(c_\ell^{i_j}) = c^{-1}$ for $I = (1, \ldots, r)$ (in other words: $\det(c_\ell^j) = c^{-1}$), while $\det(c_\ell^{i_j}) = 0$ for all $I \in \mathcal{I} \setminus \{(1, \ldots, r)\}$. In other words, in the $r \times n$ matrix

$$\begin{pmatrix} c_1^1 & c_1^2 & \cdots & c_1^n \\ \vdots & \vdots & & \vdots \\ c_r^1 & c_r^2 & \cdots & c_r^n \end{pmatrix},$$

the $r \times r$ minor which is furthest to the left equals $c^{-1} \neq 0$, while all other $r \times r$ minors vanish! This implies that the first r columns of the above matrix (viewed as vectors in $\mathbb{R}^r$) form a basis for $\mathbb{R}^r$. Furthermore, it follows that *every other column vanishes,* i.e. $c_\ell^i = 0$ for all $i > r$ and $\ell \in \{1, \ldots, r\}$. (Proof: Suppose that there is some $i > r$ such that the

ith column is not 0. Then there exists a subset of $r-1$ among the first r columns which together with the ith column form a basis for $\mathbb{R}^r$. This implies that the corresponding $r \times r$ minor is non-zero, a contradiction.) Therefore $w_\ell = c_\ell^k v_k \in \text{Span}\{v_1, \ldots, v_r\}$ for each $\ell \in \{1, \ldots, r\}$; furthermore since the matrix $(c_\ell^k)_{\ell,k\in\{1,\ldots,r\}}$ is invertible we get $v_\ell \in \text{Span}\{w_1, \ldots, w_r\}$ for each $\ell \in \{1, \ldots, r\}$; hence $\text{Span}\{v_1, \ldots, v_r\} = \text{Span}\{w_1, \ldots, w_r\}$, as we wanted to prove!

Conversely, now assume that $v_1, \ldots, v_r$ are linearly independent and $v_1, \ldots, v_r$ and $w_1, \ldots, w_r$ span the same r-dimensional linear subspace of V. This means that $w_\ell = c_\ell^k v_k$ for some constants $c_\ell^k \in \mathbb{R}$ ($\ell, k \in \{1, \ldots, r\}$) such that the $r \times r$ matrix (c_ℓ^k) is non-singular. Then $v_1 \wedge \cdots \wedge v_r \neq 0$ in $\bigwedge^r(V)$, since $v_1 \wedge \cdots \wedge v_r$ can be part of a basis for $\bigwedge^r(V)$ by "Prop. 3" in Sec. 7.2 in the lecture notes. Let $(\gamma_i^j) := (c_\ell^k)^{-1} \in M_r(\mathbb{R})$; then $v_k = \gamma_k^\ell w_\ell$ and so

$$v_1 \wedge \cdots \wedge v_r = (\gamma_1^{\ell_1} w_{\ell_1}) \wedge \cdots \wedge (\gamma_r^{\ell_r} w_{\ell_r}) = \det(\gamma_k^\ell) \cdot w_1 \wedge \cdots \wedge w_r,$$

and $\det(\gamma_k^\ell) \neq 0$. Done! □

(b).

(c). If $v_1, \ldots, v_r$ are not linearly independent then the parallelotope in question is contained in some $r-1$ dimensional subspace and so has r-dimensional volume 0; also $v_1 \wedge \cdots \wedge v_r = 0$ and so $\|v_1 \wedge \cdots \wedge v_r\| = 0$, i.e. the formula holds.

Now assume that $v_1, \ldots, v_r$ are linearly independent. Pick an ON-basis $e_1, \ldots, e_r$ for the r-dimensional subspace spanned by $v_1, \ldots, v_r$, and choose $e_{r+1}, \ldots, e_n$ such that $e_1, \ldots, e_n$ is an ON-basis for V. Take $c_j^k \in \mathbb{R}$ so that $v_j = c_j^k e_k$. Then the volume of the r-dimensional parallelotope spanned by $v_1, \ldots, v_r$ equals $|\det(c_j^k)|$ (basic fact about volumes). On the other hand $v_1 \wedge \cdots \wedge v_r = \det(c_j^k) \cdot e_1 \wedge \cdots \wedge e_r$ (by a computation similar to a step in the solution to part a) and hence

$$\begin{aligned}\langle v_1 \wedge \cdots \wedge v_r, v_1 \wedge \cdots \wedge v_r \rangle &= (\det(c_j^k))^2 \cdot \langle e_1 \wedge \cdots \wedge e_r, e_1 \wedge \cdots \wedge e_r \rangle \\ &= (\det(c_j^k))^2,\end{aligned}$$

where the last equality holds by property (i) in part b. Hence

$$\|v_1 \wedge \cdots \wedge v_r\| = \sqrt{(\det(c_j^k))^2} = |\det(c_j^k)|.$$

Done! □

Problem 65:

(a). (i) $\Rightarrow$ (ii): Let $\mathcal{A}$ be an oriented atlas for M. By a simple modification of the standard C^∞ charts for TM (cf. Lecture #2, pp. 10–11, and Problem 16), one proves that for any C^∞ chart (U, x) for M, (U, η_x) is a bundle chart for (TM, π, M), where η_x is the map

$$\eta_x : TU \to U \times \mathbb{R}^d;$$
$$\eta_x(w) = (\pi(w), dx_{\pi(w)}(w)).$$

In particular the following is an atlas of bundle charts for (TM, π, M):

$$\mathcal{A}' := \{(U, \eta_x) \,:\, (U, x) \in \mathcal{A}\}.$$

We claim that $\mathcal{A}'$ makes (TM, π, M) an oriented vector bundle. To prove this consider any two charts $(U, x), (V, y) \in \mathcal{A}$, and any point $p \in U \cap V$. Our task is to prove that the linear map

$$\eta_{y,p} \circ \eta_{x,p}^{-1} : \mathbb{R}^d \to \mathbb{R}^d$$

is in $\mathrm{GL}_d^+(\mathbb{R})$. However $\eta_{x,p} = dx_p$ and $\eta_{y,p} = dy_p$ (both are linear maps from $T_p(M)$ to $\mathbb{R}^d$); hence $\eta_{y,p} \circ \eta_{x,p}^{-1} = dy_p \circ dx_p^{-1} = d(y \circ x^{-1})_{x(p)}$. However $y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$ is the chart transition map between (U, x) and (V, y); hence by our assumption on $\mathcal{A}$, $\det d(y \circ x^{-1})_{x(p)} > 0$, i.e.

$$\eta_{y,p} \circ \eta_{x,p}^{-1} = d(y \circ x^{-1})_{x(p)} \in \mathrm{GL}_d^+(\mathbb{R}).$$

Hence $\mathcal{A}'$ indeed makes (TM, π, M) an oriented vector bundle.

(ii) $\Rightarrow$ (i): Let $\mathcal{A}'$ be an atlas of bundle charts for (TM, π, M) with respect to which (TM, π, M) is an oriented vector bundle. We will prove that M possesses an oriented atlas. Let $\mathcal{A}_1$ be an *arbitrary* C^∞ atlas for M. Set

$$\mathcal{A}_2 := \Big\{(W, x_{|W}) \,:\, (U,x) \in \mathcal{A}_1,\ (V,\varphi) \in \mathcal{A}',\ \text{and } W \text{ is a path-connected component of } U \cap V\Big\}.$$

Then $\mathcal{A}_2$ is also a C^∞ atlas for M, and it has the convenient property that whenever (U,x) is a chart in $\mathcal{A}_2$, U is path-connected and there is a bundle chart (V,φ) in $\mathcal{A}'$ such that $U \subset V$.

Now consider any $(U,x) \in \mathcal{A}_2$ and $(V,\varphi) \in \mathcal{A}'$ subject to $U \subset V$. (More generally, the following argument applies to any C^∞ chart (U,x) on M such that U is path-connected and $U \subset V$ for some $(V,\varphi) \in \mathcal{A}'$.) Then for any $p \in U$ both dx_p and φ_p are linear isomorphisms from T_pU onto $\mathbb{R}^d$; hence $dx_p \circ \varphi_p^{-1}$ is a linear isomorphism of $\mathbb{R}^d$ onto itself, and so the determinant $\det(dx_p \circ \varphi_p^{-1})$ is well-defined and non-zero. Hence by continuity (crucially using the fact that U is path-connected),[42] we either have $\det(dx_p \circ \varphi_p^{-1}) > 0$ for all $p \in U$ or $\det(dx_p \circ \varphi_p^{-1}) < 0$ for all $p \in U$. Let us define the *"sign of (U,x) wrt (V,φ)"* to be $s = +1$ in the first case and $s = -1$ in the second case. In this situation, we note that: for *every* $(W,\eta) \in \mathcal{A}'$ and *every* $p \in U \cap W$, we have $\det(dx_p \circ \eta_p^{-1}) = s$. (Proof: For each $p \in U$ we have $dx_p \circ \eta_p^{-1} = dx_p \circ \varphi_p^{-1} \circ (\varphi_p \circ \eta_p^{-1})$ and $\det(\varphi_p \circ \eta_p^{-1}) > 0$ since $\mathcal{A}'$ makes (TM, π, M) oriented; hence $\det(dx_p \circ \eta_p^{-1})$ and $\det(dx_p \circ \varphi_p^{-1})$ have the same sign.)

From the previous discussion we conclude: Every $(U,x) \in \mathcal{A}_2$ (and more generally every C^∞ chart (U,x) on M such that U is path-connected and $U \subset V$ for some $(V,\varphi) \in \mathcal{A}'$) has a well-defined *sign* $s \in \{-1, 1\}$ *wrt* $\mathcal{A}'$, with the property that

$$\forall (W,\eta) \in \mathcal{A}',\ \forall p \in U \cap W: \qquad \det(dx_p \circ \eta_p^{-1}) = s. \tag{172}$$

Now let us *fix* a non-singular linear map $R \in \mathrm{GL}_d(\mathbb{R})$ with $\det R < 0$ (e.g. a reflection). For any $(U,x) \in \mathcal{A}_2$ we define

$$\widehat{x} := \begin{cases} x & \text{if } (U,x) \text{ has sign } +1 \text{ wrt } \mathcal{A}' \\ R \circ x & \text{if } (U,x) \text{ has sign } -1 \text{ wrt } \mathcal{A}'. \end{cases}$$

[42] some more details: we have to prove that $\det(dx_p \circ \varphi_p^{-1})$ is a continuous function of $p \in U$. But we know that $\alpha := dx \circ \varphi_{|TU}^{-1}$ is a C^∞ diffeomorphism from $U \times \mathbb{R}^d$ onto $T(x(U)) = x(U) \times \mathbb{R}^d$, and for any $m, n \in \{1, \ldots, d\}$, the (m,n)-entry of the matrix representing the linear map $dx_p \circ \varphi_p^{-1}$ equals $e_m \cdot \mathrm{pr}_2(\alpha(p, e_n))$, where e_m is the mth standard unit vector in $\mathbb{R}^d$, $\cdot$ is the standard scalar product on $\mathbb{R}^d$, and $\mathrm{pr}_2 : x(U) \times \mathbb{R}^d \to \mathbb{R}^d$ is the projection onto the second factor. From this we see that each matrix entry of (the matrix representing) $dx_p \circ \varphi_p^{-1}$ depends continuously on p; hence also the determinant of $dx_p \circ \varphi_p^{-1}$ depends continuously on p.

Note that then also $(U, \widehat{x})$ is a C^∞ chart on M, and $(U, \widehat{x})$ has *sign* $+1$ wrt $\mathcal{A}'$. We set:

$$\mathcal{A}_3 := \big\{(U, \widehat{x}) \,:\, (U, x) \in \mathcal{A}_2\big\}.$$

Then $\mathcal{A}_3$ is also a C^∞ atlas for M, and it has the property that for any chart $(U, x) \in \mathcal{A}_3$, U is path-connected, there is some $(V, \varphi) \in \mathcal{A}'$ with $U \subset V$, and (U, x) has sign $+1$ wrt $\mathcal{A}'$. We claim that $\mathcal{A}_3$ is an *oriented* atlas. To prove this, consider any two charts $(U, x), (V, y) \in \mathcal{A}_3$, and any point $p \in U \cap W$. We have to prove that $\det(dx_p \circ dy_p^{-1}) > 0$. Take a bundle chart $(W, \eta) \in \mathcal{A}'$ with $p \in W$. Since both (U, x) and (V, y) have sign $+1$ wrt $\mathcal{A}'$, we have both $\det(dx_p \circ \eta_p^{-1}) = +1$ and $\det(dy_p \circ \eta_p^{-1}) = +1$ (cf. (172)). Hence

$$\begin{aligned}\det(dx_p \circ dy_p^{-1}) &= \det\big((dx_p \circ \eta_p^{-1}) \circ (dy_p \circ \eta_p^{-1})^{-1}\big)\\ &= \det\big(dx_p \circ \eta_p^{-1}\big) \cdot \det\big(dy_p \circ \eta_p^{-1}\big)^{-1} = 1.\end{aligned}$$

Done! □

(i) ⇔ (iii): Cf., e.g., [1, Def. V.7.5, Thm. V.7.6].

(b). Let $\mathcal{A}$ be any C^∞ atlas for M; then we know from Lecture #2, pp. 10–11 (cf. also Problem 16) that the family

$$\mathcal{A}' := \{(TU, dx) \,:\, (U, x) \in \mathcal{A}\}$$

is a C^∞ atlas for TM. Let us prove that $\mathcal{A}'$ is an oriented atlas! Thus fix any two charts $(U, x), (V, y) \in \mathcal{A}$. Set $W := U \cap V$ and $\eta := y \circ x^{-1}$; then η is a C^∞ diffeomorphism from $x(W)$ onto $y(W)$. We have to prove that the diffeomorphism $dy \circ (dx)^{-1} = d\eta$ from $T(x(W)) = x(W) \times \mathbb{R}^d$ onto $T(y(W)) = y(W) \times \mathbb{R}^d$ has everywhere positive Jacobian determinant. For any $(p, v) \in x(W) \times \mathbb{R}^d$ we have

$$d\eta(p, v) = (\eta(p), d\eta_p(v)).$$

Hence the Jacobian matrix of $d\eta$ at (p, v) has a block decomposition

$$\begin{pmatrix} d\eta_p & 0 \\ * & d\eta_p \end{pmatrix},$$

where "$d\eta_p$", "0" and "$*$" are $d \times d$ matrices (here $*$ stands for a matrix which we don't care exactly what it is; note also that the bottom right $d \times d$ matrix equals $d\eta_p$ since the differential of a linear map at any point equals the map itself). The determinant of the above $2d \times 2d$-matrix is $(\det(d\eta_p))^2$, which is everywhere positive. Done! □

Problem 66:

(a). Let us use the short-hand notation

$$\partial_i := \frac{\partial}{\partial x^i} \in \Gamma(TU).$$

By definition we have

$$A^j_{i;k} = (\nabla_{\partial_k} A)(\partial_i \otimes dx^j),$$

where the right hand side stands for the contraction of $\nabla_{\partial_k} A \in \Gamma(T^1_1 U)$ against $\partial_i \otimes dx^j \in T^1_1 U$. This gives, via Problem 59:

$$\begin{aligned} A^j_{i;k} &= \partial_k \Big(A(\partial_i \otimes dx^j) \Big) - A\Big((\nabla_{\partial_k} \partial_i) \otimes dx^j \Big) - A\Big(\partial_i \otimes (\nabla_{\partial_k}(dx^j)) \Big) \\ &= \partial_k A^j_i - A\Big((\Gamma^\ell_{ki} \partial_\ell) \otimes dx^j \Big) - A\Big(\partial_i \otimes (-\Gamma^j_{k\ell} \, dx^\ell) \Big) \\ &= \partial_k A^j_i - \Gamma^\ell_{ki} \cdot A(\partial_\ell \otimes dx^j) + \Gamma^j_{k\ell} \cdot A(\partial_i \otimes dx^\ell) \\ &= \partial_k A^j_i - \Gamma^\ell_{ki} \cdot A^j_\ell + \Gamma^j_{k\ell} \cdot A^\ell_i. \end{aligned}$$

(In the second equality we used [12, (4.1.22)] for the last term.) Done! □

(b). (Cf. [14, Lemma 4.8].) Suppose that

$$A_{|U} = A^{j_1 \cdots j_r}_{i_1 \cdots i_s} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}},$$

and write

$$\nabla_{\frac{\partial}{\partial x^k}} A = A^{j_1 \cdots j_r}_{i_1 \cdots i_s ; k} \cdot dx^{i_1} \otimes \cdots \otimes dx^{i_s} \otimes \frac{\partial}{\partial x^{j_1}} \otimes \cdots \otimes \frac{\partial}{\partial x^{j_r}}$$

in U. Then by the same type of computation as in part a we find:

$$A^{j_1 \cdots j_r}_{i_1 \cdots i_s ; k} = \frac{\partial}{\partial x^k} A^{j_1 \cdots j_r}_{i_1 \cdots i_s} - \sum_{p=1}^{s} \Gamma^\ell_{k \, i_p} \cdot A^{j_1 \cdots j_r}_{i_1 \cdots \ell \cdots i_s} + \sum_{p=1}^{r} \Gamma^{j_p}_{k \, \ell} \cdot A^{j_1 \cdots \ell \cdots j_r}_{i_1 \cdots i_s}.$$

□

Problem 67:

We first make some computations useful for all parts of the problem: As in [12, p. 3, Ex. 1]), we also introduce the chart (V, z) on S^d, with

$$V = S^d \setminus \{(0, \ldots, 0, 1)\}; \qquad z(x) = \Big(\frac{x_1}{1 - x_{d+1}}, \ldots, \frac{x_d}{1 - x_{d+1}}\Big)$$

Note that both y and z are diffeomorphism onto all of $\mathbb{R}^d$. We compute that the inverse map of y is given by

$$x = \Big(\frac{2y_1}{1 + \|y\|^2}, \cdots, \frac{2y_d}{1 + \|y\|^2}, \frac{1 - \|y\|^2}{1 + \|y\|^2}\Big), \qquad \forall y \in \mathbb{R}^d.$$

Here $\|y\|$ is the standard Euclidean norm; thus $\|y\|^2 = y_1^2 + \cdots y_d^2$. Similarly, the inverse map of z is given by

$$x = \Big(\frac{2z_1}{1 + \|z\|^2}, \cdots, \frac{2z_d}{1 + \|z\|^2}, \frac{\|z\|^2 - 1}{\|z\|^2 + 1}\Big), \qquad \forall z \in \mathbb{R}^d.$$

Note also that $U \cap V = S^d \setminus \{(0, \ldots, 0, \pm 1)\}$ and

$$y(U_1 \cap U_2) = z(U_1 \cap U_2) = \mathbb{R}^d \setminus \{0\},$$

and so $y \circ z^{-1}$ is a diffeomorphism from $\mathbb{R}^d \setminus \{0\}$ onto itself. We compute that $y \circ z^{-1}$ is explicitly given by

$$y_j = \frac{\big(\frac{2z_j}{\|z\|^2+1}\big)}{1 + \frac{\|z\|^2-1}{\|z\|^2+1}} = \frac{z_j}{\|z\|^2} \qquad (z \in \mathbb{R}^d \setminus \{0\},\ j = 1, \ldots, d).$$

Note also that

$$\|y\|^2 = \frac{\|z\|^2}{\|z\|^4} = \frac{1}{\|z\|^2},$$

and hence

$$z_j = \|z\|^2 y_j = \frac{y_j}{\|y\|^2} \qquad (y \in \mathbb{R}^d \setminus \{0\},\ j = 1, \ldots, d).$$

(Note that $y \circ z^{-1}$ and $z \circ y^{-1}$ are in fact *the same map* from $\mathbb{R}^d \setminus \{0\}$ onto $\mathbb{R}^d \setminus \{0\}$. Geometrically this map is *inversion in the sphere* $S^{d-1} \subset \mathbb{R}^d$.)

Next we compute, for all $y \in \mathbb{R}^d \setminus \{0\}$:

$$\frac{\partial z_k}{\partial y_j} = \frac{\partial}{\partial y_j}\Big(\frac{y_k}{\|y\|^2}\Big) = \frac{\delta_{jk}\|y\|^2 - y_k \cdot 2y_j}{\|y\|^4} = \delta_{jk}\|z\|^2 - 2z_k z_j \qquad (j, k \in \{1, \ldots, d\}).$$

Hence

$$\frac{\partial}{\partial y_j} = \frac{\partial z_k}{\partial y_j}\frac{\partial}{\partial z_k} = \|z\|^2 \frac{\partial}{\partial z_j} - 2z_j z_k \frac{\partial}{\partial z_k} \qquad (j \in \{1, \ldots, d\}). \tag{173}$$

(The last relation is an equality between vector fields on $U \cap V \subset S^d$.)

(a). Assume the opposite, i.e. that there exists a vector field $X \in \Gamma(TS^2)$ such that $X_{|U} = y_1 \frac{\partial}{\partial y_1}$ (wrt the chart (U, y)). Then there exist unique functions $\alpha_1, \alpha_2 \in C^\infty(V)$ such that

$$X_{|V} = \alpha_1 \frac{\partial}{\partial z_1} + \alpha_2 \frac{\partial}{\partial z_2}$$

(wrt the chart (V, z)). However by (173) we have on $U \cap V$:

$$\frac{\partial}{\partial y_1} = \|z\|^2 \frac{\partial}{\partial z_1} - 2z_1^2 \frac{\partial}{\partial z_1} - 2z_1 z_2 \frac{\partial}{\partial z_2} = \left(-z_1^2 + z_2^2\right) \frac{\partial}{\partial z_1} - 2z_1 z_2 \frac{\partial}{\partial z_2},$$

and $y_1 = z_1/\|z\|^2$. Hence we must have

$$\alpha_1(z) = \frac{z_1\left(-z_1^2 + z_2^2\right)}{\|z\|^2}, \qquad \alpha_2(z) = -\frac{2z_1^2 z_2}{\|z\|^2},$$

for all $z \in \mathbb{R}^2 \setminus \{0\}$. We will now prove that the above formula implies that α_2 cannot be extended to a smooth function on all of $\mathbb{R}^2$; this gives a contradiction against $\alpha_2 \in C^\infty(V)$ and so the solution to part a will be complete.

The above formula implies

$$\frac{\partial \alpha_2}{\partial z_1} = -\frac{4z_1 z_2^3}{\|z\|^4} = -\frac{4z_1 z_2^3}{z_1^2 + z_2^2}, \qquad \forall z \in \mathbb{R}^2 \setminus \{0\},$$

and the limit of this function as $z \to 0$ in $\mathbb{R}^2$ *does not exist!* (Indeed, for $z = t(a, b) \neq 0$ the above expression equals $-\frac{4ab^3}{(a^2+b^2)^2}$, and for any fixed $(a, b) \in \mathbb{R}^2 \setminus \{0\}$ this tends to $-\frac{4ab^3}{(a^2+b^2)^2}$ as $t \to 0$. Now one immediately verifies that $-\frac{4ab^3}{(a^2+b^2)^2}$ can take different values for different choices of $(a, b) \in \mathbb{R}^2 \setminus \{0\}$. This shows that $\frac{\partial \alpha_2}{\partial z_1}$ has different limits as z approaches the origin along different lines in $\mathbb{R}^2$, and therefore the 'full 2-dim limit' of $\frac{\partial \alpha_2}{\partial z_1}$ as $z \to (0, 0)$ does not exist.) This proves that we cannot have $\alpha_2 \in C^1(\mathbb{R}^2)$, and hence, afortiori, we cannot have $\alpha_2 \in C^\infty(\mathbb{R}^2)$. (In fact the same argument applies to any of the partial derivatives $\frac{\partial \alpha_j}{\partial z_k}$, $j, k \in \{1, 2\}$, and in particular we cannot have $\alpha_1 \in C^\infty(\mathbb{R}^2)$ either.) $\square$

(b). In $U \cap V$ we have

(174)

$$
\begin{aligned}
&(y_1y_3 - y_2)\frac{\partial}{\partial y_1} + (y_2y_3 + y_1)\frac{\partial}{\partial y_2} + \frac{1 - y_1^2 - y_2^2 + y_3^2}{2}\,\frac{\partial}{\partial y_3} \\
&= \frac{z_1z_3 - z_2\|z\|^2}{\|z\|^4} \cdot \Big(\|z\|^2\frac{\partial}{\partial z_1} - 2z_1z_k\frac{\partial}{\partial z_k}\Big) + \frac{z_2z_3 + z_1\|z\|^2}{\|z\|^4} \cdot \Big(\|z\|^2\frac{\partial}{\partial z_2} - 2z_2z_k\frac{\partial}{\partial z_k}\Big) \\
&\qquad + \frac{\|z\|^4 - z_1^2 - z_2^2 + z_3^2}{2\|z\|^4} \cdot \Big(\|z\|^2\frac{\partial}{\partial z_3} - 2z_3z_k\frac{\partial}{\partial z_k}\Big) \\
&= \frac{1}{\|z\|^2}\bigg((z_1z_3 - z_2\|z\|^2)\frac{\partial}{\partial z_1} + (z_2z_3 + z_1\|z\|^2)\frac{\partial}{\partial z_2} + \frac{\|z\|^4 - z_1^2 - z_2^2 + z_3^2}{2}\frac{\partial}{\partial z_3}\bigg) \\
&\quad + \frac{1}{\|z\|^4}\bigg(\underbrace{-2z_1(z_1z_3 - z_2\|z\|^2) - 2z_2(z_2z_3 + z_1\|z\|^2) - z_3\big(\|z\|^4 - z_1^2 - z_2^2 + z_3^2\big)}_{(*)}\bigg)z_k\,\frac{\partial}{\partial z_k}
\end{aligned}
$$

Here the expression called "(*)" equals:

$$
\begin{aligned}
-2z_1^2z_3 - 2z_2^2z_3 - z_3\big(\|z\|^4 - z_1^2 - z_2^2 + z_3^2\big) &= -z_3\Big(z_1^2 + z_2^2 + z_3^2 + \|z\|^4\Big) \\
&= -z_3\|z\|^2(1 + \|z\|^2).
\end{aligned}
$$

Hence we can continue; the vector field in (174) equals

$$
\begin{aligned}
&\frac{1}{\|z\|^2}\bigg((z_1z_3 - z_2\|z\|^2 - z_3z_1(1 + \|z\|^2))\frac{\partial}{\partial z_1} + (z_2z_3 + z_1\|z\|^2 - z_3z_2(1 + \|z\|^2))\frac{\partial}{\partial z_2} \\
&\qquad + \Big(\frac{\|z\|^4 - z_1^2 - z_2^2 + z_3^2}{2} - z_3^2(1 + \|z\|^2)\Big)\frac{\partial}{\partial z_3}\bigg)
\end{aligned}
$$

Here

$$
\begin{aligned}
\frac{1}{\|z\|^2}\Big(\frac{\|z\|^4 - z_1^2 - z_2^2 + z_3^2}{2} - z_3^2(1 + \|z\|^2)\Big) &= \frac{1}{\|z\|^2}\Big(\frac{\|z\|^4 - \|z\|^2}{2} - z_3^2\|z\|^2\Big) \\
&= \frac{\|z\|^2 - 1 - 2z_3^2}{2} = \frac{z_1^2 + z_2^2 - z_3^2 - 1}{2},
\end{aligned}
$$

and hence we finally conclude that the above vector field equals

$$
-(z_2 + z_1z_3)\frac{\partial}{\partial z_1} + (z_1 - z_2z_3)\frac{\partial}{\partial z_2} + \frac{z_1^2 + z_2^2 - z_3^2 - 1}{2}\frac{\partial}{\partial z_3}. \tag{175}
$$

Recall that this computation was performed in the set $U \cap V$; however the expression in (175) clearly defines a (C^∞) vector field on all of V. Hence we can define the (C^∞) vector field $X \in \Gamma(TS^3)$ to be given by the expression in (the first line of) (174) in U, and by the expression in (175) in V; the above computation shows that this vector field is well-defined, i.e. the two formulas really give the same vector field in the region of overlap, $U \cap V$. □

(c). In $U \cap V$ we have

$$dy_j = \frac{\partial y_j}{\partial z_k} dz_k = \frac{\delta_{jk}\|z\|^2 - 2z_j z_k}{\|z\|^4} dz_k,$$

i.e.

$$dy_1 = \frac{z_2^2 - z_1^2}{\|z\|^4}\, dz_1 - \frac{2z_1 z_2}{\|z\|^4}\, dz_2$$

and

$$dy_2 = -\frac{2z_1 z_2}{\|z\|^4}\, dz_1 + \frac{z_1^2 - z_2^2}{\|z\|^4}\, dz_2.$$

Hence:

$$dy_1 \otimes dy_1 = \frac{1}{\eta^4}\Big(\big(z_2^2 - z_1^2\big)\, dz_1 - 2z_1 z_2\, dz_2\Big) \otimes \Big(\big(z_2^2 - z_1^2\big)\, dz_1 - 2z_1 z_2\, dz_2\Big)$$

$$= \frac{1}{\eta^4}\Big(\big(z_2^2 - z_1^2\big)^2\, dz_1 \otimes dz_1 - 2z_1 z_2 \big(z_2^2 - z_1^2\big)\big(dz_1 \otimes dz_2 + dz_2 \otimes dz_1\big) + 4(z_1 z_2)^2\, dz_2 \otimes dz_2\Big).$$

The formula for $dy_2 \otimes dy_2$ is exactly the same except that all z_1's and z_2's are swapped. Hence, noticing also

$$\big(z_2^2 - z_1^2\big)^2 + 4(z_1 z_2)^2 = \big(z_2^2 + z_1^2\big)^2 = \|z\|^4$$

and

$$\frac{1}{(1 + y_1^2 + y_2^2)^4} \cdot \frac{1}{\|z\|^8} = \frac{1}{(1 + \|y\|^2)^4 \|z\|^8} = \frac{1}{(1 + \|z\|^{-2})^4 \|z\|^8} = \frac{1}{(1 + \|z\|^2)^4},$$

we obtain:

$$\frac{1}{(1 + y_1^2 + y_2^2)^4}\big(dy_1 \otimes dy_1 + dy_2 \otimes dy_2\big)$$

$$= \frac{\|z\|^4}{(1 + \|z\|^2)^4}\big(dz_1 \otimes dz_1 + dz_2 \otimes dz_2\big). \tag{176}$$

Recall that this computation was performed in the set $U \cap V$; however the last expression clearly defines a (C^∞) section of all of $T_2^0(V)$. Hence we can define the (C^∞) section

$$m \in \Gamma(T_2^0(S^2))$$

to be given by the expression in the left hand side of (176) in U, and by the expression in the right hand side of (176) in V; the above equality (which is valid in $U \cap V$) shows that this section in m is well-defined.

For each $p \in S^2$, $m(p)$ is a vector in $T_2^0(S^2)_p = T_p^*(S^2) \otimes T_p^*(S^2)$ and can thus be viewed as a bilinear form on $T_p(S^2)$. We see by inspection in (176) that this bilinear form is *symmetric* at every $p \in S^2$. Furthermore it is *positive definite* at every $p \in U$, since its matrix with respect to the basis $\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}$ equals the positive number $(1 + \|y\|^2)^{-4}$ times the 2×2 identity matrix. Hence $m_{|U}$ indeed defines a Riemannian metric on U. However at the point $(0,0,-1) \in S^2$, which corresponds to $z = (0,0) \in V$, $m(p)$ is the *zero* form, thus not positive definite. Hence m does not define a Riemannian metric on S^2. $\square$

(d). Such a vector bundle can in fact be constructed for *any* given C^∞ function $\mu : U \cap V \to \mathrm{GL}_2(\mathbb{R})$; and similarly a rank n vector bundle over S^2 can be constructed having any given C^∞ function $\mu : U \cap V \to \mathrm{GL}_n(\mathbb{R})$ as transition function; there is simply no obstruction present!

The easiest solution is to simply refer to Jost's [12, Thm. 2.1.1]. However that theorem is not very clearly formulated; let us attempt to give an alternative, more precise statement here:

Theorem. *Let M be a C^∞ manifold, let $(U_\alpha)_{\alpha\in A}$ be a covering of M by open sets, and for any $\alpha, \beta \in A$ let $\varphi_{\beta\alpha}$ be a C^∞ function from $U_\alpha \cap U_\beta$ to $GL(n, \mathbb{R})$. Assume that for all $\alpha, \beta, \gamma \in A$, the following holds:*

$$\begin{aligned} &\varphi_{\alpha\alpha}(x) = \mathrm{id}_{\mathbb{R}^n}, && \forall x \in U_\alpha; \\ &\varphi_{\alpha\beta}(x)\varphi_{\beta\alpha}(x) = \mathrm{id}_{\mathbb{R}^n}, && \forall x \in U_\alpha \cap U_\beta; \\ &\varphi_{\alpha\gamma}(x)\varphi_{\gamma\beta}(x)\varphi_{\beta\alpha}(x) = \mathrm{id}_{\mathbb{R}^n}, && \forall x \in U_\alpha \cap U_\beta \cap U_\gamma. \end{aligned}$$

Then there exists a vector bundle E over M (unique up to isomorphism of vector bundles over M) which has a bundle atlas $\{(U_\alpha, \varphi_\alpha)\}_{\alpha\in A}$ for which the transition functions are given by the above $\varphi_{\beta\alpha}$'s.

Using the above theorem, the existence of the desired vector bundle E over S^2 is *immediate;* simply take $A = \{1, 2\}$; $U_1 = U$, $U_2 = V$, $\varphi_{12} \equiv \mu$ and $\varphi_{21} \equiv \mu^{-1}$ in $U \cap V$, $\varphi_{11} \equiv \mathrm{id}_{\mathbb{R}^2}$ in U and $\varphi_{22} \equiv \mathrm{id}_{\mathbb{R}^2}$ in V. One verifies that these $\varphi_{\alpha\beta}$'s satisfy all conditions in the above theorem; and the vector bundle which the theorem gives has the desired property! □

Exercise: Prove the above theorem, e.g. using Problem 36! (Cf. also [15, Exc. 10-6].)

Alternative: We will construct the desired vector bundle E over S^2 using Problem 36. As a *set* we define

$$E := S^2 \times \mathbb{R}^2.$$

We also set $\pi := \mathrm{pr}_1 : E \to S^2$, i.e. projection onto the first coordinate. Note, though, that (unless $m = 0$) E will *not* become equipped with the standard product C^∞ manifold structure of $S^2 \times \mathbb{R}^2$! Note that $\pi^{-1}(U) = U \times \mathbb{R}^2$ and $\pi^{-1}(V) = V \times \mathbb{R}^2$. Also $E_p = \pi^{-1}(p) = \{p\} \times \mathbb{R}^2$ for every $p \in S^2$, and we equip each such fiber with the standard vector space structure of $\mathbb{R}^2$. Let ϕ be the *identity map* on $U \times \mathbb{R}^2$, and let ψ be the map

$$\psi : V \times \mathbb{R}^2 \to V \times \mathbb{R}^2,$$

$$\psi(p, v) := \begin{cases} (p, \mu(p) \cdot v) & \text{if } p \in U \cap V \\ (p, v) & \text{if } p = (0, 0, -1). \end{cases}$$

(This map is well-defined since V is the disjoint union of $U \cap V$ and $\{(0, 0, -1)\}$. Note that "$\mu(p) \cdot v$" denotes the product of the matrix $\mu(p) \in \mathrm{GL}_2(\mathbb{R})$ and the vector $v \in \mathbb{R}^2$ viewed as a 2×1 column matrix.)

Now $\pi : E \to M$ toghether with the family $\{(U, \phi), (V, \psi)\}$ is easily seen to satisfy all the assumptions of Problem 36. (In particular $\psi \circ \phi^{-1}(p, v) = (p, \mu(p) \cdot v)$ and $\phi \circ \psi^{-1}(p, v) = (p, \mu(p)^{-1} \cdot v)$ for all $(p, v) \in (U \cap V) \times \mathbb{R}^2$, from which we see[43] that both the maps $\psi \circ \phi^{-1}$ and $\phi \circ \psi^{-1}$ are C^∞ maps from $(U \cap V) \times \mathbb{R}^2$ onto itself.)

Hence by Problem 36, E possesses a unique C^∞ manifold structure such that (E, π, M) is a vector bundle of rank 2 and (U, ϕ) and (V, ψ) are bundle charts. Note that the transition function from (U, ϕ) to (V, ψ) equals μ by construction!

Remark: From a conceptual point of view the *"discontinuity"* of the above map ψ at $p = (0, 0, -1)$ is confusing and ugly! The way to think about this is that our initial definition of E as a *set*, "$S^2 \times \mathbb{R}^2$" is only a technical device used to fit the construction into the result from Problem 36, where we need from start E to be a given (well-defined) *set!* Note that this "$S^2 \times \mathbb{R}^2$" carries *no topology* from start, so it is actually meaningless to speak about continuity/discontinuity of the maps ψ and ϕ! The *only* topology and differential structure which we endow "$S^2 \times \mathbb{R}^2$" with, is the one imposed by requiring that the two bundle charts should be diffeomorphisms! Hence conceptually it is much better to think of E as *the result of gluing the two vector bundles* $V \times \mathbb{R}^2$ *and* $U \times \mathbb{R}^2$ *together* in line with the above description — and *forget* about the set "$S^2 \times \mathbb{R}^2$" used in the construction.

[43]Of course here it is crucial to note that μ is a C^∞ map from $U \cap V$ to $\mathrm{GL}_2(\mathbb{R})$. This is clear from the formula defining μ, if we view $\alpha(y) = \arg(y^1 + iy^2)$ as a C^∞ function from $\mathbb{R}^2 \setminus \{0\}$ to the circle $\mathbb{R}/2\pi\mathbb{Z}$ and then use the fact that both $\cos(m\alpha)$ and $\sin(m\alpha)$ are well-defined C^∞ functions on $\mathbb{R}/2\pi\mathbb{Z}$.

Problem 68:

(a). By the definition of $\widetilde{\Gamma}_{ij}^k$ we have

$$\widetilde{\Gamma}_{ij}^k \cdot (s_k \circ f) = (f^* D)_{\frac{\partial}{\partial y^i}} (s_j \circ f) \qquad \text{in } \Gamma(f^* E)_{|V},$$

for all $i \in \{1, \ldots, d'\}$ and $j \in \{1, \ldots, n\}$. Let us evaluate the above at an arbitrary point $p \in V$, using the identity from Problem 57(a); this gives:

$$\widetilde{\Gamma}_{ij}^k(p) \cdot s_k(f(p)) = D_v(s_j), \tag{177}$$

where $v := df_p(\frac{\partial}{\partial y^i}) \in T_{f(p)} N$. We have

$$v = \frac{\partial f^\ell}{\partial y^i}(p) \cdot \frac{\partial}{\partial x^\ell}(f(p)).$$

(Here we write $f^\ell := x^\ell \circ f$, as usual.) Hence the right hand side of (177) can be evaluated as

$$D_v(s_j) = \frac{\partial f^\ell}{\partial y^i}(p) \cdot \Gamma_{\ell j}^k(f(p)) \cdot s_k(f(p)).$$

Comparing with (177), and using the fact that $s_1(f(p)), \ldots, s_n(f(p))$ is a basis of $E_{f(p)}$, we conclude:

$$\widetilde{\Gamma}_{ij}^k(p) = \frac{\partial f^\ell}{\partial y^i}(p) \cdot \Gamma_{\ell j}^k(f(p)).$$

This can also be expressed as:

$$\widetilde{\Gamma}_{ij}^k = \frac{\partial f^\ell}{\partial y^i} \cdot (\Gamma_{\ell j}^k \circ f).$$

□

(b). **Proof of existence of $f^*: \Omega^r(E) \to \Omega^r(f^*E)$, first alternative:** We wish to prove that there exists a unique $\mathbb{R}$-linear map $f^*: \Omega^r(E) \to \Omega^r(f^*E)$ satisfying

$$f^*(\mu \otimes \omega) = (\mu \circ f) \otimes f^*(\omega) \qquad \text{for all } \mu \in \Gamma E \text{ and } \omega \in \Omega^r(N). \tag{178}$$

It follows from $\Omega^r(E) = \Gamma(E \otimes \bigwedge^r(T^*N)) = \Gamma(E) \otimes \Gamma(\bigwedge^r(T^*N))$ (cf. Problem 43(d)) that every section in $\Omega^r(E)$ can be expressed as a finite sum of sections of the form $\mu \otimes \omega$ with $\mu \in \Gamma E$ and $\omega \in \Omega^r(N)$. Hence the formula (178) together with the $\mathbb{R}$-linearity certainly makes the map f^* uniquely defined, if it exists at all. The problem is that different decompositions of a given section $s \in \Omega^r(E)$ as a sum of "pure" sections $\mu \otimes \omega$ might apriori lead to different answers for what "$f^*(s)$" should be. To resolve this we will give an alternative, "pointwise", definition for $f^*(s)$.

For each $p \in M$, let $A_p : \bigwedge^r(T^*_{f(p)}N) \to \bigwedge^r(T^*_pM)$ be the map given by

$$A_p(\alpha)(v_1, \ldots, v_r) := \alpha(df_p(v_1), \ldots, df_p(v_r)), \qquad \forall \alpha \in \textstyle\bigwedge^r(T^*_{f(p)}N),\ v_1, \ldots, v_r \in T_pM.$$

This map A_p is clearly $\mathbb{R}$-linear. (Note that this map A_p in principle appears in Definition 5 in Lecture #8; namely we have $f^*(\omega)_p = A_p(\omega_{f(p)})$ for any $\omega \in \Omega^r(N)$ and any $p \in M$.) Hence for each $p \in M$ there is a unique $\mathbb{R}$-linear map

$$B_p := 1_{E_{f(p)}} \otimes A_p : \ E_{f(p)} \otimes \textstyle\bigwedge^r(T^*_{f(p)}N) \to E_{f(p)} \otimes \bigwedge^r(T^*_pM)$$

satisfying $B_p(v \otimes \alpha) = v \otimes A_p(\alpha)$ for all $v \in E_{f(p)}$ and $\alpha \in \bigwedge^r(T^*_{f(p)}N)$. Note that under standard identifications, B_p can equivalently be viewed as a map

$$B_p : \ (E \otimes \textstyle\bigwedge^r T^*N)_{f(p)} \to (f^*E \otimes \bigwedge^r T^*M)_p.$$

Now let us *define* $f^*(s)$, for any $s \in \Omega^r(E)$, by[44]

$$(f^*(s))(p) := B_p(s(f(p))), \qquad \forall p \in M. \tag{179}$$

Then for every $s \in \Omega^r(E)$, $f^*(s)$ is a function from M to $f^*E \otimes \bigwedge^r(T^*M)$, mapping each $p \in M$ into the fiber $(f^*E \otimes \bigwedge^r(T^*M))_p$. It is also clear from (179) that $f^*(s)$ is C^∞ [45] and hence $f^*(s) \in \Omega^r(f^*E)$. Therefore f^* is a map from $\Omega^r(E)$ to $\Omega^r(f^*E)$, and it is immediate from (179) that this map is $\mathbb{R}$-linear. Finally, for any $s = \mu \otimes \omega$ with $\mu \in \Gamma E$ and $\omega \in \Omega^r(N)$, we have, for all $p \in M$:

$$(f^*(s))(p) = B_p(s(f(p))) = B_p\Big(\mu(f(p)) \otimes \omega(f(p))\Big) = \mu(f(p)) \otimes A_p(\omega(f(p)))$$
$$= \mu(f(p)) \otimes f^*(\omega)(p) = f^*(\mu \otimes \omega)(p), \tag{180}$$

and hence our map f^* satisfies (178). Done! □

[44]The way one initially finds out that this formula (179) should/must hold, is by a computation similar to (180), but "in the other direction".

[45]The fact that $f^*(s)$ is C^∞ can also be seen as follows: Decompose s in some way as a finite sum $s = \mu_1 \otimes \omega_1 + \cdots + \mu_m \otimes \omega_m$ with $\mu_1, \ldots, \mu_m \in \Gamma E$ and $\omega_1, \ldots, \omega_m \in \Omega^r(N)$. Then similarly as in the computation (180) we have $f^*(s) = \sum_{j=1}^m (\mu_j \circ f) \otimes f^*(\omega_j)$, and the right hand side is C^∞ by inspection.

Proof of existence of $f^*: \Omega^r(E) \to \Omega^r(f^*E)$, second alternative: Using $\Omega^r(E) = \Gamma E \otimes \Omega^r(N)$ and $\Omega^r(f^*E) = \Gamma f^*E \otimes \Omega^r(M)$, it is tempting to simply say that, by the machinery from Problem 43 etc., "it suffices to show that the corresponding map from $\Gamma E \times \Omega^r(N)$ to $\Gamma f^*E \otimes \Omega^r(M)$ is bilinear" (cf. (182)). However there are complications due to the fact that we here have a mix of $C^\infty(N)$-modules and $C^\infty(M)$-modules, and one has to be careful about what "bilinear" really means... One can fill in the details as follows.

On $C^\infty(N) = \Omega^0(N)$ the map f^* is

$$f^*: C^\infty(N) \to C^\infty(M); \qquad f^*(g) = g \circ f \quad (\forall g \in C^\infty(N)), \tag{181}$$

and one verifies that f^* is a *ring homomorphism* from $C^\infty(N)$ to $C^\infty(M)$. Using this ring homomorphism, any $C^\infty(M)$-module gets an induced structure of a $C^\infty(N)$-module.

Now consider the map

$$J: \Gamma E \times \Omega^r(N) \to \Gamma f^*E \otimes \Omega^r(M); \qquad J(\mu, \omega) = (\mu \circ f) \otimes f^*(\omega). \tag{182}$$

Note that J is a map from a Cartesian product of two $C^\infty(N)$-modules to the $C^\infty(M)$-module $\Gamma f^*E \otimes \Omega^r(M)$; [46] however by what we have said above, $\Gamma f^*E \otimes \Omega^r(M)$ also has an induced structure of a $C^\infty(N)$-module, via the homomorphism f^* in (181). Now one verifies that *the map J is $C^\infty(N)$-bilinear.* [Details: One immediately verifies that $J(\mu_1 + \mu_2, \omega) = J(\mu_1, \omega) + J(\mu_2, \omega)$ and $J(\mu, \omega_1 + \omega_2) = J(\mu, \omega_1) + J(\mu, \omega_2)$ for all $\mu_1, \mu_2, \mu \in \Gamma E$ and $\omega_1, \omega_2, \omega \in \Omega^r(M)$. Next for arbitrary $\mu \in \Gamma E$, $\omega \in \Omega^r(M)$ and $g \in C^\infty(N)$ we have $(g \cdot \mu) \circ f = (g \circ f) \cdot (\mu \circ f)$, and therefore

$$J(g \cdot \mu, \omega) = (g \circ f) \cdot J(\mu, \omega) = f^*(g) \cdot J(\mu, \omega).$$

Also $f^*(g \cdot \omega) = f^*(g) \cdot f^*(\omega)$ and therefore

$$J(\mu, g \cdot \omega) = f^*(g) \cdot J(\mu, \omega).$$

Hence J is $C^\infty(N)$-bilinear.]

The fact that the map (182) is $C^\infty(N)$-bilinear now implies, via the defining property of tensor product (of $C^\infty(N)$-modules) that there exists a unique $C^\infty(N)$-linear map

$$f^*: \Gamma E \otimes \Omega^r(N) \to \Gamma f^*E \otimes \Omega^r(M)$$

such that

$$f^*(\mu \otimes \omega) = J(\mu, \omega) = (\mu \circ f) \otimes f^*(\omega).$$

This is the desired map! (Indeed recall $\Gamma E \otimes \Omega^r(N) = \Omega^r(E)$ and $\Gamma f^*E \otimes \Omega^r(M) = \Omega^r(f^*E)$. The fact that f^* is $C^\infty(N)$-linear implies in particular that f^* is $\mathbb{R}$-linear, as desired.) □

[46] Recall that the tensor product in "$\Gamma f^*E \otimes \Omega^r(M)$" always stands for *tensor product of $C^\infty(M)$-modules.* A more precise notation is "$\Gamma f^*E \otimes_{C^\infty(M)} \Omega^r(M)$".

Proof of the formula involving d^{f^*D}. We now turn to the second task of the problem, i.e. to prove that for any $s \in \Omega^r(E)$ we have

$$(d^{f^*D})(f^*(s)) = f^*(d^D s). \tag{183}$$

We first prove the auxiliary result that the map $f^* : \Omega^r(E) \to \Omega^r(f^*E)$ respects wedge product, i.e.

(184)
$$f^*(\sigma \wedge \eta) = f^*(\sigma) \wedge f^*(\eta) \text{ in } \Omega^{r+s}(E), \qquad \forall \sigma \in \Omega^r(E),\ \eta \in \Omega^s(N).$$

By $\mathbb{R}$-linearity in σ it suffices to prove (184) for $\sigma = \sigma_1 \otimes \eta_1$ with $\sigma_1 \in \Gamma E$ and $\eta_1 \in \Omega^r(N)$. In this case,

$$\begin{aligned} f^*(\sigma \wedge \eta) &= f^*(\sigma_1 \otimes (\eta_1 \wedge \eta)) = (\sigma_1 \circ f) \otimes f^*(\eta_1 \wedge \eta) \\ &= (\sigma_1 \circ f) \otimes f^*(\eta_1) \wedge f^*(\eta) = f^*(\sigma) \wedge f^*(\eta), \end{aligned}$$

where in the third equality we used the fact that $f^* : \Omega(N) \to \Omega(M)$ respects wedge product (cf. #8, p. 9). Hence (184) is proved.

Now we prove (183). In the case $r = 0$ (i.e., $s \in \Gamma E$ and $d^D = D$ and $d^{f^*D} = f^*D$) we see that (183) is equivalent with the identity in Problem 57(a) if we can only prove that

$$D_{df(\cdot)}(s) = f^*(Ds), \tag{185}$$

and assuming $Ds = \sum_{j=1}^m \mu_j \otimes \omega_j$ with $\mu_1, \ldots, \mu_m \in \Gamma E$ and $\omega_1, \ldots, \omega_m \in \Omega^1(N)$ we have, for every $p \in M$ and $X \in T_pM$:

$$\begin{aligned} \big(f^*(Ds)\big)(X) = \sum_{j=1}^m \big((\mu_j \circ f) \otimes f^*(\omega_j)\big)(X) &= \sum_{j=1}^m \omega_j(df(X)) \cdot \mu_j(f(p) \\ &= (Ds)(df(X)) = D_{df(X)}(s). \end{aligned}$$

Hence (185) holds, and so we have proved that (183) holds when $r = 0$.

Finally we prove (183) for $r \geq 1$. By $\mathbb{R}$-linearity, it is enough to check that (183) holds when $s = \mu \otimes \omega$ for some $\mu \in \Gamma E$, $\omega \in \Omega^r(N)$. Then:

$$\begin{aligned} f^*(d^D s) &= f^*(d^D(\mu \otimes \omega)) \\ &= f^*(D\mu \wedge \omega + \mu \otimes d\omega) \\ &\overset{[1]}{=} f^*(D\mu) \wedge f^*(\omega) + (\mu \circ f) \otimes f^*(d\omega) \\ &\overset{[2]}{=} \big((f^*D)(\mu \circ f)\big) \wedge f^*(\omega) + (\mu \circ f) \otimes d\, f^*(\omega) \\ &= (d^{f^*D})\big((\mu \circ f) \otimes f^*\omega\big) \\ &= (d^{f^*D})\big(f^*(\mu \otimes \omega)\big) \\ &= (d^{f^*D})(f^*(s)). \end{aligned}$$

(Here equality [1] holds by $\mathbb{R}$-linearity and (184), and equality [2] holds by [(183) for $r = 0$].) Hence we have proved that (183) holds for any $r \geq 0$. $\square$

Problem 70: Let ∇ be the Levi-Civita connection for $(M, \langle\cdot,\cdot\rangle)$. Then ∇ is metric also wrt $[\cdot,\cdot]$, since for any vector fields $X, Y \in \Gamma(TM)$ we have

$$\begin{aligned} d[X,Y] = d(c\langle X,Y\rangle) = c\cdot d\langle X,Y\rangle &= c\cdot\langle\nabla X, Y\rangle + c\cdot\langle X, \nabla Y\rangle \\ &= [\nabla X, Y] + [X, \nabla Y] \end{aligned}$$

in $\Omega^1(M)$. Also ∇ is torsion free (recall that this notion is independent of the Riemannian metric). Hence, by the uniqueness in Theorem 1 in #13, ∇ is the Levi-Civita connection also for $(M, [\cdot,\cdot])$.

Hence $(M, \langle\cdot,\cdot\rangle)$ and $(M, [\cdot,\cdot])$ also have the same curvature tensor, $R = \nabla\circ\nabla \in \Omega^2(\operatorname{End} TM)$. (However the tensor field "Rm" – cf. p. 1 in Lecture #14 – is *not* the same for $(M, \langle\cdot,\cdot\rangle)$ and $(M, [\cdot,\cdot])$, since its definition involves the inner product.) Now from Definition 1 in #15 it follows that, if K and $\widetilde{K}$ denote sectional curvature on $(M, \langle\cdot,\cdot\rangle)$ and on $(M, [\cdot,\cdot])$, respectively, then for any $p \in M$ and any linearly independent $X, Y \in T_pM$,

$$\widetilde{K}(X\wedge Y) = \frac{[R(X,Y)Y, X]}{[X\wedge Y, X\wedge Y]} = \frac{c\,\langle R(X,Y)Y, X\rangle}{c^2\,\langle X\wedge Y, X\wedge Y\rangle)} = c^{-1}\,K(X\wedge Y).$$

□

Example: Let $(M, \langle\cdot,\cdot\rangle)$ be the standard unit sphere S^d in $\mathbb{R}^{d+1}$, and let $[\cdot,\cdot]$ be the Riemannian metric obtained by instead using the embedding $x \mapsto Rx$ of S^d into $\mathbb{R}^{d+1}$, for some fixed $R > 0$ (still using the standard Riemannian metric on $\mathbb{R}^{d+1}$). In other words $(M, [\cdot,\cdot])$ is the sphere of radius R in $\mathbb{R}^{d+1}$. Then $[\cdot,\cdot] = R^2\langle\cdot,\cdot\rangle$ and thus

$$\widetilde{K}(X\wedge Y) = R^{-2}\,K(X\wedge Y)$$

for any X, Y as above. This agrees with the fact that the sphere of radius R has constant sectional curvature R^{-2}.

Problem 71: Let $p \in M$ and $X \in T_pM$ with $\|X\| = 1$. Choose an ON basis $X_1, \ldots, X_d$ of T_pM with $X_d = X$. Then the uniform average of the sectional curvatures of all planes in T_pM containing X equals:

$$A = \frac{1}{\omega(S^{d-2})} \int_{S^{d-2}} K\big(X_d \wedge (\alpha_1 X_1 + \cdots + \alpha_{d-1} X_{d-1})\big)\, d\omega(\boldsymbol{\alpha}),$$

where $S^{d-2} = \{\boldsymbol{\alpha} \in \mathbb{R}^{d-1} \,:\, \alpha_1^2 + \cdots + \alpha_{d-1}^2 = 1\}$ is the standard $d-2$ dimensional unit sphere and ω is the is its standard volume measure (cf., e.g., [6, Thm. 2.49]). Using the fact that for any $\boldsymbol{\alpha} \in S^{d-2}$, the two vectors X_d and $\alpha_1 X_1 + \cdots + \alpha_{d-1} X_{d-1}$ in T_pM are orthogonal and have unit length, we get

$$\begin{aligned} A &= \frac{1}{\omega(S^{d-2})} \int_{S^{d-2}} \Big\langle R\Big(X_d, \sum_{j=1}^{d-1} \alpha_j X_j\Big) \sum_{j=1}^{d-1} \alpha_j X_j, X_d \Big\rangle\, d\omega(\boldsymbol{\alpha}) \\ &= \frac{1}{\omega(S^{d-2})} \sum_{j=1}^{d-1} \sum_{k=1}^{d-1} \int_{S^{d-2}} \Big\langle R\Big(X_d, \alpha_j X_j\Big) \alpha_k X_k, X_d \Big\rangle\, d\omega(\boldsymbol{\alpha}). \\ &= \sum_{j=1}^{d-1} \sum_{k=1}^{d-1} \langle R(X_d, X_j) X_k, X_d \rangle \frac{1}{\omega(S^{d-2})} \int_{S^{d-2}} \alpha_j \alpha_k\, d\omega(\boldsymbol{\alpha}). \end{aligned}$$

Here we note that for any $j \neq k \in \{1, \ldots, d-1\}$ we have $\int_{S^{d-2}} \alpha_j \alpha_k\, d\omega(\boldsymbol{\alpha}) = 0$, since the measure ω is preserved by the reflection in the hyperplane $\alpha_j = 0$. On the other hand for each $j \in \{1, \ldots, d-1\}$, the integral $\omega(S^{d-2})^{-1} \int_{S^{d-2}} \alpha_j^2\, d\omega(\boldsymbol{\alpha})$ equals a constant which is independent of j, since ω is invariant under any permutation of the coordinates $\alpha_1, \ldots, \alpha_{d-1}$. Let us define C_d by

$$C_d^{-1} := \frac{1}{\omega(S^{d-2})} \int_{S^{d-2}} \alpha_j^2\, d\omega(\boldsymbol{\alpha})$$

(any $j \in \{1, \ldots, d-1\}$); this is a positive number which only depends on the dimension d.

(For $d = 2$ we immediately compute $C_2 = 1$; indeed note that in this case $S^{d-2} = \{1, -1\} \subset \mathbb{R}$ and $\omega(\{1\}) = \omega(\{-1\}) = \frac{1}{2}$. Furthermore for $d = 3$ we have $C_3^{-1} = (2\pi)^{-1} \int_0^{2\pi} (\cos\varphi)^2\, d\varphi = \frac{1}{2}$, i.e. $C_3 = 2$.)

We get:

$$\begin{aligned} A = C_d^{-1} \sum_{j=1}^{d-1} \langle R(X_d, X_j) X_j, X_d \rangle &= C_d^{-1} \sum_{j=1}^{d} \langle R(X_d, X_j) X_j, X_d \rangle \\ &= C_d^{-1} \cdot \mathrm{Ric}(X, X), \end{aligned}$$

where in the second equality we used the fact that $\langle R(X_d, X_d) X_d, X_d \rangle = 0$, and in the last equality we used the definition of the Ricci tensor, Def. 2 in Lecture #15. (Details for the last step: Fix any chart (U, x) on

M with $p \in U$, such that $X_j = \frac{\partial}{\partial x^j}$ at p, for $j = 1, \ldots, d$. Then (g_{ij}) equals the identity matrix at p, and hence also the inverse matrix, (g^{ij}) equals the identity matrix. Hence $\mathrm{Ric}(X, X) = g^{j\ell}\langle R(X, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^\ell}, X\rangle = \sum_{j=1}^d \langle R(X, \frac{\partial}{\partial x^j})\frac{\partial}{\partial x^j}, X\rangle = \sum_{j=1}^d \langle R(X_d, X_j)X_j, X_d\rangle$, as claimed.)

Summing up, we have proved

$$\mathrm{Ric}(X, X) = C_d \cdot A,$$

and this is the desired formula. It only remains to compute C_d. Since C_d only depends on the dimension d, it can be conveniently computed by considering any manifold of *constant* sectional curvature ($\neq 0$). For example let M be the unit sphere S^d with its standard Riemannian metric. Then the sectional curvature is everywhere equal to 1, and so $A = 1$ for any $p \in M$ and any unit vector $X \in T_pM$. On the other hand by again choosing an ON-basis $X_1, \ldots, X_d \in T_pM$ with $X_d = X$ then as above we have $\mathrm{Ric}(X, X) = \sum_{j=1}^d \langle R(X_d, X_j)X_j, X_d\rangle = \sum_{j=1}^{d-1} 1 = d - 1$. Hence:

$$C_d = d - 1.$$

□

Remark: It is also somewhat satisfactory to compute C_d directly from its definition. Using basic properties of ω, in particular $\omega(S^{d-2}) = \frac{2\pi^{(d-1)/2}}{\Gamma((d-1)/2)}$, we get:

$$\begin{aligned} C_d^{-1} &= \frac{\Gamma(\frac{d-1}{2})}{2\pi^{(d-1)/2}} \int_{-1}^1 \alpha_1^2 \cdot \frac{2\pi^{(d-2)/2}}{\Gamma(\frac{d-2}{2})} (1 - \alpha_1^2)^{\frac{d-4}{2}} \, d\alpha_1 \\ &= \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\,\Gamma(\frac{d-2}{2})} \cdot 2 \int_0^1 x(1-x)^{\frac{d-4}{2}} \, \frac{dx}{2\sqrt{x}} \\ &= \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\,\Gamma(\frac{d-2}{2})} \cdot \int_0^1 x^{\frac{1}{2}}(1-x)^{\frac{d-4}{2}} \, dx \\ &= \frac{\Gamma(\frac{d-1}{2})}{\sqrt{\pi}\,\Gamma(\frac{d-2}{2})} \cdot \frac{\Gamma(\frac{3}{2})\Gamma(\frac{d-2}{2})}{\Gamma(\frac{d+1}{2})} \\ &= \frac{1}{d-1}. \end{aligned}$$

□

Problem 72: First note that, for any $X, Y, Z \in V$:

$$2\mathcal{R}(X, Z, Z, Y) = K(X + Y, Z) - K(X, Z) - K(Y, Z).$$

Next, for any $X, Y, Z, W \in V$:

$$\begin{aligned}
&\mathcal{R}(X, Z, W, Y) + \mathcal{R}(X, W, Z, Y) \\
&= \mathcal{R}(X, Z + W, Z + W, Y) - \mathcal{R}(X, Z, Z, Y) - \mathcal{R}(X, W, W, Y) \\
&= \tfrac{1}{2}\big(K(X + Y, Z + W) - K(X, Z + W) - K(Y, Z + W)\big) \\
&\qquad - \tfrac{1}{2}\big(K(X + Y, Z) - K(X, Z) - K(Y, Z)\big) \\
&\qquad - \tfrac{1}{2}\big(K(X + Y, W) - K(X, W) - K(Y, W)\big).
\end{aligned}$$

Using also $\mathcal{R}(X, W, Z, Y) = -\mathcal{R}(W, X, Z, Y)$, the above identity can be rewritten as

$$\begin{aligned}
2\mathcal{R}(X, Z, W, Y) =& 2\mathcal{R}(W, X, Z, Y) \\
& + \big(K(X + Y, Z + W) - K(X, Z + W) - K(Y, Z + W)\big) \\
& - \big(K(X + Y, Z) - K(X, Z) - K(Y, Z)\big) \\
& - \big(K(X + Y, W) - K(X, W) - K(Y, W)\big).
\end{aligned} \tag{186}$$

Here is the same identity with X, Z, W cyclically permuted:

$$\begin{aligned}
2\mathcal{R}(W, X, Z, Y) =& 2\mathcal{R}(Z, W, X, Y) \\
& + \big(K(W + Y, X + Z) - K(W, X + Z) - K(Y, X + Z)\big) \\
& - \big(K(W + Y, X) - K(W, X) - K(Y, X)\big) \\
& - \big(K(W + Y, Z) - K(W, Z) - K(Y, Z)\big).
\end{aligned} \tag{187}$$

We rewrite (186) by solving for $\mathcal{R}(W, X, Z, Y)$:

$$\begin{aligned}
2\mathcal{R}(W, X, Z, Y) =& 2\mathcal{R}(X, Z, W, Y) \\
& - \big(K(X + Y, Z + W) - K(X, Z + W) - K(Y, Z + W)\big) \\
& + \big(K(X + Y, Z) - K(X, Z) - K(Y, Z)\big) \\
& + \big(K(X + Y, W) - K(X, W) - K(Y, W)\big).
\end{aligned} \tag{188}$$

Now add (187) and (188) and add an extra $2\mathcal{R}(W,X,Z,Y)$ on both sides, and then use the first Bianchi identity in the right hand side. This gives:

$$
\begin{aligned}
6\mathcal{R}(W,X,Z,Y) = {} & \big(K(W+Y,X+Z) - K(W,X+Z) - K(Y,X+Z)\big) \\
& - \big(K(W+Y,X) - K(W,X) - K(Y,X)\big) \\
& - \big(K(W+Y,Z) - K(W,Z) - K(Y,Z)\big) \\
& - \big(K(X+Y,Z+W) - K(X,Z+W) - K(Y,Z+W)\big) \\
& + \big(K(X+Y,Z) - K(X,Z) - K(Y,Z)\big) \\
& + \big(K(X+Y,W) - K(X,W) - K(Y,W)\big) \\
= {} & K(W+Y,X+Z) - K(W,X+Z) - K(Y,X+Z) \\
& - K(W+Y,X) + K(Y,X) \\
& - K(W+Y,Z) + K(W,Z) \\
& - K(X+Y,Z+W) + K(X,Z+W) + K(Y,Z+W) \\
& + K(X+Y,Z) - K(X,Z) \\
& + K(X+Y,W) - K(Y,W).
\end{aligned}
\tag{189}
$$

This is an explicit formula for $\mathcal{R}$ in terms of K! Changing letters ($W \to X$, $X \to Y$, $Y \to W$) the formula reads:

$$
\begin{aligned}
6 \cdot \mathcal{R}(X,Y,Z,W) = {} & K(X+W,Y+Z) - K(X,Y+Z) - K(W,Y+Z) \\
& - K(X+W,Y) + K(W,Y) \\
& - K(X+W,Z) + K(X,Z) \\
& - K(Y+W,Z+X) + K(Y,Z+X) + K(W,Z+X) \\
& + K(Y+W,Z) - K(Y,Z) \\
& + K(Y+W,X) - K(W,X),
\end{aligned}
$$

which is exactly the formula which Jost states in his [12, Lemma 4.3.3], except for the factor "6" in the left hand side. □

Problem 73: Cf., e.g., [13, p. 292, Thm. 1] or [14, Prop. 7.8].

Assume that there is a function $c : M \to \mathbb{R}$ such that (wrt any C^∞ chart (U, x) on M):

$$R_{ik} = c \cdot g_{ik}. \tag{190}$$

(Cf. the note on p. 5 in Lecture #15.)

Fix a point $p \in M$ and assume that (U, x) are normal coordinates around p. Then by the second Bianchi identity,

$$\partial_h R_{ijk\ell} + \partial_k R_{ij\ell h} + \partial_\ell R_{ijhk} = 0 \qquad \text{at } p.$$

(Here $\partial_h := \frac{\partial}{\partial x^h}$.) Multiply the above relation with $g^{ik} g^{j\ell}$ and add over all i, k, j, ℓ; this gives:

$$g^{ik} g^{j\ell} \cdot \partial_h R_{ijk\ell} + g^{ik} g^{j\ell} \cdot \partial_k R_{ij\ell h} + g^{ik} g^{j\ell} \cdot \partial_\ell R_{ijhk} = 0 \qquad \text{at } p.$$

However we have $\partial_h g^{ik} = 0$ at p, for all h, i, k; hence the above relation is equivalent with:

$$\partial_h \big(g^{ik} g^{j\ell} R_{ijk\ell} \big) + \partial_k \big(g^{ik} g^{j\ell} R_{ij\ell h} \big) + \partial_\ell \big(g^{ik} g^{j\ell} R_{ijhk} \big) = 0 \qquad \text{at } p. \tag{191}$$

Recall now that $g^{j\ell} R_{ijk\ell} = R_{ik}$, by definition. Hence using also (190) we get

$$g^{ik} g^{j\ell} R_{ijk\ell} = g^{ik} R_{ik} = g^{ik} \cdot c \cdot g_{ik} = c \cdot \delta^i_i = d \cdot c,$$

where $d := \dim M$. Also

$$g^{ik} g^{j\ell} R_{ij\ell h} = -g^{ik} g^{j\ell} R_{ijh\ell} = -g^{ik} R_{ih} = -g^{ik} \cdot c \cdot g_{ih} = -\delta^k_h \cdot c.$$

and

$$g^{ik} g^{j\ell} R_{ijhk} = -g^{ik} g^{j\ell} R_{jihk} = -g^{j\ell} R_{jh} = -g^{j\ell} \cdot c \cdot g_{jh} = -\delta^\ell_h \cdot c$$

Substituting these relations in (191) we obtain, at p:

$$0 = d \cdot \partial_h c - \partial_h c - \partial_h c = (d-2) \partial_h c.$$

Hence since we are assuming $d \geq 3$, we conclude that $\partial_h c = 0$ at p. This is true for all h; hence $dc_p = 0$. This is true for all $p \in M$; hence c is *constant,* qed. □

Problem 76:

(a) By Problem 35(c), every vector in $E_{f(p)}$ can be obtained as $s(f(p))$ for some $s \in \Gamma E$; hence it suffices to prove that for any $s \in \Gamma E$:

$$\widetilde{R}(s \circ f)(X_1, X_2) = (Rs)(df(X_1), df(X_2)) \qquad \text{in } (f^*E)_p = E_{f(p)}, \quad \forall p \in M,\ X, Y \in T_p(M).$$

Using (for $r = 2$) the map $f^* : \Omega^r(E) \to \Omega^r(f^*E)$ defined in Problem 68(b), the above relation can be expressed:

$$\widetilde{R}(s \circ f) = f^*(Rs) \qquad \text{in } \Omega^2(f^*E).$$

By the definition of $f^* : \Omega^0(E) \to \Omega^0(f^*E)$, this can also be expressed (slightly more nicely?) as

$$\widetilde{R}(f^*(s)) = f^*(Rs) \qquad \text{in } \Omega^2(f^*E). \tag{192}$$

Recall that, by definition,

$$\widetilde{R} = d^{f^*D} \circ f^*D \qquad \text{and} \qquad R = d^D \circ D.$$

(Here, as in Problem 68(b), we write d^{f^*D} and d^D for the exterior covariant derivatives, and not just "f^*D" and "D" as we usually do.) Now we compute, for any $s \in \Gamma E$:

$$\widetilde{R}(f^*(s)) = d^{f^*D}\big((f^*D)(f^*s)\big)$$

$\big[$Apply Problem 57(a); cf. also Problem 68(b).$\big]$

$$= d^{f^*D}\big(f^*(Ds)\big)$$

$\big[$Apply Problem 68(b).$\big]$

$$= f^*\Big(d^D(Ds)\Big)$$
$$= f^*(R(s)).$$

Hence (192) is proved! □

(b). Here we are considering a C^∞ map $c \equiv F : H \to M$, where

$$H = [a,b] \times (-\varepsilon, \varepsilon),$$

and where M is a Riemannian manifold with ∇ being the Levi-Civita connection on TM. (Actually, in order not to have to consider a manifold with boundary, we should instead take $H = (a - \varepsilon', b + \varepsilon') \times (-\varepsilon, \varepsilon)$; cf. section 3.1 in the lecture notes.) Also (t, s) are the standard coordinates on H; thus $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial s}$ are well-defined vector fields in $\Gamma(TH)$. Finally recall that in "$\nabla_{\frac{\partial}{\partial t}}$" and "$\nabla_{\frac{\partial}{\partial s}}$", the "$\nabla$" is really a short-hand notation for the pullback connection $F^*\nabla$ on $\Gamma(F^*(TM))$. Hence our task is to prove:

$$R\Big(\frac{\partial}{\partial t}F, \frac{\partial}{\partial s}F\Big)\Big(\frac{\partial}{\partial s}F\Big)$$

$$= (F^*\nabla)_{\frac{\partial}{\partial s}}(F^*\nabla)_{\frac{\partial}{\partial t}}\Big(\frac{\partial}{\partial s}F\Big) - (F^*\nabla)_{\frac{\partial}{\partial t}}(F^*\nabla)_{\frac{\partial}{\partial s}}\Big(\frac{\partial}{\partial s}F\Big). \tag{193}$$

(Here $\frac{\partial}{\partial s}F = dF \circ \frac{\partial}{\partial s}$ and $\frac{\partial}{\partial t}F = dF \circ \frac{\partial}{\partial t}$ are sections in $\Gamma_F(TM) = \Gamma(F^*(TM))$, so that the expression in the right-hand side makes sense. Note also that the expressions on both sides of the equality are sections in $\Gamma_F(TM) = \Gamma(F^*(TM))$.)

From now on let us use the short-hand notation $\partial_t := \frac{\partial}{\partial t}$, $\partial_s := \frac{\partial}{\partial s}$.

Note that for every $p \in H$ we have $(\partial_t F)(p) = dF(\partial_t(p))$ and $(\partial_s F)(p) = dF(\partial_s(p))$ in $T_{F(p)}(M)$. Hence by part a,

$$R(\partial_t F, \partial_s F)(\partial_s F) = \widetilde{R}\big(\partial_t, \partial_s\big)(\partial_s F) \qquad \text{in } \Gamma_F(TM),$$

where $\widetilde{R}$ is the curvature of the connection $F^*\nabla$ on $F^*(TM)$. However by Theorem 1 in Lecture #11, using $[\partial_t, \partial_s] \equiv 0$, we have

$$\widetilde{R}\big(\partial_t, \partial_s\big)(\partial_s F) = (F^*\nabla)_{\partial_t}\big((F^*\nabla)_{\partial_s}(\partial_s F)\big) - (F^*\nabla)_{\partial_s}\big((F^*\nabla)_{\partial_t}(\partial_s F)\big).$$

This proves (193)! $\square$

Problem 78: (Cf. [14, Prop. 10.9].) Recall that the fact that the chart (U,x) gives normal coordinates means that there is some $r>0$ such that $\exp_{p|B_r(0)}$ is a diffeomorphism onto U, where $B_r(0)$ is the open ball of radius r about the origin in $T_p(M)$ (wrt the Riemannian metric $\langle\cdot,\cdot\rangle$); also we fix an identification $T_p(M)=\mathbb{R}^d$ which carries $\langle\cdot,\cdot\rangle$ on $T_p(M)$ to the standard scalar product on $\mathbb{R}^d$; thus $B_r(0)$ is now the open ball of radius r about the origin in $\mathbb{R}^d$; and finally $x := (\exp_{p|B_r(0)})^{-1} : U \to \mathbb{R}^d$, with image $x(U)=B_r(0)$.

Now fix a point $x \in B_r(0)\setminus\{0\}$. Set $T := \|x\| \in (0,r)$ and consider the geodesic

$$c : [0,T] \to M, \qquad c(t) = \exp_p\big(t\|x\|^{-1}x\big).$$

Note that c is parametrized by arc length, i.e. $\|\dot c(t)\| = 1$ for all $t\in[0,T]$. Using the chart (U,x) to *identify* U and $B_r(0)$, the map $\exp_p : B_r(0)\to U$ becomes simply the *identity map* on $B_r(0)$; the geodesic c becomes $c(t)=tx$, and also for any $w\in B_r(0)$ the differential of $\exp_p : T_p(M)\to M$ at w,

$$(d\exp_p)_w :\, T_w(T_p(M)) = \mathbb{R}^d \to T_{\exp_p(w)}(M) = \mathbb{R}^d, \tag{194}$$

gets[47] identified with the identity map on $\mathbb{R}^d$. Hence by Cor. 1 in Lecture #17, for any $v\in\mathbb{R}^d$ the formula

$$X(t) := t\cdot v \in T_{c(t)}(M) = \mathbb{R}^d \qquad (t\in[0,T]) \tag{195}$$

defines a Jacobi field along c.

The Riemannian metric $\langle\cdot,\cdot\rangle$ on U carries over to a Riemannian metric $\langle\cdot,\cdot\rangle$ on $B_r(0)$, which is given by

$$\langle v,w\rangle = g_{ij}(x)v^i w^j$$

for any $x\in B_r(0)$ and $v,w\in\mathbb{R}^d$.

Let us first assume that the vector v in (195) satisfies $v\cdot x = 0$. Since $g_{ij}(0)=\delta_{ij}$ (by Lemma 1 in #4), this implies that v and $\|x\|^{-1}x$ are orthogonal when viewed as tangent vectors in $T_p(M)$. Note also $\dot c(0) = \|x\|^{-1}x$; hence by "Gauss' Lemma" (Cor. 2 in Lecture #17), $\langle X(t),\dot c(t)\rangle = 0$ for all $t\in[0,T]$, i.e. X is a *normal* Jacobi field along c. Hence by the discussion on pp. 5–6 in Lecture #17 we have

$$X(t) = s_\rho(t)\cdot X_1(t), \qquad \forall t\in[0,T], \tag{196}$$

where $X_1(t)$ is a *parallel* vector field along c, and

$$s_\rho(t) = \begin{cases} \rho^{-1/2}\sin(\rho^{1/2}t) & \text{if } \rho>0 \\ t & \text{if } \rho = 0 \\ |\rho|^{-1/2}\sinh(|\rho|^{1/2}t) & \text{if } \rho<0. \end{cases}$$

[47]In (194), the last identification "$T_{\exp_p(w)}(M)=\mathbb{R}^d$" of course comes from using the basis of sections $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^d}\in\Gamma(TU)$, at the point $\exp_p(w)\in U$.

Using $X(t) = t \cdot v$ and $g_{ij}(0) = \delta_{ij}$ we see that $\langle t^{-1}X(t), t^{-1}X(t)\rangle \to \|v\|^2$ as $t \to 0^+$. Combining this with (196) and $t^{-1}s_\rho(t) \to 1$ as $t \to 0^+$ we conclude that $\langle X_1(t), X_1(t)\rangle \to \|v\|^2$ as $t \to 0^+$. However $\langle X_1(t), X_1(t)\rangle$ is independent of t since X_1 is parallel along c; hence

$$\langle X_1(t), X_1(t)\rangle = \|v\|^2, \qquad \forall t \in [0, T].$$

In particular, since $T \cdot v = X(T) = s_\rho(T) \cdot X_1(T)$, it follows that for v viewed as a vector in $T_x(M) = T_{c(T)}(M)$ (recall $T = \|x\|$):

$$\langle v, v\rangle = T^{-2}s_\rho(T)^2\|v\|^2 = \left\{\begin{array}{ll} \dfrac{\sin^2(\rho^{1/2}\|x\|)}{\rho\,\|x\|^2} & \text{if } \rho > 0 \\ 1 & \text{if } \rho = 0 \\ \dfrac{\sinh^2(|\rho|^{1/2}\|x\|)}{|\rho|\,\|x\|^2} & \text{if } \rho < 0. \end{array}\right\} \cdot \|v\|^2.$$

On the other hand if v is proportional to x then we know (by Gauss' Lemma or by Problem 23 = [12]) that for v viewed as a vector in $T_{c(t)}(M)$ for any $t \in [0, T]$,

$$\langle v, v\rangle = \|v\|^2.$$

In particular this holds at $x = c(T)$.

Finally let v be an arbitrary vector in $\mathbb{R}^d$. We can then write $v = ax + w$ where $a = \|x\|^{-2}(v \cdot x)$ and $w = v - ax$; then ax is proportional to x while $v \cdot w = 0$. It follows from Gauss' Lemma (or Problem 23 = [12]) that $\langle ax, w\rangle = 0$ when ax and w are viewed as vectors in $T_x(M)$. Hence

$$\langle v, v\rangle = \langle ax, ax\rangle + \langle w, w\rangle = \frac{(v \cdot x)^2}{\|x\|^2} + \left\{\begin{array}{ll} \dfrac{\sin^2(\rho^{1/2}\|x\|)}{\rho\,\|x\|^2} & \text{if } \rho > 0 \\ 1 & \text{if } \rho = 0 \\ \dfrac{\sinh^2(|\rho|^{1/2}\|x\|)}{|\rho|\,\|x\|^2} & \text{if } \rho < 0. \end{array}\right\} \cdot \|w\|^2.$$

Note here that

$$\|w\|^2 = \|v\|^2 - \frac{(v \cdot x)^2}{\|x\|^2}.$$

By polarization (i.e. using $\langle v, v'\rangle = \frac{1}{2}(\langle v+v', v+v'\rangle - \langle v, v\rangle - \langle v', v'\rangle))$, the above formula leads to

$$\langle v, v'\rangle = \frac{(v\cdot x)(v'\cdot x)}{\|x\|^2} + \left\{\begin{array}{ll} \dfrac{\sin^2(\rho^{1/2}\|x\|)}{\rho\,\|x\|^2} & \text{if } \rho > 0 \\ 1 & \text{if } \rho = 0 \\ \dfrac{\sinh^2(|\rho|^{1/2}\|x\|)}{|\rho|\,\|x\|^2} & \text{if } \rho < 0. \end{array}\right\} \cdot \Big(v\cdot v' - \frac{(v\cdot x)(v'\cdot x)}{\|x\|^2}\Big)$$

for any $v, v' \in \mathbb{R}^d$ viewed as vectors in $T_x(M)$. Inserting here $v = e_i$, $v' = e_j$ we conclude:

$$g_{ij}(x) = \langle e_i, e_j\rangle = \begin{cases} \dfrac{x_i x_j}{\|x\|^2} + \dfrac{\sin^2(\rho^{1/2}\|x\|)}{\rho\,\|x\|^2}\Big(\delta_{ij} - \dfrac{x_i x_j}{\|x\|^2}\Big) & \text{if } \rho > 0 \\ \delta_{ij} & \text{if } \rho = 0 \\ \dfrac{x_i x_j}{\|x\|^2} + \dfrac{\sinh^2(|\rho|^{1/2}\|x\|)}{|\rho|\,\|x\|^2}\Big(\delta_{ij} - \dfrac{x_i x_j}{\|x\|^2}\Big) & \text{if } \rho < 0, \end{cases}$$

which is the desired formula.

(Using the fact that the Taylor series for $(\frac{\sin r}{r})^2$ and for $(\frac{\sinh r}{r})^2$ has the form $1 + c_1 r^2 + c_2 r^4 + \cdots$, which converges for all $r \in \mathbb{R}$, one immediately verifies that the last expression is C^∞ also at $x = 0$, if extended by continuity to this point.) □

Problem 79:

For symmetry reasons we may assume $i = 1$, $j = 2$. Since (U, x) gives normal coordinates, we know that $x(U)$ is an open ball in $\mathbb{R}^d$ centered at the origin; take $R > 0$ so that $x(U) = B_R(0)$. As usual we identify U and $B_R(0)$ via x. Let us furthermore introduce the short-hand notation "(x, y)" for $(x, y, 0, \ldots, 0) \in \mathbb{R}^d$.

Given a point (x, y) with $0 < \|(x, y)\| < R$, we consider the two tangent vectors $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ and $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ in $T_{(x,y)}(M)$. We note that for an arbitrary choice of polar coordinates $(r, \theta_1, \ldots, \theta_{d-1})$ on $\mathbb{R}^d$ [48], these two vectors are given by

$$x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} = r\frac{\partial}{\partial r}, \quad \text{and} \quad -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} = \sum_{j=1}^{d-1} \alpha_j \frac{\partial}{\partial \theta_j} \tag{197}$$

for some $\alpha_1, \ldots, \alpha_{d-1} \in \mathbb{R}$ (which depend on (x, y) and on the choice of polar coordinates). The first formula in (197) follows from the fact that $x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ is the tangent vector of the curve $c(t) := (tx, ty, 0, \ldots, 0)$ at $t = 1$, and in polar coordinates this curve is given by $c(t) = (tr, \theta_1, \ldots, \theta_{d-1})$ for some *fixed* $\theta_1, \ldots, \theta_{d-1}$. Similarly the second formula in (197) follows from the fact that $-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$ is the tangent vector of the curve $\gamma(t) = (r\cos t, r\sin t)$ at a certain $t = t_0$, where $r := \|(x, y)\| = \sqrt{x^2 + y^2}$, and in polar coordinates this curve takes the form $\gamma(t) = (r, \theta_1(t), \ldots, \theta_{d-1}(t))$ where $r = \|(x, y)\|$ is fixed and $\theta_1(t), \ldots, \theta_{d-1}(t)$ are some smooth real-valued functions of t.

It follows from (197) and Problem 23 that, for any point (x, y) in the punctured disc $0 < \|(x, y)\| < R$,

$$\Big\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} \Big\rangle = r^2 = x^2 + y^2 \tag{198}$$

and

$$\Big\langle x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}, -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \Big\rangle = 0, \tag{199}$$

as stated in the hint to the problem. Note that (198) and (199) are also valid at $(x, y) = (0, 0)$, by inspection (or by continuity). [Remark on notation: In (198) and (199), "$\langle \cdot, \cdot \rangle$" denotes the Riemannian scalar product on $T_{(x,y)}(M)$, whereas we use "$\|(x, y)\|$" to denote the standard Euclidean norm, $\sqrt{x^2 + y^2}$.]

[48]by this we mean: We have fixed a chart (V, φ) on S^{d-1} and we then consider the corresponding chart $(\mathbb{R}^+ V, (r, \theta_1, \ldots, \theta_{d-1}))$ on $\mathbb{R}^d$, where $r(z) = \|z\|$ (standard Euclidean length of the vector z) and $(\theta_1(z), \ldots, \theta_{d-1}(z)) = \varphi(\|z\|^{-1} z)$ for all z in the open cone $\mathbb{R}^+ V$. This is just as in Problem 23, but with different variable names. Of course, we assume that the given point $(x, y) = (x, y, 0, \ldots, 0)$ lies in the cone $\mathbb{R}^+ V$.

Expanding the left hand sides of (198) and (199) we get

$$x^2 g_{11} + 2xy g_{12} + y^2 g_{22} = x^2 + y^2 \tag{200}$$

and

$$-xy g_{11} + (x^2 - y^2) g_{12} + xy g_{22} = 0, \tag{201}$$

where g_{ij} of course stands for $g_{ij}(x, y) = g_{ij}(x, y, 0, \ldots, 0)$. The relations (200) and (201) are valid for all (x, y) with $\|(x, y)\| < R$.

Taking $y = 0$ in (200) we obtain $g_{11}(x, 0) = 1$ for all $x \in (-R, R) \setminus \{0\}$; and by continuity this is also valid for $x = 0$. This implies that all iterated derivatives $g_{11,1}(x, 0)$, $g_{11,11}(x, 0)$, $g_{11,111}(x, 0)$, ..., vanish identically for all $x \in (-R, R)$. In particular

$$g_{11,11}(0) = 0,$$

as desired. Note that a symmetric argument (exchanging the roles of x and y) also gives $g_{22}(0, y) = 1$ and $g_{22,2}(0, y) = g_{22,22}(0, y) = \ldots = 0$ for all $y \in (-R, R)$.

Differentiating (201) with respect to x we get

$$-y g_{11} - xy g_{11,1} + 2x g_{12} + (x^2 - y^2) g_{12,1} + y g_{22} + xy g_{22,1} = 0,$$

and differentiating this three times with respect to y gives[49]

$$y \cdot \boxed{*} + x \cdot \boxed{*} - 3 g_{11,22} - 6 g_{12,12} + 3 g_{22,22} = 0,$$

where each "$\boxed{*}$" stands for a sum where each term is a polynomial in x and y times a partial derivative of g_{11} or g_{12} or g_{22}. The above is valid for all (x, y) in the disc $\|(x, y)\| < R$. Setting now $(x, y) = (0, 0)$, and using $g_{22,22}(0, 0) = 0$, we conclude that

$$g_{11,22}(0, 0) = -2 g_{12,12}(0).$$

By the symmetric argumment ($x \leftrightarrow y$) we also have

$$g_{22,11}(0, 0) = -2 g_{12,12}(0).$$

□

[49] using the general Leibniz rule; specifically
$\left(\frac{\partial}{\partial y}\right)^3 \big(a(y)b(y)\big) = a'''(y)b(y) + 3a''(y)b'(y) + 3a'(y)b''(y) + a(y)b'''(y).$

Problem 80:

(See wikipedia; Bertrand-Diquet-Puiseux Theorem.)

Let the chart (U, z) on M be normal coordinates centered at p such that $X := \frac{\partial}{\partial z^1}(p)$ and $Y := \frac{\partial}{\partial z^2}(p)$ form an ON-basis of Π. (Such a chart is easily obtained by choosing the identification between T_pM and $\mathbb{R}^d$ appropriately in the construction of normal coordinates.) As usual, we will identify U with the open ball $z(U) \subset \mathbb{R}^d$ (via z). In particular the submanifold $\exp_p(D_r)$ gets identified with

$$D_r = \{(x, y, 0, \ldots, 0) \,:\, x^2 + y^2 < r^2\}.$$

From now on we will write "(x, y)" as a short-hand for "$(x, y, 0, \ldots, 0)$" (denoting either a point in $\mathbb{R}^d = T_pM$ or a point in M).

Let the Riemannian metric be represented by $(g_{ij}(z))$ with respect to (U, z). Then the induced Riemannian metric on the submanifold $\exp_p(D_r)$ is represented by the 2×2 matrix function

$$\begin{pmatrix} g_{11}(x,y) & g_{12}(x,y) \\ g_{21}(x,y) & g_{22}(x,y) \end{pmatrix}, \qquad \forall (x, y) \in D_r.$$

(this is immediate from the definition of the induced Riemannian metric; cf. Problem 18). Hence by the definition of the volume measure on a Riemannian manifold (cf. #12, p. 1), we have

$$A_r = \int_{D_r} \sqrt{g_{11}(x,y)g_{22}(x,y) - g_{12}(x,y)^2}\, dx\, dy. \tag{202}$$

Recall that $g_{ij}(0) = \delta_{ij}$ and $g_{ij,k}(0) = 0$ for all $i, j, k \in \{1, \ldots, d\}$ (cf. Lemma 1 in Lecture #4). Hence we have the Taylor expansion

$$g_{ij}(x, y) = \delta_{ij} + \frac{g_{ij,11}(0)}{2}x^2 + g_{ij,12}(0)xy + \frac{g_{ij,22}(0)}{2}y^2 + O((x^2 + y^2)^{3/2})$$

for all (x, y) near $(0, 0)$. This gives

$$\begin{aligned} &g_{11}(x,y)g_{22}(x,y) - g_{12}(x,y)^2 \\ &= \Big(1 + \frac{g_{11,11}}{2}x^2 + g_{11,12}xy + \frac{g_{11,22}}{2}y^2\Big)\Big(1 + \frac{g_{22,11}}{2}x^2 + g_{22,12}xy + \frac{g_{22,22}}{2}y^2\Big) \\ &\qquad + O\Big((x^2 + y^2)^{3/2}\Big), \end{aligned}$$

where all the $g_{ij,kl}$'s in the right hand side are evaluated at 0. Multiplying out and using $g_{11,11}(0) = g_{22,22}(0) = 0$ (cf. Problem 79), we get

$$\begin{aligned} &g_{11}(x,y)g_{22}(x,y) - g_{12}(x,y)^2 \\ &= 1 + \frac{g_{22,11}}{2}x^2 + \big(g_{11,12} + g_{22,12}\big)xy + \frac{g_{11,22}}{2}y^2 + O\Big((x^2 + y^2)^{3/2}\Big). \end{aligned}$$

Hence, using the fact that $\sqrt{1+\alpha} = 1+\frac{1}{2}\alpha+O(\alpha^2)$ for α near 0, we conclude that for all (x,y) sufficiently near 0:

$$\sqrt{g_{11}(x,y)g_{22}(x,y)-g_{12}(x,y)^2}$$
$$= 1+\frac{g_{22,11}}{4}x^2+\frac{g_{11,12}+g_{22,12}}{2}xy+\frac{g_{11,22}}{4}y^2+O\Big((x^2+y^2)^{3/2}\Big).$$

Inserting this in (202), we note that the xy-term gives a 0-contribution, since the function xy is odd wrt x. Passing to polar coordinates we now get, for $r>0$ sufficiently small

$$A_r = \int_0^r \int_0^{2\pi} \Big(1+r_1^2\Big(\frac{g_{22,11}}{4}\cos^2\varphi+\frac{g_{11,22}}{4}\sin^2\varphi\Big)+O(r_1^3)\Big)\, r_1\, d\varphi\, dr_1$$
$$= \pi r^2 + \frac{\pi r^4}{16}\big(g_{22,11}+g_{11,22}\big)+O(r^5).$$

Hence

$$\lim_{r\to 0^+} 12\,\frac{\pi r^2 - A_r}{\pi r^4} = -\frac{3}{4}\big(g_{22,11}+g_{11,22}\big) = -\frac{3}{2}g_{11,22},$$

where the last equality holds by Problem 79. On the other hand we have

$$K(\Pi) = K(X\wedge Y) = K(X,Y) = \langle R(X,Y)Y,X\rangle = \Big\langle R^k_{212}\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^1}\Big\rangle$$
$$= R^1_{212} = \frac{1}{2}\big(g_{12,12}+g_{12,12}-g_{22,11}-g_{11,22}\big) = -\frac{3}{2}g_{11,22},$$

where at the end we used Lemma 3 of Lecture #14 and then the relations from Problem 79. Hence

$$\lim_{r\to 0^+} 12\,\frac{\pi r^2 - A_r}{\pi r^4} = K(\Pi),$$

as desired! □

Problem 81: Let $I : S \to \mathbb{R}^3$ be the inclusion map, i.e. $I(x,\alpha) = (x, f(x)\cos\alpha, f(x)\sin\alpha)$. Then

$$dI_{(x,\alpha)}\Big(\frac{\partial}{\partial x}\Big) = (1, f'(x)\cos\alpha, f'(x)\sin\alpha)$$

and

$$dI_{(x,\alpha)}\Big(\frac{\partial}{\partial \alpha}\Big) = (0, -f(x)\sin\alpha, f(x)\cos\alpha).$$

Hence on S we have (cf. Problem 18):

$$\begin{aligned}
\Big\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial x}\Big\rangle &= 1^2 + (f'(x)\cos\alpha)^2 + (f'(x)\sin\alpha)^2 = 1 + f'(x)^2;\\
\Big\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial \alpha}\Big\rangle &= f'(x)f(x)\big(-\cos\alpha\sin\alpha + \cos\alpha\sin\alpha\big) = 0;\\
\Big\langle \frac{\partial}{\partial \alpha}, \frac{\partial}{\partial \alpha}\Big\rangle &= f(x)^2.
\end{aligned}$$

In other words, the matrix representing the Riemannian metric on S wrt the (x,α) coordinates is:

$$g(x,\alpha) = \begin{pmatrix} 1 + f'(x)^2 & 0 \\ 0 & f(x)^2 \end{pmatrix}.$$

We also note that the inverse matrix is:

$$g(x,\alpha)^{-1} = \begin{pmatrix} (1 + f'(x)^2)^{-1} & 0 \\ 0 & f(x)^{-2} \end{pmatrix}.$$

From this, using the formula for the Christoffel symbols of the Levi-Civita connection,

$$\Gamma^i_{jk} = \tfrac{1}{2} g^{il}(g_{jl,k} + g_{kl,j} - g_{jk,l}),$$

we compute:

$$\Gamma^1_{11}(x,\alpha) = \frac{f'(x)f''(x)}{1+f'(x)^2}; \qquad \Gamma^2_{12}(x,\alpha) = \Gamma^2_{21}(x,\alpha) = \frac{f'(x)}{f(x)};$$
$$\Gamma^1_{22}(x,\alpha) = -\frac{f(x)f'(x)}{1+f'(x)^2},$$

while all other functions Γ^i_{jk} are identically zero. Suppose now that $c(t) = (x(t), \alpha(t))$ is a C^∞ curve on S and $s(t)$ is a vector field along c; we write

$$s(t) = a^1(t)\frac{\partial}{\partial x} + a^2(t)\frac{\partial}{\partial \alpha}. \tag{203}$$

Then by the formula for $\dot{s}(t)$ in local coordinates (cf. Lecture #9, p. 8):

$$\dot{s}(t) = \dot{a}^1(t)\frac{\partial}{\partial x} + \dot{a}^2(t)\frac{\partial}{\partial \alpha} + \dot{c}^j(t)a^k(t)\Big(\Gamma^1_{jk}(c(t))\frac{\partial}{\partial x} + \Gamma^2_{jk}(c(t))\frac{\partial}{\partial \alpha}\Big)$$

$$= \dot{a}^1\frac{\partial}{\partial x} + \dot{a}^2\frac{\partial}{\partial \alpha} + \frac{f'f''}{1+f'^2}\dot{x}a^1\frac{\partial}{\partial x} + \frac{f'}{f}\Big(\dot{x}a^2 + \dot{\alpha}a^1\Big)\frac{\partial}{\partial \alpha} - \frac{ff'}{1+f'^2}\dot{\alpha}a^2\frac{\partial}{\partial x}$$

(204)

$$= \Big(\dot{a}^1 + \frac{f'f''}{1+f'^2}\dot{x}a^1 - \frac{ff'}{1+f'^2}\dot{\alpha}a^2\Big)\frac{\partial}{\partial x} + \Big(\dot{a}^2 + \frac{f'}{f}\Big(\dot{x}a^2 + \dot{\alpha}a^1\Big)\Big)\frac{\partial}{\partial \alpha}.$$

(a). The equation for c being a geodesic is $\nabla_{\dot{c}}\dot{c} = 0$. But the vector field $s(t) = \dot{c}(t)$ is given by (203) with $a^1 = \dot{x}$ and $a^2 = \dot{\alpha}$. Hence the equation becomes (cf. (204)):

$$\begin{cases} \ddot{x} + \dfrac{f'(x)f''(x)}{1+f'(x)^2}\dot{x}^2 - \dfrac{f(x)f'(x)}{1+f'(x)^2}\dot{\alpha}^2 = 0 \\ \ddot{\alpha} + 2\dfrac{f'(x)}{f(x)}\dot{x}\dot{\alpha} = 0. \end{cases} \tag{205}$$

To prove that $f(x(t))^2\dot{\alpha}(t)$ remains constant along any geodesic, we simply note that

$$\frac{d}{dt}\Big(f(x)^2\dot{\alpha}\Big) = 2f(x)f'(x)\dot{x}\dot{\alpha} + f(x)^2\ddot{\alpha} = 0,$$

where the last equality holds by the second equation in (205).

Remark: The fact that $f(x)^2\dot{\alpha}$ remains constant around any geodesic is called *Clairaut's relation.* Note that $f(x(t))\dot{\alpha}(t) = \|\dot{c}(t)\|\sin\psi(t)$, where $\psi(t)$ is the angle between $\dot{c}(t)$ and the *meridians* of S. Hence (since $\|\dot{c}(t)\|$ is constant along any geodesic) an equivalent formulation of the relation is to say that $f(x)\sin\psi$ remains constant along any geodesic.

From (205) we see that the geodesics with $x \equiv$ constant are exactly the curves $c(t) = (k_1, k_2 + k_3t)$ with $k_1, k_2, k_3 \in \mathbb{R}$, and $[k_3 = 0$ or $f'(k_1) = 0]$. Any curve $c(t) = (k_1, k_2 + k_3t)$ with $k_3 \neq 0$ is called a *parallel* of S, and what we have just shown is that a parallel of S is a geodesic iff its x-value satisfies $f'(x) = 0$.

Finally, let us consider a curve with $\alpha \equiv$ constant, i.e. $c(t) = (x(t), \alpha)$. By (205), this is a geodesic iff

$$\ddot{x} + \frac{f'(x)f''(x)}{1+f'(x)^2}\dot{x}^2 \equiv 0. \tag{206}$$

We may note that the function $x(t)$ satisfies (206) iff

$$\dot{x}(t) \equiv \frac{C}{\sqrt{1+f'(x(t))^2}} \qquad \text{for some constant } C \in \mathbb{R}. \tag{207}$$

[Proof: Note that for *every* real-valued C^∞ function $x(t)$ we have

$$\frac{d}{dt}\Big(\dot{x}(t)\sqrt{1+f'(x(t))^2}\Big) = \ddot{x}\sqrt{1+f'(x)^2} + \dot{x}\cdot\frac{f'(x)}{\sqrt{1+f'(x)^2}}\cdot f''(x)\cdot\dot{x}$$
$$= \sqrt{1+f'(x)^2}\Big(\ddot{x} + \frac{f'(x)f''(x)}{1+f'(x)^2}\cdot\dot{x}^2\Big).$$

This implies that the function $\dot{x}(t)\sqrt{1+f'(x(t))^2}$ is constant iff (206) holds. This is the desired equivalence.]

The equation (207) is seen to be equivalent with the statement that $c(t) = (x(t),\alpha)$ viewed as a curve in $\mathbb{R}^3$ (that is, $c(t) = (x(t), f(x(t))\cos\alpha, f(x(t))\sin\alpha)$), is *parametrized proportionally to arc length.* In particular, up to rescaling the parametrization and changing direction, *there exists a unique geodesic* $\alpha \equiv$ *constant* for *every* choice of the constant α! Such a curve is called a *meridian* of S. □

(b). The equation for parallel transport along a given curve $c(t) = (x(t),\alpha(t))$ is $\dot{s}(t) = 0$, i.e., by (204):

$$\begin{cases} \dot{a}^1 + \dfrac{f'(x)f''(x)}{1+f'(x)^2}\dot{x}a^1 - \dfrac{f(x)f'(x)}{1+f'(x)^2}\dot{\alpha}a^2 = 0 \\ \dot{a}^2 + \dfrac{f'(x)}{f(x)}\Big(\dot{x}a^2 + \dot{\alpha}a^1\Big) = 0. \end{cases}$$

We should consider this for the curve $c(t) = (x,t)$, $t \in [0,2\pi]$. Then the above equation becomes:

$$\begin{cases} \dot{a}^1 - \dfrac{f(x)f'(x)}{1+f'(x)^2}a^2 = 0 \\ \dot{a}^2 + \dfrac{f'(x)}{f(x)}a^1 = 0. \end{cases} \tag{208}$$

If $f'(x) = 0$ then the equation implies that both a^1 and a^2 are *constant.*

Now assume $f'(x) \neq 0$. Then differentiating the first equation and substituting the second into the result gives:

$$\ddot{a}^1(t) = -\frac{f'(x)^2}{1+f'(x)^2}a^1(t).$$

Here $-\dfrac{f'(x)^2}{1+f'(x)^2} < 0$; hence the general solution is

$$a^1(t) = C_1\sin\Big(C_2 + \frac{f'(x)}{\sqrt{1+f'(x)^2}}\,t\Big)$$

with $C_1, C_2 \in \mathbb{R}$. Then from the first equation in (208) we get

$$a^2(t) = \frac{1+f'(x)^2}{f(x)f'(x)} C_1 \frac{f'(x)}{\sqrt{1+f'(x)^2}} \cos\Big(C_2 + \frac{f'(x)}{\sqrt{1+f'(x)^2}} t\Big)$$
$$= C_1 \frac{\sqrt{1+f'(x)^2}}{f(x)} \cos\Big(C_2 + \frac{f'(x)}{\sqrt{1+f'(x)^2}} t\Big).$$

It is nicer to express $s(t)$ in terms of the following ON basis for $T_{(x,\alpha)}S$:

$$b_1 := \frac{1}{\sqrt{1+f'(x)^2}} \frac{\partial}{\partial x}, \qquad b_2 := \frac{1}{f(x)} \frac{\partial}{\partial \alpha}. \tag{209}$$

We get:

$$s(t) = \widetilde{C}_1 \Big(\sin\big(C_2 + \sigma_x\, t\big)\, b_1 + \cos\big(C_2 + \sigma_x\, t\big)\, b_2 \Big) \tag{210}$$

where $\sigma_x := \frac{f'(x)}{\sqrt{1+f'(x)^2}}$ and $\widetilde{C}_1 = C_1\sqrt{1+f'(x)^2}$. Thus in these ON coordinates the vector is simply *rotating at constant speed along the curve.* □

Alternative; a more geometrical solution (outline) (cf., e.g., stackexchange): Let S' be the *cone* (or cylinder, if $f'(x) = 0$) in $\mathbb{R}^3$ which is tangent to S along c. It follows from [12, Thm. 4.7.1] that the Levi-Civita connections for S and S' are *equal* at every point along c; hence also parallel transport along c is the same in S and S'. But we can imagine the cone S' being constructed by rolling a paper; unrolling the paper then gives a (local) isometry between S' and $\mathbb{R}^2$ with its standard Riemannian metric; and in $\mathbb{R}^2$ parallel transport of any vector along any curve means simply keeping the vector *constant* in the standard coordinates on $T_{c(t)}\mathbb{R}^2 = \mathbb{R}^2$. Our curve c is mapped by the isometry to an arc of angle $\frac{|f'(x)|}{\sqrt{f'(x)^2+1}} 2\pi$ along a circle with radius $f(x) \cdot \frac{\sqrt{f'(x)^2+1}}{|f'(x)|}$. Parallel transport of any vector $v \in \mathbb{R}^2$ along this circle means that the angle between v and $\dot{c}$ increases/decreases with a constant rate, with the total change being $\frac{|f'(x)|}{\sqrt{f'(x)^2+1}} 2\pi$ as t goes from 0 to 2π. Hence in the basis b_1, b_2 (209) The general form of such a motion is indeed given by (210), with $\sigma_x := \pm\frac{|f'(x)|}{\sqrt{1+f'(x)^2}}$. Further inspection of the cone unfolding argument shows that in the (x, α) coordinates, v rotates in *positive* direction if $f'(x) < 0$, and *negative* direction if $f'(x) > 0$; thus in fact in (210) we have $\sigma_x := \frac{f'(x)}{\sqrt{1+f'(x)^2}}$. □

(c). Since $\dim T_pS = \dim S = 2$ we can indeed speak of *the* sectional curvature of S at a point $p \in S$, and this sectional curvature equals $K(X \wedge Y)$ where X, Y is any basis for T_pS; cf. Problem 69. We compute this with $X = \frac{\partial}{\partial x}$, $Y = \frac{\partial}{\partial \alpha}$ at the point $p = (x, \alpha)$. First note:

$$K\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial \alpha}\Big) = \Big\langle R\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial \alpha}\Big)\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial x}\Big\rangle = \Big\langle R^1_{212}\frac{\partial}{\partial x} + R^2_{212}\frac{\partial}{\partial \alpha}, \frac{\partial}{\partial x}\Big\rangle$$
$$= (1 + f'(x)^2) \cdot R^1_{212}(x, \alpha).$$

Now use the formula from Lecture #11, p. 3:

$$\begin{aligned} R^1_{212} &= \frac{\partial \Gamma^1_{22}}{\partial x} - \frac{\partial \Gamma^1_{12}}{\partial \alpha} + \Gamma^1_{1j}\Gamma^j_{22} - \Gamma^1_{2j}\Gamma^j_{12} \\ &= -\frac{\partial}{\partial x}\Big(\frac{f(x)f'(x)}{1+f'(x)^2}\Big) - \frac{f'(x)f''(x)}{1+f'(x)^2} \cdot \frac{f(x)f'(x)}{1+f'(x)^2} \\ &\qquad + \frac{f(x)f'(x)}{1+f'(x)^2} \cdot \frac{f'(x)}{f(x)} \\ &= -\frac{(f'^2 + ff'')(1+f'^2) - 2ff'^2f''}{(1+f'^2)^2} - \frac{ff'^2f''}{(1+f'^2)^2} + \frac{f'^2}{1+f'^2} \\ &= -\frac{f(x)f''(x)}{(1+f'(x)^2)^2}. \end{aligned}$$

Hence

$$K\Big(\frac{\partial}{\partial x}, \frac{\partial}{\partial \alpha}\Big) = -\frac{f(x)f''(x)}{1+f'(x)^2}.$$

Also, since $\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial \alpha}\rangle = 0$,

$$\Big\|\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \alpha}\Big\|^2 = \Big\|\frac{\partial}{\partial x}\Big\|^2 \cdot \Big\|\frac{\partial}{\partial \alpha}\Big\|^2 = (1+f'(x)^2)f(x)^2.$$

Hence

$$K\Big(\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \alpha}\Big) = -\frac{f(x)f''(x)}{1+f'(x)^2} \cdot \frac{1}{(1+f'(x)^2)f(x)^2} = -\frac{f''(x)}{f(x)(1+f'(x)^2)^2}.$$

Answer: The sectional curvature at (x, α) is $-\dfrac{f''(x)}{f(x)(1+f'(x)^2)^2}$.

In particular we note that the sectional curvature at (x, α) is positive iff $f''(x) < 0$ and negative iff $f''(x) > 0$.

Finally for $f(x) = \sqrt{r^2 - x^2}$ we compute that

$$-\frac{f''(x)}{f(x)(1+f'(x)^2)^2} = \frac{1}{r^2} \qquad \text{for all } x \in (-r, r).$$

This is indeed the scalar curvature at any point of a sphere of radius r. □

Problem 82:

(Cf. [14, p. 128, Problem 7-3].)

We have

$$\begin{aligned}(\nabla^2\eta)(X,Y,Z) &= (\nabla_Z(\nabla\eta))(X,Y)\\ &= Z\big((\nabla\eta)(X,Y)\big) - (\nabla\eta)(\nabla_Z X, Y) - (\nabla\eta)(X, \nabla_Z Y)\\ &= Z\big((\nabla_Y\eta)(X)\big) - (\nabla_Y\eta)(\nabla_Z X) - (\nabla_{\nabla_Z Y}(\eta))(X)\\ &= Z\Big(Y(\eta(X)) - \eta(\nabla_Y X)\Big) - Y\big(\eta(\nabla_Z X)\big) + \eta(\nabla_Y\nabla_Z X)\\ &\qquad - (\nabla_Z Y)(\eta(X)) + \eta(\nabla_{\nabla_Z Y} X).\end{aligned}$$

(In the above computation, in the first and the third equalities we used the definition of $\nabla : \Gamma T^r_s(M) \to \Gamma T^r_{s+1}(M)$ given in the problem formulation, while in the second and the fourth equalities we used the formula from Problem 59(a).) Subtracting the corresponding expression for $(\nabla^2\eta)(X,Z,Y)$ from the above expression, and using

$$\begin{aligned}Z(Y(\eta(X))) - Y(Z(\eta(X))) - (\nabla_Z Y)(\eta(X)) + (\nabla_Y Z)(\eta(X))\\ = \Big([Z,Y] - \nabla_Z Y + \nabla_Y Z\Big)(\eta(X)) = 0,\end{aligned}$$

we obtain:

$$\begin{aligned}&(\nabla^2\eta)(X,Y,Z) - (\nabla^2\eta)(X,Z,Y)\\ &= \eta\Big(\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X + \nabla_{\nabla_Z Y} X - \nabla_{\nabla_Y Z} X\Big)\\ &= \eta\Big(\nabla_Y\nabla_Z X - \nabla_Z\nabla_Y X - \nabla_{[Y,Z]} X\Big)\\ &= \eta\Big(R(Y,Z)X\Big).\end{aligned}$$

Done! $\square$

Alternative: It suffices to prove that the two functions $(\nabla^2\eta)(X,Y,Z)-(\nabla^2\eta)(X,Z,Y)$ and $\eta(R(Y,Z)X)$ have the identical restrictions to the set U for any chart (U,x). Given such a chart, by expanding each of X,Y,Z in the basis of sections $\frac{\partial}{\partial x^1},\ldots,\frac{\partial}{\partial x^d}\in\Gamma(TU)$, and using the fact that both $(\nabla^2\eta)(X,Y,Z)-(\nabla^2\eta)(X,Z,Y)$ and $\eta(R(Y,Z)X)$ are $C^\infty(M)$-linear in each of X,Y,Z, we see that it suffices to prove that, for any i,j,k,

(211)
$$(\nabla^2\eta)\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\Big)-(\nabla^2\eta)\Big(\frac{\partial}{\partial x^i},\frac{\partial}{\partial x^k},\frac{\partial}{\partial x^j}\Big)=\eta\Big(R\Big(\frac{\partial}{\partial x^j},\frac{\partial}{\partial x^k}\Big)\frac{\partial}{\partial x^i}\Big).$$

From now on let us use the short-hand notation $\partial_i:=\frac{\partial}{\partial x^i}$. Take $\eta_1,\ldots,\eta_d\in C^\infty(U)$ so that
$$\eta_{|U}=\eta_l\,dx^l.$$
Then in U we have
$$\nabla\eta=\nabla(\eta_l\,dx^l)=dx^l\otimes d\eta_l+\eta_l\,\nabla(dx^l)=dx^l\otimes\big((\partial_m\eta_l)dx^m\big)-\eta_l\,\Gamma^l_{ab}\,dx^b\otimes dx^a$$
$$=(\partial_a\eta_b-\eta_l\Gamma^l_{ab})\,dx^b\otimes dx^a,$$
where we used [12, (4.1.22)]. It follows that, for any k,
$$\begin{aligned}\nabla_{\partial_k}(\nabla\eta)&=\nabla_{\partial_k}\Big((\partial_a\eta_b-\eta_l\Gamma^l_{ab})\,dx^b\otimes dx^a\Big)\\
&=\Big(\partial_k(\partial_a\eta_b-\eta_l\Gamma^l_{ab})\Big)\,dx^b\otimes dx^a+(\partial_a\eta_b-\eta_l\Gamma^l_{ab})\,\nabla_{\partial_k}(dx^b)\otimes dx^a\\
&\qquad+(\partial_a\eta_b-\eta_l\Gamma^l_{ab})\,dx^b\otimes\nabla_{\partial_k}(dx^a)\\
&=\Big(\partial_k\partial_a\eta_b-(\partial_k\eta_l)\Gamma^l_{ab}-(\partial_k\Gamma^l_{ab})\eta_l\Big)\,dx^b\otimes dx^a\\
&\quad-(\partial_a\eta_b-\eta_l\Gamma^l_{ab})\,\Gamma^b_{kc}\,dx^c\otimes dx^a-(\partial_a\eta_b-\eta_l\Gamma^l_{ab})\,\Gamma^a_{kc}\,dx^b\otimes dx^c\\
&=\Big(\partial_k\partial_a\eta_b-(\partial_k\eta_l)\Gamma^l_{ab}-(\partial_k\Gamma^l_{ab})\eta_l-(\partial_a\eta_c)\,\Gamma^c_{kb}+\eta_l\Gamma^l_{ac}\Gamma^c_{kb}\\
&\qquad-(\partial_c\eta_b)\,\Gamma^c_{ka}+\eta_l\Gamma^l_{cb}\Gamma^c_{ka}\Big)\,dx^b\otimes dx^a.\end{aligned}$$
This means that
$$(\nabla^2\eta)\Big(\partial_i,\partial_j,\partial_k\Big)=\Big(\nabla_{\partial_k}(\partial\eta)\Big)\Big(\partial_i,\partial_j\Big)$$
$$=\partial_k\partial_j\eta_i-(\partial_k\eta_l)\Gamma^l_{ji}-(\partial_k\Gamma^l_{ji})\eta_l-(\partial_j\eta_c)\,\Gamma^c_{ki}+\eta_l\Gamma^l_{jc}\Gamma^c_{ki}-(\partial_c\eta_i)\,\Gamma^c_{kj}+\eta_l\Gamma^l_{ci}\Gamma^c_{kj}.$$
Subtracting the corresponding expression with j and k swapped (and using $\Gamma^c_{kj}=\Gamma^c_{jk}$, which holds since the Levi-Civita connection is torsion free; cf. Lemma 2 in #13), we obtain
$$(\nabla^2\eta)\Big(\partial_i,\partial_j,\partial_k\Big)-(\nabla^2\eta)\Big(\partial_i,\partial_k,\partial_j\Big)=\Big(-\partial_k\Gamma^l_{ji}+\Gamma^l_{jc}\Gamma^c_{ki}+\partial_j\Gamma^l_{ki}-\Gamma^l_{kc}\Gamma^c_{ji}\Big)\eta_l.$$
Comparing with the formula for R on p. 3 in Lecture #11, we get
$$=R^l_{ijk}\,\eta_l=\eta(R(\partial_j,\partial_k)\partial_i).$$
(The last equality holds since $R(\partial_j,\partial_k)\partial_i=R^l_{ijk}\partial_l$ and $\eta=\eta_l\,dx^l$.) Hence we have proved (211)! □

Problem 83:

(a). One simple construction is as follows: Equip M with an arbitrary Riemannian metric. This is possible by [12, Thm. 1.4.1]. Let $\exp : \mathcal{D} \to M$ be the corresponding exponential map with its maximal domain $\mathcal{D}$ (an open subset of TM). By a simple compactness argument (using the fact that $0_p \in \mathcal{D}$ for all $p \in M$), there exists some $\varepsilon > 0$ such that $s \cdot Y(t) \in \mathcal{D}$ for all $t \in [0,1]$ and all $s \in (-\varepsilon, \varepsilon)$. Now define

$$c(t,s) := \exp(s \cdot Y(t)), \qquad (t \in [0,1], \ s \in (-\varepsilon, \varepsilon)). \tag{212}$$

This is easily verified to be a smooth variation of the given curve c with $c' = Y$, which is furthermore proper if $Y(0) = 0 = Y(1)$. $\square$

(b). (Outline.) First assume $c(0) \neq c(1)$. Then it is possible to choose the Riemannian metric on M so that for each $j \in \{0,1\}$, either $Y(j) = 0$ or else γ_j is a geodesic. [Indeed, inspecting the construction in [12, Thm. 1.4.1] we see that it suffices to prove that if $\dot{\gamma}_j(0) = Y(j) \neq 0$ then there exist $\varepsilon > 0$ and an open neighborhood U of $\gamma_j(0)$ which can be equipped with a Riemannian metric such that $\gamma_{j|[-\varepsilon,\varepsilon]}$ is a geodesic in U. And this can be constructed by letting U be the domain of a chart (U,x) such that $x(\gamma_j(t)) = (t, 0, \ldots, 0)$ for all t near 0 (as is possible by Problem 12), and then equipping U with the Riemannian metric inherited from the standard Riemannian metric on $\mathbb{R}^d$ via $x : U \to \mathbb{R}^d$.] With this choice, the variation in (212) again has all the desired properties.

If $c(0) = c(1)$ then the above construction can be modified e.g. as follows: For $j = 0,1$, choose a Riemannian metric m_j on M such that if $Y(j) \neq 0$ then γ_j is a geodesic wrt m_j. Then also $u\mathsf{m}_0 + (1-u)\mathsf{m}_1$ is a Riemannian metric on M for each $u \in [0,1]$, and we denote by $\exp(\,\cdot\,; u) : \mathcal{D}_u \to M$ the corresponding exponential map. Now one can prove that there is some $\varepsilon > 0$ such that $s \cdot Y(t) \in \mathcal{D}_t$ for all $t \in [0,1]$ and $s \in (-\varepsilon, \varepsilon)$. Then define

$$c(t,s) := \exp(s \cdot Y(t); t).$$

This is a variation having the required properties. $\square$

Remark: Note that when we apply the result of this Problem 83 in practice, M often comes already equipped with a Riemannian metric; however the Riemannian metric which is chosen in the above construction may well be *another* ("completely unrelated") Riemannian metric!

Problem 84: Lemma 1 in Lecture #16 implies that, in our situation, $E'(s) = L'(s) = 0$ for all s. Hence $E(s)$ and $L(s)$ are indeed constant functions. $\square$

Problem 85:

(a). Identify T_pM with $\mathbb{R}^d$ by some fixed linear map respecting the inner product. Let γ_1 and γ_2 be the following C^∞ curves in $T_pM = \mathbb{R}^d$:

$$\gamma_1(t) = te_1, \qquad t \in [0, \pi],$$

where $e_1 = (1, 0, \ldots, 0)$, and

$$\gamma_2(t) = (\pi \cos t, \pi \sin t, 0, 0, \ldots, 0), \qquad t \in [0, a],$$

where a is any fixed positive constant. Then set:

$$\gamma = \gamma_1 \cdot \gamma_2 \cdot \overline{\gamma_2}.$$

In order to fit into the problem formulation, this product path should be understood to be reparametrized in some way so that the domain of γ is $[0, 1]$.

Note then that

$$\gamma(1) = \gamma_1(\pi) = \pi e_1.$$

Also $\|\gamma_2(t)\| = \pi$ and thus $\exp_p(\gamma_2(t)) = -p$ for all $t \in [0, a]$! Hence

$$L(\exp_p \circ \gamma) = L(\exp_p \circ \gamma_1) = \|\gamma_1(\pi)\| = \|\gamma(1)\|,$$

where the second equality holds since $t \mapsto \exp_p \circ \gamma_1(t) = \exp_p(te_1)$ is a geodesic. However γ is certainly not a reparametrization of the curve $t \mapsto t \cdot \gamma(1) = t \cdot \pi e_1$ ($t \in [0, 1]$), since the image of γ in $\mathbb{R}^d$ contains points outside the line $\mathbb{R}e_1$. $\square$

(b). Writing $e_1, \ldots, e_d$ for the standard basis vectors in $\mathbb{R}^d$, our task is to prove that

$$\big\langle d(\exp_p \circ y^{-1})_{\tilde{y}}(e_1), d(\exp_p \circ y^{-1})_{\tilde{y}}(e_1)\big\rangle = 1 \tag{213}$$

and

$$\big\langle d(\exp_p \circ y^{-1})_{\tilde{y}}(e_1), d(\exp_p \circ y^{-1})_{\tilde{y}}(e_j)\big\rangle = 0 \qquad \text{for } j = 2, \ldots, d. \tag{214}$$

Now for any $v \in \mathbb{R}^d$ we have

$$d(\exp_p \circ y^{-1})_{\tilde{y}}(v) = (d\exp_p)_{y^{-1}(\tilde{y})} \circ (dy^{-1})_{\tilde{y}}(v).$$

Furthermore, the "polar coordinates" assumption implies that, for any $\tilde{y} \in y(W)$, if $w = y^{-1}(\tilde{y}) \in W$ then

$$\|w\| = \tilde{y}^1 > 0; \qquad (dy^{-1})_{\tilde{y}}(e_1) = \|w\|^{-1} w; \tag{215}$$

and

$$\big\langle (dy^{-1})_{\tilde{y}}(e_j), w\big\rangle = 0, \qquad \text{for } j = 2, \ldots, d. \tag{216}$$

[Proof: We have $y^1(w') = \|w'\| \geq 0$ for all $w' \in W$, and $y(W)$ is open; hence $y^1(w') > 0$ for all $w' \in W$, and in particular the first relation in (215) holds. Next consider the curve $c(t) = (1+t)w$ $(-\varepsilon < t < \varepsilon)$ in T_pM; for ε sufficiently small this curve is contained in W, and the polar coordinates assumption implies that $y(c(t)) = ((1+t)\tilde{y}^1, \tilde{y}^2, \ldots, \tilde{y}^d)$ for all $t \in (-\varepsilon, \varepsilon)$. Considering the tangent vector of this curve at $t = 0$ we find $dy_w(w) = \tilde{y}^1 e_1$, and this implies the second relation in (215). Finally for given $j \in \{2, \ldots, d\}$ consider the curve $\gamma(t) = \tilde{y} + te_j$ $(-\varepsilon < t < \varepsilon)$ in $\mathbb{R}^d$; for ε sufficiently small this curve is contained in $y(W)$, and the polar coordinates assumption implies that the curve $y^{-1} \circ \gamma$ is contained in the sphere $\{w' \in W : \|w'\| = \|w\| > 0\}$. Note also $y^{-1}(\gamma(0)) = w$. Hence the tangent vector of $y^{-1}(\gamma(t))$ at $t = 0$ is orthogonal to w, and since $\dot{\gamma}(0) = e_j$ this implies (216).]

In view of the above, (213) is equivalent with[50]

$$\big\langle (d\exp_p)_w(\|w\|^{-1}w), (d\exp_p)_w(\|w\|^{-1}w)\big\rangle = 1, \qquad \forall w \in W \tag{217}$$

and in order to prove (214) it suffices to prove that

$$\big\langle (d\exp_p)_w(\|w\|^{-1}w), (d\exp_p)_w(v)\big\rangle = 0, \qquad \text{whenever } w \in W,\ v \in T_pM,\ \langle w, v\rangle = 0. \tag{218}$$

However these two statements are clearly implied by "Gauss Lemma"! □

[50] Recall that we saw in (215) that $\|w\| > 0$ for all $w \in W$; thus the statements (217) and (218) make sense.

(c). *Outline:* We follow the argument on pp. 8–9 in Lecture #4 (mutatis mutandis) using the fact proved in part b (where of course the bottom right $(d-1)\times(d-1)$ submatrix must be everywhere positive semidefinite). Note that in that argument, in

$$L(\gamma) = \int_0^1 \|(\exp_p \circ\gamma)^{\cdot}(t)\|\,dt \geq \int_0^1 |\dot r(t)|\,dt \geq |r(1)-r(0)| = \|v\|,$$

we now *cannot* claim that equality in the first inequality holds iff $\dot\varphi(t)=0$ $\forall t\in(0,1)$, namely since $(d\exp_p)_{\gamma(t)}$ may be singular. However equality in the *second* equality holds iff $\dot r(t)\geq 0$ $\forall t\in(0,1)$. Thus, since we are assuming $L(\gamma)=\|v\|$, the function $r(t)=\|\gamma(t)\|$ must be increasing, and so $r(t)\leq\|\gamma(1)\|=\|v\|$ for all $t\in[0,1]$.

Now *assume* that there is some $t\in(0,1)$ with $\gamma(t)\notin[0,1]\cdot\|v\|$, and let $t_0\in(0,1]$ be the *supremum* of the set of such t. By continuity, $\gamma(t_0)\in[0,1]\cdot\|v\|$, say $\gamma(t_0)=t_1\|v\|$ $(t_1\in[0,1])$. Now *since the point* $c(t_1)=\exp(t_1v)$ *is not conjugate to* $c(0)$ *along* c, we have that $(d\exp_p)_{\gamma(t_0)}$ is non-singular, and hence $(d\exp_p)_w$ is non-singular for all w in some open neighborhood $\Omega\subset\mathcal{D}_p$ of $\gamma(t_0)$. On the other hand it follows from the definition of t_0 that there exist t-values $<t_0$ arbitrarily near t_0 where $\gamma(t)\notin[0,1]\cdot\|v\|$. Hence there also exist t-values $<t_0$ arbitrarily near t_0 where $\dot\varphi(t)\neq 0$. Since such a t can be found arbitrarily near t_0, we can ensure that $\gamma(t)\in\Omega$. Both $\dot\varphi(t)\neq 0$ and $\gamma(t)\in\Omega$ are "open" conditions; hence there must in fact exist a whole open interval $(t_2,t_2+\eta)\subset(0,t_0)$ $(\eta>0)$ such that $\dot\varphi(t)\neq0$ and $\gamma(t)\in\Omega$ for all $t\in(t_2,t_2+\eta)$. *Now* we can conclude

$$\int_{t_2}^{t_2+\eta} \|(\exp_p\circ\gamma)^{\cdot}(t)\|\,dt > \int_{t_2}^{t_2+\eta} \|\dot r(t)|\,dt,$$

since $\|(\exp_p\circ\gamma)^{\cdot}(t)\| > \|\dot r(t)|$ for all $t\in(t_2,t_2+\eta)$, and so in total we must have a strict inequality $L(\gamma)>\|v\|$, contradicting our assumption.

Hence we must have $\gamma(t)\in[0,1]\cdot\|v\|$ for all t; and since also $r(t)$ is increasing, it follows that γ is a reparametrization of the curve $t\mapsto tv$. $\square$

Problem 86: As stated in the lecture notes, this is clear from Cor. 1 in Lecture #17. Indeed, after possibly shrinking and reparametrizing the geodesic, and possibly changing its direction, we may assume $t_0 = a = 0$, $t_1 = b = T > 0$. Set also $p = c(0)$ and $q = c(T)$. Then our task is to prove that $c(0)$ and $c(T)$ are conjugate along c iff

$$(d\exp_p)_{T\cdot\dot{c}(0)} : T_p(M) \to T_q(M)$$

is singular. By Cor. 1 in #17, for every $v \in T_p(M)$ we have that $(d\exp_p)_{T\cdot\dot{c}(0)}(v)$ equals $X(T)$ when X is the unique Jacobi field along c with $X(0) = 0$, $\dot{X}(0) = v$. [51] Hence, since also $T_p(M)$ and $T_q(M)$ have the same dimension, $(d\exp_p)_{T\cdot\dot{c}(0)}$ is singular iff there is some $v \neq 0$ in $T_p(M)$ with such that the unique Jacobi field along c with $X(0) = 0$ and $\dot{X}(0) = v$ satisfies $X(T) = 0$. In other words, $(d\exp_p)_{T\cdot\dot{c}(0)}$ is singular iff there is some Jacobi field $X \not\equiv 0$ along c with $X(0) = 0$ and $X(T) = 0$, i.e. iff $c(0)$ and $c(T)$ are conjugate along c. □

[51] The fact that there indeed exists a unique such Jacobi field is provided by Lemma 1 in #17.

Problem 88: As in the proof of Lemma 1 in #17, let $X_1, \ldots, X_d \in \mathcal{V}_c$ be *parallel* vector fields along c such that $X_1(t), \ldots, X_d(t)$ forms an ON-basis $T_{c(t)}(M)$ for each $t \in [a,b]$, and define $\rho_i^k \in C^\infty([a,b])$ by $R(X_i, \dot{c})\dot{c} = \rho_i^k X_k$. Then any pw C^∞ vector field Y along c can be (uniquely) expressed as

$$Y = \xi^i X_i,$$

where $\xi_1, \ldots, \xi_d$ are pw C^∞ functions $[a,b] \to \mathbb{R}$. We then also have

$$\begin{aligned} I(Y,Y) &= \int_a^b \Big(\big\| \nabla_{\frac{\partial}{\partial t}} Y \|^2 - \langle R(Y, \dot{c})\dot{c}, Y \rangle \Big) \, dt \\ &= \int_a^b \Big(\sum_{i=1}^d \dot{\xi}_i(t)^2 - \sum_{i=1}^d \sum_{k=1}^d \rho_i^k(t) \xi_i(t) \xi_k(t) \Big) \, dt. \end{aligned} \tag{219}$$

Hence the problem is reduced to a problem in real analysis: We see that it now suffices to prove that if f is any pw C^∞ function $f : [a,b] \to \mathbb{R}$, then there exists a sequence $f_1, f_2, \ldots$ of functions in $C^\infty([a,b])$ such that

$$\begin{aligned} & \|f_k - f\|_{L^\infty} \to 0 \quad \text{and} \quad \|f_k' - f'\|_{L^1} \to 0, \\ & \text{and} \quad \|f_k'\|_{L^\infty} \text{ stays bounded as } k \to \infty, \end{aligned} \tag{220}$$

and that we may furthermore choose this sequence so that $f_k(t) = f(t)$ for all k and all $t \in [a,b] \setminus \cup_{j=1}^{m-1} (t_j - \varepsilon, t_j + \varepsilon)$, where $\varepsilon > 0$ is any fixed constant and $t_1 < t_2 < \cdots < t_{m-1}$ are any given numbers in (a,b) including all 'break-points' of f (viz., $f_{|[t_{j-1}, t_j]} \in C^\infty([t_{j-1}, t_j])$ for $j = 1, \ldots, m$, where $t_0 = a$ and $t_m = b$).

(Indeed, if we can prove the statement of the last sentence, then we apply it to each of the functions $\xi_1, \ldots, \xi_d$ describing the given pw C^∞ vector field Y, with fixed $\varepsilon > 0$ and with $t_1 < \cdots < t_{m-1}$ being all the 'break-points' of Y in (a,b). This gives a sequence of C^∞ vector fields $Z_1, Z_2, \ldots$ along c each satisfying $Z_k(t) = Y(t)$ for all $t \in [a,b] \setminus \cup_{j=1}^{m-1} (t_j - \varepsilon, t_j + \varepsilon)$, and, as one verifies using (219): $I(Z_k, Z_k) \to I(Y,Y)$ as $k \to \infty$.)

Outline of real analysis argument: Thus assume that $f : [a,b] \to \mathbb{R}$ is pw C^∞ and let $a = t_0 < t_1 < \cdots < t_m = b$ be such that $f_{|[t_{j-1}, t_j]} \in C^\infty([t_{j-1}, t_j])$ for $j = 1, \ldots, m$. In fact we may extend f to a pw C^∞ function $(a - \varepsilon, b + \varepsilon) \to \mathbb{R}$ so that $f_{|(a-\varepsilon, t_1]}$ and $f_{|[t_{m-1}, b+\varepsilon)}$ are C^∞. (Cf. the lecture notes, Sec. 3.1.) Fix a C^∞ function $\phi : \mathbb{R} \to \mathbb{R}_{\geq 0}$ with support contained in $(-1,1)$ and $\int_{\mathbb{R}} \phi = 1$. For every $\eta > 0$ set $\phi_\eta(t) := \eta^{-1} \phi(\eta^{-1} t)$; then ϕ_η has support contained in $(-\eta, \eta)$ and $\int_{\mathbb{R}} \phi_\eta = 1$. Now choose any sequence $\varepsilon > \eta_1 > \eta_2 > \cdots > 0$ with $\eta_k \to 0$, and set $f_k := \phi_{\eta_k} * f$ (convolution), i.e.

$$f_k(t) = \int_{\mathbb{R}} f(t - x) \phi_{\eta_k}(x) \, dx.$$

It follows from $\operatorname{supp}(\phi_{\eta_k}) \subset (-\varepsilon, \varepsilon)$ that f_k is well-defined and C^∞ for $t \in [a,b]$. One now verifies that (220) holds (in fact $|f_k'|_{L^\infty} \leq \|f'\|_{L^\infty}$ for all k).

Finally one can use a partition of unity argument, creating f_k^{new} weighting appropriately between f_k and f, to *also* ensure $f_k(t) = f(t)$ for all k and all $t \in [a,b] \setminus \cup_{j=1}^{m-1}(t_j - \varepsilon, t_j + \varepsilon)$ (while (220) remains true). $\square$

Problem 89: We are assuming that $\exp_p$ is defined and injective on the open ball $B_r(0)$ in $T_p(M)$. We claim that $(d\exp_p)_v$ is non-singular for every $v \in B_r(0)$. Assume the opposite, i.e. assume that $v \in B_r(0)$ is such that $(d\exp_p)_v$ is singular. Take $b > 1$ so that $bv \in B_r(0)$, and let c be the geodesic

$$c : [0,b] \to M; \qquad c(t) := \exp_p(tv).$$

Our assumption that $(d\exp_p)_v$ is singular implies that the point $c(1)$ is conjugate to $c(0)$ along c (by Problem 86). Hence by Theorem 1 in Lecture #18, c is *not* a local minimum for L among pw C^∞ curves from p to $q := c(b)$. This implies in particular that $d(p,q) < L(c)$. Now by Problem 28 there is a geodesic from p to q realizing the distance $d(p,q)$, i.e. there is some $w \in T_p(M)$ such that $q = \exp_p(w)$ and $\|w\| = d(p,q)$. Of course $w \in B_r(0)$ and $w \neq bv$, since $\|w\| = d(p,q) < L(c) = \|bv\| < r$. Now we have $\exp_p(bv) = q = \exp_p(w)$, contradicting the fact that $\exp_{p|B_r(0)}$ is injective. This completes the proof that $(d\exp_p)_v$ is non-singular for every $v \in B_r(0)$.

From the non-singularity just proved it follows, via the Inverse Function Theorem, that $\exp_{p|B_r(0)}$ is a *local* diffeomorphism and that $U := \exp_p(B_r(0))$ is an open subset of M. Since $\exp_{p|B_r(0)}$ is injective, this map is a bijection of $B_r(0)$ onto U. Let $f : U \to B_r(0)$ be the inverse map. The fact that $\exp_{p|B_r(0)}$ is a local diffeomorphism implies that f is C^∞ in all U. Hence $\exp_{p|B_r(0)}$ is a diffeomorphism onto the open set U. $\square$

Problem 90: Let (U,x) and (V,y) be two charts with $p \in U$, $p \in V$, and assume that

$$\frac{\partial}{\partial x^{j_1}} \cdots \frac{\partial}{\partial x^{j_r}} f = 0 \qquad \text{at } \ p, \tag{221}$$

for any $1 \leq r \leq k$ and any $j_1, \ldots, j_r \in \{1, \ldots, d\}$. We know that

$$\frac{\partial}{\partial y^i} = \varphi_i^j \frac{\partial}{\partial x^j} \qquad (\forall i \in \{1, \ldots, d\})$$

(equality of vector fields in $\Gamma T(U \cap V)$), where

$$\varphi_i^j := \frac{\partial x^j}{\partial y^i} \in C^\infty(U \cap V).$$

Now take any $r \in \{1, \ldots, k\}$ and $i_1, \ldots, i_r \in \{1, \ldots, d\}$. Then in $U \cap V$ we have

$$\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_r}} f = \left(\varphi_{i_1}^{j_1} \frac{\partial}{\partial x^{j_1}} \right) \cdots \left(\varphi_{i_r}^{j_r} \frac{\partial}{\partial x^{j_r}} \right) f,$$

and this can be expanded as a sum where each term is of the form "$A \cdot B$" where each "A" is a product of partial derivatives of some of the functions $\varphi_{i_l}^{j_l}$, and each "B" equals $\frac{\partial}{\partial x^{j_{l(1)}}} \cdots \frac{\partial}{\partial x^{j_{l(s)}}} f$ for some $1 \leq l(1) < l(2) < \cdots < l(s) \leq r$ (with $1 \leq s \leq r$). Evaluating this sum *at* p, each B-factor *vanishes*, because of (221) (and since $s \leq r \leq k$). Hence

$$\frac{\partial}{\partial y^{i_1}} \cdots \frac{\partial}{\partial y^{i_r}} f = 0 \qquad \text{at } \ p.$$

□

Problem 91:

(a). Let (U, x) be any chart on S^d and let $(g_{ij}(x))$ give the standard Riemannian metric $\langle \cdot, \cdot \rangle$ with respect to (U, x). Then the Riemannian metric $[\cdot, \cdot]$ is given by $(h_{ij}(x))$ where[52]

$$h_{ij}(x) = f(x) \cdot g_{ij}(x), \qquad \forall i, j \in \{1, \ldots, d\},\ x \in x(U).$$

Now for any j, l, k, and for all $x \in x(U)$:

$$\frac{\partial}{\partial x_k} h_{jl}(x) = \frac{\partial}{\partial x_k}\Big(f(x) g_{jl}(x)\Big) = \Big(\frac{\partial}{\partial x_k} f(x)\Big) \cdot g_{jl}(x) + f(x) \cdot \frac{\partial}{\partial x_k} g_{jl}(x).$$

If x lies on the curve c then $f(x) = 1$ and $\frac{\partial}{\partial x_k} f(x) = 0$ (since $f \in \mathcal{F}_1$), and hence

$$\frac{\partial}{\partial x_k} h_{jl}(x) = \frac{\partial}{\partial x_k} g_{jl}(x). \tag{222}$$

Also of course $h_{jl}(x) = f(x) g_{jl}(x) = g_{jl}(x)$ for all x along c. Let ∇ and $\widetilde{\nabla}$ be the Levi-Civita connections on $T(S^d)$ corresponding to the Riemannian metrics $\langle \cdot, \cdot \rangle$ and $[\cdot, \cdot]$, respectively, and let Γ^i_{jk} and $\widetilde{\Gamma}^i_{jk}$ be the Christoffel symbols for ∇ and $\widetilde{\nabla}$, respectively, with respect to the basis of sections $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^d} \in \Gamma(TU)$. Then

$$\Gamma^i_{jk}(x) = \frac{1}{2} g^{il}(x) \bigg(\frac{\partial}{\partial x^k} g_{jl}(x) + \frac{\partial}{\partial x^j} g_{kl}(x) - \frac{\partial}{\partial x^l} g_{jk}(x) \bigg) \tag{223}$$

and

$$\widetilde{\Gamma}^i_{jk}(x) = \frac{1}{2} h^{il}(x) \bigg(\frac{\partial}{\partial x^k} h_{jl}(x) + \frac{\partial}{\partial x^j} h_{kl}(x) - \frac{\partial}{\partial x^l} h_{jk}(x) \bigg)$$

(for all $x \in x(U)$). It follows from the observations made above that

$$\widetilde{\Gamma}^i_{jk}(x) = \Gamma^i_{jk}(x) \qquad \text{for all } x \text{ along } c.$$

Since c is a geodesic on S^d we have

$$\ddot{c}^i(t) + \Gamma^i_{jk}(c(t))\, \dot{c}^j(t) \dot{c}^k(t) = 0,$$

for all $i \in \{1, \ldots, d\}$ and all $t \in [0, \pi]$ with $c(t) \in x(U)$. (Here $c^i := x^i \circ c$.) It follows that also

$$\ddot{c}^i(t) + \widetilde{\Gamma}^i_{jk}(c(t))\, \dot{c}^j(t) \dot{c}^k(t) = 0$$

for all $t \in [0, \pi]$ with $c(t) \in x(U)$. The fact that this holds for every chart (U, x) on S^d implies that c is a geodesic in S^d_f. $\square$

[52] As usual, "x" denotes *two* things, namely a map from U to $\mathbb{R}^d$ and also a general point in $x(U)$; also "$f(x)$" really stands for "$f(x^{-1}(x))$".

(b). Let $f \in \mathcal{F}_2$. Then c is a geodesic in S^d_f, by part (a). We also know that *in* S^d, $c(0)$ and $c(\pi)$ are conjugate along c, and there is no point before $c(\pi)$ conjugate to $c(0)$ along c. (This can for example be easily verified from the explicit formula for a general Jacobi field along a geodesic in constant curvature; cf. pp. 5–6 in #17. Alternatively, the statement can be proved using Theorem 1 in #18 combined with the known facts that any arc of length $< \pi$ of a great circle on S^d is a strict local minimum for L, while any arc of length $> \pi$ of a great circle on S^d is not a local minimum for L.)

Hence it now suffices to prove that an arbitrary vector field along c is a Jacobi field in S^d_f iff it is a Jacobi field in S_f.

Thus consider an arbitrary vector field X along c. Note that apriori "$\dot{X}(t)$" stands for different things in S^d and S^d_f, since it is defined in terms of the Levi-Civita connection. However in local coordinates, the expression for $\dot{X}(t)$ only involves the Christoffel symbols *evaluated at points along c;*[53] and we know from part (a) that these agree for S^d and S^d_f (since $f \in \mathcal{F}_2 \subset \mathcal{F}_1$). Hence "$\dot{X}(t)$" means the same thing in S^d and S^d_f, for any vector field X along c. Repeated use of this fact implies that also $\ddot{X}(t)$ means the same thing in S^d and S^d_f, for any vector field X along c.

Recall that by definition, X is a Jacobi field iff "$\ddot{X} + R(X, \dot{c})\dot{c} \equiv 0$"; hence it now only remains to prove that "$R(X, \dot{c})\dot{c}$" stands for the same thing in S^d and S^d_f. However, if (U, x) is an arbitrary chart on M, and $J := \{t \in [0, \pi] \ : \ c(t) \in U\}$, and if X is represented by the functions $a^1, \ldots, a^d \in C^\infty(J)$ (viz., $X(t) = a^j(t) \cdot (\frac{\partial}{\partial x^j})_{c(t)}$, $\forall t \in J$), then

$$R(X(t), \dot{c}(t))\dot{c}(t) = R^k_{jim}(c(t)) \cdot a^i(t)\, \dot{c}^m(t)\, \dot{c}^j(t) \left(\frac{\partial}{\partial x^k}\right)_{c(t)}, \qquad \forall t \in J.$$

Hence it suffices to prove that "$R^k_{jim}(c(t))$" is the same for S^d and S^d_f, for all $t \in J$. In view of the formula

$$R^k_{jim} = \frac{\partial \Gamma^k_{mj}}{\partial x^i} - \frac{\partial \Gamma^k_{ij}}{\partial x^m} + \Gamma^k_{il}\Gamma^l_{mj} - \Gamma^k_{ml}\Gamma^l_{ij} \qquad \text{(valid in all } U),$$

it suffices to prove that "Γ^k_{mj}" and "$\frac{\partial}{\partial x^i}\Gamma^k_{mj}$" are the same for S^d and S^d_f, at every point $c(t)$, $t \in J$ (and for all $m, j, k, i \in \{1, \ldots, d\}$). For Γ^k_{mj} this

[53] Indeed, let (U, x) be an arbitrary chart on M and set $J := \{t \in [0, \pi] \ : \ c(t) \in U\}$; then there exist unique functions $a^1, \ldots, a^d \in C^\infty(J)$ such that

$$X(t) = a^j(t) \cdot \left(\frac{\partial}{\partial x^j}\right)_{c(t)}, \qquad \forall t \in J.$$

Then if $\Gamma^l_{jk} \in C^\infty(x(U))$ are the Christoffel symbols for the Levi-Civita connection, we have

$$\dot{X}(t) = \nabla_{\dot{c}(t)} X(t) = \left(\dot{a}^l(t) + \dot{c}^j(t)\, a^k(t)\, \Gamma^l_{jk}(c(t))\right) \left(\frac{\partial}{\partial x^l}\right)_{c(t)}, \qquad \forall t \in J.$$

Cf. Lecture # 9, p. 8.

was proved in the solution to part (a). In order to prove it also for the derivative "$\frac{\partial}{\partial x^i}\Gamma^k_{mj}$", we see from the formula (223) that it suffices to prove that $g_{jl}(x)$ and all its first and second derivatives "are the same" for S^d and S^d_f, at every point $c(t)$ ($t \in J$), and also that the corresponding thing holds for $g^{il}(x)$ and all its first derivatives. We already know this fact for $g_{jl}(x)$ and $g^{il}(x)$ themselves, and also for all the first derivatives of $g_{jl}(x)$; cf. part (a). Thus, in the notation from the solution to part (a), what remains to prove is that, for any $k, i, l, j \in \{1, \ldots, d\}$,

$$\frac{\partial}{\partial x^k} g^{il}(x) = \frac{\partial}{\partial x^k} h^{il}(x) \tag{224}$$

and also

$$\frac{\partial^2}{\partial x^i \partial x^k} g_{jl}(x) = \frac{\partial^2}{\partial x^i \partial x^k} h_{jl}(x) \tag{225}$$

for all x along c. Note that $h^{il}(x) = f(x)^{-1} \cdot g^{il}(x)$ throughout $x(U)$, and also that for every x along c we have $f(x)^{-1} = 1$ and $\frac{\partial}{\partial x^k}(f(x)^{-1}) = -f(x)^{-2}\frac{\partial}{\partial x^k} f(x) = 0$; with these observations, (224) follows in the same way as (222). On the other hand, (225) follows from

$$\frac{\partial^2}{\partial x^i \partial x^k} g_{jl}(x) = \Big(\frac{\partial^2}{\partial x^i \partial x^k} f(x)\Big) \cdot g_{jl}(x) + \Big(\frac{\partial}{\partial x_i} f(x)\Big) \cdot \frac{\partial}{\partial x_k} g_{jl}(x) + \Big(\frac{\partial}{\partial x_k} f(x)\Big) \cdot \frac{\partial}{\partial x_i} g_{jl}(x) + f(x) \cdot \frac{\partial^2}{\partial x^i \partial x^k} g_{jl}(x),$$

and the fact that if x lies on c then $f(x) = 1$ and $\frac{\partial^2}{\partial x^i \partial x^k} f(x) = \frac{\partial}{\partial x_i} f(x) = \frac{\partial}{\partial x_k} f(x) = 0$ (since $f \in \mathcal{F}_2$). Done! □

(c). Take $t_1 \in [0, \pi]$ so that $q := c(t_1)$ lies in U. Then also $c(t) \in U$ for all t in some neighborhood of t_1 in $[0, \pi]$. Hence we may assume from start that $t_1 \in (0, \pi)$. Now take $r > 0$ so small that $B_q(r) \subset U$, where $B_q(r)$ is the open ball in S^d_f of radius r about the point q. Let $c_1 : [0, \pi] \to S^d_f$ be any pw C^∞ curve with $c_1(0) = c(0)$ and $c_1(\pi) = c(\pi)$ and $d_f(c_1(t), c(t)) < r$, $\forall t \in [0, \pi]$. We then claim that $L_f(c_1) \geq L_f(c)$ with equality only if c_1 is a reparametrization of c. Here and in the following we write d_f and L_f for the metric and length of curves in S^d_f, and we will write d and L for the corresponding things in S^d.

Note that $[v, w] \geq \langle v, w \rangle$ for all $p \in S^d$, $v, w \in T_p S^d$, since $f \geq 1$ everywhere. Hence

$$L_f(c_1) = \int_0^\pi \sqrt{[\dot c_1(t), \dot c_1(t)]}\, dt \geq \int_0^\pi \sqrt{\langle \dot c_1(t), \dot c_1(t)\rangle}\, dt = L(c_1) \geq L(c) = L_f(c), \tag{226}$$

where $L(c_1) \geq L(c)$ holds since we know that c is a distance minimizing geodesic in S^d, and $L(c) = L_f(c)$ holds since $f = 1$ along c. Hence we have proved the desired inequality, $L_f(c_1) \geq L_f(c)$, and it only remains to prove the statement about when equality holds.

Thus assume $L_f(c_1) = L_f(c)$. Then equality must hold in both "$\geq$" in (226). The fact that equality holds in the second "$\geq$" in (226) implies that c_1 is a geodesic in S^d, up to reparametrization (cf. Problem 24). However we know that the geodesics in S^d are exactly the (pieces of) great circles in S^d; hence we conclude that c_1 is an arc of a great circle between the antipodal points $c(0)$ and $c(\pi)$. If this great circle is not equal to (the image of) c itself then $c_1(t) \notin c([0, \pi])$ for all $t \in (0, \pi)$. In particular $c_1(t_1) \notin c([0, \pi])$. But we have $d_f(c_1(t_1), c(t_1)) < r$ by assumption, i.e. $c_1(t_1) \in B_q(r)$. Hence $c_1(t_1) \in U \setminus c([0, \pi])$, and therefore $f(c_1(t_1)) < 1$. This implies that

$$[\dot{c}_1(t), \dot{c}_1(t)] > \langle \dot{c}_1(t), \dot{c}_1(t) \rangle$$

for all t in some neighborhood of t_1, and therefore the first "$\geq$" in (226) must be a strict inequality, contradicting $L_f(c_1) = L_f(c)$! Hence the great circle $c_1([0, \pi])$ must be equal to $c([0, \pi])$, i.e. c_1 is a reparametrization of c, qed. $\square$

(d). Take $t_1 \in (0, \pi)$ and r as in part (c). Let $c_1 : [0, \pi] \to S^d$ be any great circle from $c(0)$ to $c(\pi)$, parametrized by arc length, not equal to c itself and satisfying $d_f(c_1(t_1), c(t_1)) < r$. We will then prove that $L_f(c_1) < L_f(c)$. Since such curves c_1 can be chosen with $\sup_{t \in [0,\pi]} d_f(c_1(t), c(t))$ arbitrarily small[54] this will complete the proof that c is not a local minimum for L in S_f^d among pw C^∞ curves with fixed endpoints (in fact it even follows that there exists a proper variation $c(t, s)$ of c such that $L(c_s) < L(c)$ for all $s \neq 0$ near 0).

We have:

$$L_f(c_1) = \int_0^\pi \sqrt{[\dot{c}_1(t), \dot{c}_1(t)]}\, dt < \int_0^\pi \sqrt{\langle \dot{c}_1(t), \dot{c}_1(t) \rangle}\, dt = L(c_1) = \pi$$
$$= L(c) = L_f(c).$$

The "$<$" in the above computation holds since $f \leq 1$ throughout S^d and since by an argument as in part (c) we have

$$[\dot{c}_1(t), \dot{c}_1(t)] < \langle \dot{c}_1(t), \dot{c}_1(t) \rangle$$

for all t in some neighborhood of t_1. Hence we have proved $L_f(c_1) < L_f(c)$, as desired! $\square$

[54]Indeed, note that $d_f(p, q) \leq (\sup_{S^d} \sqrt{f}) \cdot d(p, q)$, $\forall p, q \in S^d$, and we can make $\sup_{t \in [0,\pi]} d(c_1(t), c(t))$ arbitrarily small.

Problem 92: (Cf. Cheeger & Ebin [2, Cor. 1.30].)

In view of the definition of the length of a curve in a Riemannian manifold, and the fact that $\frac{d}{dt}(\exp_p(c(t))) = d(\exp_p)_{c(t)}(\dot{c}(t))$ (and similarly for $\exp_{p_0}$), it suffices to prove that

$$\|d(\exp_p)_{c(t)}(\dot{c}(t))\| \geq \|d(\exp_{p_0})_{c(t)}(\dot{c}(t))\|, \qquad \forall t \in [a,b].$$

(Here in the left hand side $\|\cdot\|$ is the norm on $T_{\exp_p(c(t))}(M)$ coming from the Riemannian metric on M, while in the right hand side $\|\cdot\|$ is the norm on $T_{\exp_{p_0}(c(t))}(M_0)$ coming from the Riemannian metric on M_0.) We will prove the stronger statement that

$$(227) \quad \|d(\exp_p)_x(v)\| \geq \|d(\exp_{p_0})_x(v)\|, \qquad \forall x \in B_r(0),\ v \in T_x\mathbb{R}^d = \mathbb{R}^d.$$

If $v = 0$ then (227) is trivial. If $x = 0$ then $d(\exp_p)_x$ is the identity map on $T_p(M) = \mathbb{R}^d = T_0(\mathbb{R}^d)$, and similarly for $d(\exp_{p_0})_x$, and therefore (227) holds with equality for all $v \in \mathbb{R}^d$. From now on we assume both $v \neq 0$ and $x \neq 0$. Let us decompose v as $v = u + w$ where $u \in \mathbb{R}x$ and $w \cdot x = 0$. (Thus $u = \dfrac{v \cdot x}{x \cdot x} \cdot x$.) Then by Gauss' Lemma (= Cor. 2 in Lecture #17), $d(\exp_p)_x(u)$ and $d(\exp_p)_x(w)$ are orthogonal in $T_{\exp_p(x)}(M)$, and $\|d(\exp_p)_x(u)\| = \|u\|$ (the standard length of u as a vector in $\mathbb{R}^d$); hence

$$(228) \qquad \|d(\exp_p)_x(v)\| = \sqrt{\|u\|^2 + \|d(\exp_p)_x(w)\|^2}.$$

The analogous formula holds for $\|d(\exp_{p_0})_x(v)\|$, and hence, in order to prove (227), it suffices to prove the corresponding inequality with w in place of v. In other words, it suffices to prove (227) under the extra assumption that $v \cdot x = 0$. We impose this assumption from now on.

Set

$$\hat{x} := \|x\|^{-1}x \qquad \text{and} \qquad \hat{v} := \|x\|^{-1}v.$$

Let $\gamma : [0, \|x\|] \to M$ be the geodesic $\gamma(t) = \exp_p(t\hat{x})$; note that γ is parametrized by arc length, i.e. $\|\dot{\gamma}(t)\| = 1$ for all t, since $\|\hat{x}\| = 1$. Set

$$J(t) := (d\exp_p)_{t\hat{x}}(t\hat{v}) \qquad \text{for } t \in [0, \|x\|].$$

Then by Corollary 1 (and Lemma 1) in Lecture #17, J is a Jacobi field along γ with $J(0) = 0$, $\dot{J}(0) = \hat{v}$. Note that $J^{\tan}(0) = 0$, and $\dot{J}^{\tan}(0) = 0$ since $\hat{v} \cdot \hat{x} = 0$, i.e. $\langle \hat{v}, \hat{x} \rangle = 0$ in T_pM. Also $J^{\tan}$ is a Jacobi field along γ by Lemma 3 in Lecture #17; hence $J^{\tan} \equiv 0$ by Lemma 1 in Lecture #17.

Similarly if we also set $\gamma_0(t) = \exp_{p_0}(t\hat{x})$ and

$$J_0(t) := (d\exp_{p_0})_{t\hat{x}}(t\hat{v}) \qquad \text{for } t \in [0, \|x\|]$$

then γ_0 is a geodesic in M_0 and J_0 is a Jacobi field along γ_0 with $J_0(0) = 0$, $\dot{J}_0(0) = \hat{v}$ and $J_0^{\tan} \equiv 0$.

Now let $X_1, \ldots, X_d$ be parallel vector fields along γ_0 which form an ON-basis in $T_{\gamma(0)}(M_0)$ (and hence in each tangent space $T_{\gamma(t)}(M_0)$), and which are chosen so that $X_1(0) = \dot{\gamma}_0(0)$ (thus $X_1(t) = \dot{\gamma}_0(t)$ for all $t \in [0,1]$) and

$$\hat{v} = k \cdot X_2(0) \qquad \text{for some } \; k > 0$$

(recall that we assume $v \neq 0$; thus $\hat{v} \neq 0$). Then by pp. 5–6 in Lecture #17,

$$J_0(t) = k \cdot s_\mu(t) \cdot X_2(t), \qquad \forall t \in [0, \|x\|],$$

where

$$s_\mu(t) = \begin{cases} \mu^{-1/2} \sin(\mu^{1/2} t) & (\mu > 0) \\ t & (\mu = 0) \\ |\mu|^{-1/2} \sinh(|\mu|^{1/2} t) & (\mu > 0). \end{cases}$$

Recall that we are assuming that $\exp_{p_0}$ restricted to $B_r(0)$ is a diffeomorphism. This implies that $(d \exp_{p_0})_{t\hat{x}}$ is a linear bijection for each $t \in [0, \|x\|]$, and in particular $J_0(t) \neq 0$ for all $t \in [0, \|x\|]$. This implies that $s_\mu(t) > 0$ for all $t \in (0, \|x\|]$. Now the Rauch Comparison Theorem (Theorem 1 in Lecture #19) applies to our situation, with

$$f_\mu = \|J\|^{\cdot}(0) \cdot s_\mu = k \cdot s_\mu$$

(indeed we have $\|J\|^{\cdot}(0) = \|\dot{J}(0)\| = \|\hat{v}\|$ since $J(0) = 0$), and that theorem implies

$$\|J(t)\| \geq f_\mu(t) = \|J_0(t)\|, \qquad \forall t \in [0, \|x\|].$$

Taking $t = \|x\|$ in the last inequality, we conclude that (227) holds! $\square$

Problem 93:

Assume that $\tau \in (a,b)$ and that $c(\tau)$ is a focal point of γ along c. Let X be a nontrivial Jacobi field along c satisfying $X(\tau) = 0$ and

$$X(a) \in \mathrm{Span}(\dot\gamma(0)) \quad \text{and} \quad \langle \dot X(a), \dot\gamma(0)\rangle = 0 \qquad \text{in } T_{c(a)}(M). \tag{229}$$

It follows that $X^{\tan}(a) = 0$ (since $\langle \dot c(a), \dot\gamma(0)\rangle = 0$) and $X^{\tan}(\tau) = 0$; hence by Lemmata 2,3 in Lecture #17 we must have $X^{\tan} \equiv 0$. Define the pw C^∞ vector field Y along c by

$$Y(t) = \begin{cases} X(t) & \text{if } t \in [a,\tau] \\ 0 & \text{if } t \in [\tau, b]. \end{cases}$$

(Note in particular that Y is a well-defined and continuous function, since $X(\tau) = 0$.) We have:

$$\begin{aligned} I(Y,Y) = I(X^1, X^1) &= \int_a^\tau \Big(\langle \dot X, \dot X\rangle - \langle R(\dot c, X)X, \dot c\rangle \Big)\, dt \\ &= \int_a^\tau \Big(\frac{d}{dt}\langle \dot X, X\rangle - \langle \ddot X + R(X,\dot c)\dot c, X\rangle \Big)\, dt \\ &= \langle \dot X(\tau), X(\tau)\rangle - \langle \dot X(a), X(a)\rangle = 0, \end{aligned}$$

where the last equality holds since $X(\tau) = 0$ and because of (229).

Consider any $Z \in \overset{\circ}{\mathcal{V}}_c$ (viz., $Z \in \Gamma_c(TM)$ with $Z(a) = 0 = Z(b)$). Write Z^1 for the restriction of Z to $[a,\tau]$. Then

$$\begin{aligned} I(Y+Z, Y+Z) &= I(Y,Y) + 2\cdot I(X^1, Z^1) + I(Z,Z) \\ &= 2\cdot I(X^1, Z^1) + I(Z,Z), \end{aligned}$$

and here

$$\begin{aligned} I(X^1, Z^1) &= \int_a^\tau \Big(\langle \dot X, \dot Z\rangle - \langle R(\dot c, X)Z, \dot c\rangle \Big)\, dt \\ &= \int_a^\tau \Big(\frac{d}{dt}\langle \dot X, Z\rangle - \langle \ddot X + R(X,\dot c)\dot c, Z\rangle \Big)\, dt \\ &= \langle \dot X(\tau), Z(\tau)\rangle - \langle \dot X(a), Z(a)\rangle \\ &= \langle \dot X(\tau), Z(\tau)\rangle. \end{aligned}$$

As in the proof of Theorem 1 in #18 we now *fix* a vector field $Z \in \overset{\circ}{\mathcal{V}}_c$ which is normal and which satisfies $Z(\tau) = -\dot X(\tau)$.[55] Applying the above formulas with ηZ ($\eta \in \mathbb{R}$) in place of Z gives

$$I(Y + \eta Z, Y + \eta Z) = -2\cdot \|\dot X(\tau)\|^2 \cdot \eta + I(Z,Z)\cdot \eta^2.$$

[55]To see that this can be done, note that $-\dot X(\tau)$ is normal against $\dot c(\tau)$; hence we can simply set Z equal to the parallel transport of $-\dot X(\tau)$ along c, multiplied by some smooth function $f : [a,b] \to \mathbb{R}$ with $f(\tau) = 1$ and $f(a) = f(b) = 0$.

Here $\|\dot{X}(\tau)\| > 0$ since X is nontrivial and $X(\tau) = 0$ (cf. Lemma 1 in #17). Hence for $\eta > 0$ small enough we have

$$I(Y + \eta Z, Y + \eta Z) < 0.$$

Fix such an η. Note that $Y + \eta Z$ is a normal pw C^∞ vector field along c, and by Problem 88 there is a C^∞ vector field U along c which *also* satisfies $I(U, U) < 0$, as well as $U(t) = Y(t) + \eta Z(t)$ for all t near a or b; in particular $U(a) = X(a) \in \mathrm{Span}(\dot{\gamma}(0))$ and $U(b) = 0$. Using the fact that $Y + \eta Z$ is normal along c and inspecting the solution to Problem 88, we see that we can also take U to be normal along c.

Take $k \in \mathbb{R}$ so that $U(a) = k \cdot \dot{\gamma}(0)$. Now by Problem 83(b) there exists a C^∞ variation $c : [a, b] \times (-\varepsilon, \varepsilon) \to M$ of c satisfying $c' \equiv U$ as well as $c(a, s) = \gamma(ks)$ and $c(b, s) = c(b)$ for all $s \in (-\varepsilon, \varepsilon)$. Set $L(s) := L(c_s)$. By Lemma 1 in #16 (and a remark on p. 3 of #16, and using $\langle U(a), \dot{c}(0)\rangle = 0$ and $U(b) = 0$) we have $L'(0) = 0$. Next we apply Theorem 1 in #16. Note that ${c'}^{\perp} \equiv c'$ since U is normal along c. Note also that $\nabla_{\frac{\partial}{\partial s}} c' = 0$ at $t = a$ and at $t = b$, since $c(a, s) = \gamma(ks)$ (a geodesic) and $c(b, s) = c(b)$ for all $s \in (-\varepsilon, \varepsilon)$. Hence Theorem 1 in #16 says:

$$L''(0) = \frac{1}{\|\dot{c}\|} I(U, U).$$

Hence by what we proved above, $L''(0) < 0$. This means that after shrinking ε if necessary, we have $L(s) < L(0)$ for all $s \in (-\varepsilon, \varepsilon) \setminus \{0\}$. Done! □

(Remark: An alternative solution, which is perhaps a bit simpler and even closer to the proof of Theorem 1 in #18, is to construct the vector field U without insisting that U is normal – but with all the other properties required above. Then we use Theorem 1 in #16 to deduce $E''(0) < 0$ in place of $L''(0) = 0$. We also have $E'(0) = 0$; and then the argument in the notes for #18 applies and lets us conclude what we want, i.e. that $L(s) < L(0)$ for all $s \neq 0$ sufficiently near 0.)

Problem 94: See Lee, [14, Lemma 11.6].

Problem 95: (We follow the proof of [14, Theorem 11.12].)

Let the constant sectional curvature of M be $\rho \in \mathbb{R}$.

Let us first *assume* $\rho \leq 0$. If $\rho = 0$ then set $H := \mathbb{R}^n$; if $\rho < 0$ then set $H = H^n(|\rho|)$, i.e. hyperbolic n-space scaled to have constant curvature ρ.[56] Fix any point $p \in M$. By the Cartan-Hadamard Theorem (=Theorem 2 in Lecture #19), $\exp_p : T_pM \to M$ is a surjective diffeomorphism. Hence if we fix any identification of T_pM with $\mathbb{R}^n$ respecting the inner product, then the C^∞ chart $(M, \exp_p^{-1})$ give normal coordinates on all M with center p. Similarly, for any fixed point $q \in H$, $(H, \exp_q^{-1})$ give normal coordinates on all H with center q. (Here $\exp_q$ is the exponential map from $\mathbb{R}^n = T_qH$ to H.) Now by Problem 78, since M and H have the same constant curvature ρ everywhere, the two C^∞ functions $\mathbb{R}^n \to M_n(\mathbb{R})$ which give the Riemannian metric of M wrt. $(M, \exp_p^{-1})$ and the Riemannian metric of H wrt. $(H, \exp_q^{-1})$, respectively, are *equal;* indeed this function $\mathbb{R}^n \to M_n(\mathbb{R})$ is given explicitly in Problem 78. Hence the map $\exp_p \circ \exp_q^{-1}$ is an isometry of H onto M, and we are done!

Next *assume* $\rho > 0$. Let $S \subset \mathbb{R}^{n+1}$ be the n-dimensional sphere with radius $r := \rho^{-1/2}$ about the origin, with its standard Riemannian metric. S has constant sectional curvature ρ. For any $q \in S$ we know that $\exp_q : T_qS \to S$ restricted to $B_{\pi r}(0) \subset T_qS$ is a diffeomorphism onto $S \setminus \{-q\}$; hence the chart $(S \setminus \{-q\}, \exp_q^{-1})$ give normal coordinates on S with center q. Also for any $p \in M$ it follows from Cor. 1 in #19 that $\exp_p$ restricted to $B_{\pi r}(0) \subset T_pM$ has everywhere non-singular differential, and hence is a local diffeomorphism (by the Inverse Function Theorem). Hence the ball $B_{\pi r}(0) \subset T_pM$ can be endowed with a unique Riemannian metric M such that $\exp_p : B_{\pi r}(0) \to M$ is a local isometry (cf. Problem 18). Note that $B_{\pi r}(0)$ with the Riemannian metric M has constant sectional curvature $= \rho$, since M has so. We fix identifications $T_qS = \mathbb{R}^n$ and $T_pM = \mathbb{R}^n$ respecting the inner products. Note that then $(B_{\pi r}(0), I)$ (where I is the identity map) form normal coordinates on $B_{\pi r}(0)$ wrt the metric M, since for any unit vector $v \in \mathbb{R}^n$ the curve $c : (-\pi r, \pi r) \to B_{\pi r}(0)$, $c(t) = tv$, is a geodesic. Hence the metric M is explicitly given in by the formula in Problem 78! On the other hand the expression for the metric on S wrt $(S \setminus \{-q\}, \exp_q^{-1})$ must be given by the same formula. Hence the map

$$\Phi := \exp_p \circ \exp_q^{-1}$$

from $S \setminus \{-q\}$ to M is a *local isometry.*

[56] We obtain $H^n(|\rho|)$ by replacing the Riemannian metric $\langle \cdot, \cdot \rangle$ on the standard hyperbolic n-space, H^n, by the Riemannian metric $[\cdot, \cdot] := |\rho|^{-1} \langle \cdot, \cdot \rangle$. Cf. Problem 70. Note that there's a misprint in Jost, [12, p. 228 (line -1)]; his "ρ" should be "ρ^{-1}".

Next fix any $q' \in S \setminus \{q, -q\}$ and set $p' := \Phi(q') \in M$. Then by repeating the above discussion we obtain a local isometry

$$\widetilde{\Phi} := \exp_{p'} \circ \exp_{q'}^{-1} \tag{230}$$

from $S \setminus \{-q'\}$ to M with $\widetilde{\Phi}(q') = p'$. Note that this $\widetilde{\Phi}$ depends on which identifications $T_{q'}S = \mathbb{R}^n$ and $T_{p'}M = \mathbb{R}^n$ we choose; more to the point what matters is how $T_{q'}S$ gets identified with $T_{p'}M$, since this is what is needed to make unique sense of (230). Any identification respecting the respective Riemannian scalar products on $T_{q'}S$ and $T_{p'}M$ is ok. Let us choose the identification of $T_{q'}S$ and $T_{p'}M$ to be given by $d\Phi_{q'} : T_{q'}S \to T_{p'}M$; this is indeed a linear bijection respecting the respective Riemannian scalar products, since Φ is a local isometry. Having made this choice, our map $\widetilde{\Phi}$ satisfies

$$d\widetilde{\Phi}_{q'} = d\Phi_{q'},$$

since $(d\exp_{p'})_0 : T_{p'}M = T_{p'}(T_{p'}M) \to T_{p'}M$ is the identity map, and similarly for $(d\exp_{q'})_0$. Hence by the following lemma, we actually have

$$\widetilde{\Phi}_{|S\setminus\{-q,-q'\}} \equiv \Phi_{|S\setminus\{-q,-q'\}}. \tag{231}$$

Lemma 9. *Let N and $\widetilde{N}$ be Riemannian manifolds and let $\varphi, \psi : N \to \widetilde{N}$ be local isometries. Suppose that for some point $p \in N$ we have $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$. Then $\varphi \equiv \psi$.*

Proof. First assume that $q \in N$ is such that there exists a geodesic from p to q, say $c(t) = \exp_p(tv)$, $t \in [0, T]$, for some $v \in T_p(M)$. (Thus $\exp_p(Tv) = q$.) Then since φ and ψ are local isometries, both $\varphi \circ c$ and $\psi \circ c$ are geodesics in $\widetilde{N}$. These two geodesics have $\varphi \circ c(0) = \varphi(p) = \psi(p) = \psi \circ c(0)$ and

$$(\varphi \circ c)^{\cdot}(0) = (d\varphi_p)(\dot{c}(0)) = (d\psi_p)(\dot{c}(0)) = (\psi \circ c)^{\cdot}(0);$$

hence by uniqueness of geodesics, $\varphi \circ c(t) = \psi \circ c(t)$ for all $t \in [0, T]$, and in particular

$$\varphi(q) = \varphi \circ c(T) = \psi \circ c(T) = \psi(q).$$

It follows from the above that if U is any open subset of N such that every point $q \in U$ can be reached by a geodesic from p, then $\varphi_{|U} = \psi_{|U}$.

Now let q be an arbitrary point in N. By Problem 1 there is a curve $c : [0, 1] \to M$ with $c(0) = p$, $c(1) = q$. Consider the set

$$F := \{t \in [0, 1] \;:\; \varphi(c(t)) = \psi(c(t)) \;\text{ and }\; d\varphi_{c(t)} = d\psi_{c(t)}\}.$$

This is a closed subset of $[0, 1]$, since $\varphi, \psi, d\varphi, d\psi$ are all continuous. Also $0 \in F$, since $\varphi(p) = \psi(p)$ and $d\varphi_p = d\psi_p$ by assumption. Let us prove that F is also an *open* subset of $[0, 1]$. Thus take any $t \in F$. Then $\varphi(c(t)) = \psi(c(t))$ and $d\varphi_{c(t)} = d\psi_{c(t)}$. By Theorem 3 in #4 there exists an open neighborhood U of $c(t)$ in N such that every point in U can be reached by a geodesic from

$c(t)$. Now by the above argument applied to the point $c(t)$ in place of p, $\varphi_{|U} = \psi_{|U}$, and hence $\varphi(q) = \psi(q)$ and $d\varphi_q = d\psi_q$ for all $q \in U$. Hence F contains the set $\{t' \in [0,1] \, : \, c(t') \in U\}$, and this is an open neighborhood of t in $[0,1]$. Since every point $t \in F$ has such an open neighborhood, it follows that F is open in $[0,1]$. Hence F is *connected*, being both open and closed. This together with $0 \in F$ implies $F = [0,1]$. In particular $1 \in F$, and thus $\varphi(q) = \psi(q)$. Since q was arbitrary, this completes the proof that $\varphi \equiv \psi$. □

Continuing with our proof of the Killing-Hopf Theorem, we have now proved (231) and this means that Φ and $\widetilde{\Phi}$ together define a C^∞ map Ψ from the whole of S to M, and this map Ψ is a local isometry since Φ and $\widetilde{\Phi}$ are.

Lemma 10. *Suppose that f is a local diffeomorphism from a C^∞ manifold N_1 to a C^∞ manifold N_2. Assume that N_1 is compact. Then f is a covering map.*

Proof. Since N_1 is compact and f is continuous, $f(N_1)$ is a compact subset of N_2. In particular $f(N_1)$ is a closed subset of N_2. But $f(N_1)$ is also an open subset of N_2 since f, being a local diffeomorphism, is an open map. Hence $f(N_1)$ (which is of course non-empty) is a connected component of N_2, i.e. $f(N_1) = N_2$. Hence we have proved that f is *surjective* (and also that N_2 is compact).

Now let p be an arbitrary point in N_2. Then $f^{-1}(p)$ is a closed subset of N_1, and hence compact, since N_1 is compact. Furthermore $f^{-1}(p)$ is a discrete subset of N_1, since f is a local diffeomorphism. (Indeed, for any $q \in f^{-1}(p)$ there is an open set $U \subset N_1$ with $q \in U$ such that $f_{|U}$ is a diffeomorphism; then $U \cap f^{-1}(p) = f_{|U}^{-1}(p) = \{q\}$, and this says that $\{q\}$ is an open subset of $f^{-1}(p)$ when $f^{-1}(p)$ is endowed with the relative topology as a subset of N_1. Hence $f^{-1}(p)$ with this topology is discrete.) Hence $f^{-1}(p)$, being both compact and discrete, is *finite.* Note that $f^{-1}(p) \neq \emptyset$ since f is surjective. Let us write $f^{-1}(p) = \{q_1, \ldots, q_m\}$ (with $q_1, \ldots, q_m$ pairwise distinct).

Now since f is a local diffeomorphism, there exist open subsets $U_1, \ldots, U_m \subset N_1$ such that $q_j \in U_j$ and $f_{|U_j}$ is a diffeomorphism onto an open subset of N_2, for each j. Also, since N_1 is Hausdorff, there exist open subsets $U_1', \ldots, U_m' \subset N_1$ such that $q_j \in U_j'$ for all j and $U_i' \cap U_j' = \emptyset$ for all $i \neq j$. Set $U_j'' := U_j \cap U_j'$ for $j = 1, \ldots, m$. Then for each j, U_j'' is open, $q_j \in U_j''$ and $f_{|U_j''}$ is a diffeomorphism onto an open subset of N_2, and furthermore the sets $U_1'', \ldots, U_m''$ are pairwise disjoint. Set $V := \cap_{j=1}^m f(U_j'')$. This is an open subset of N_2 containing p. Let $V = V_0 \supset V_1 \supset V_2 \supset \cdots$ be a sequence of open subsets of V which form a neighborhood basis for the point p. (Viz., $p \in V_k$ for all k, and for every open set V' containing p there is some k

such that $V_k \subset V'$. For example we can take $V_1, V_2, \ldots$ to be a decreasing sequence of open balls around p with radius tending to zero, with respect to any fixed chart containing p.) Assume that $f^{-1}(V_k) \not\subset \cup_{j=1}^m U_j''$ for every $k \geq 1$. This means that for every $k \geq 1$ there exists some point $p_k' \in N_1$, $p_k' \notin \cup_{j=1}^m U_j''$, such that $f(p_k') \in V_k$. Since N_1 is compact, after passing to a subsequence we may assume that p_k' tends to a limit point $p' \in N_1$ as $k \to \infty$. Then $f(p_k') \to f(p')$ in N_2, and $f(p_k') \in V_k$, and by our choice of $V_1, V_2, \ldots$ this implies that $f(p') = p$. On the other hand $p' \notin \{q_1, \ldots, q_m\}$, since for each j we have $q_j \in U_j''$ with U_j'' open, and $p_k' \notin U_j''$. This is a *contradiction* against $f^{-1}(p) = \{q_1, \ldots, q_m\}$. Hence we must have $f^{-1}(V_k) \subset \cup_{j=1}^m U_j''$ for some $k \geq 1$. For this k, $f^{-1}(V_k)$ equals the disjoint union of the sets $W_j := U_j'' \cap f^{-1}(V_k) = f_{|U_j''}^{-1}(V_k)$, $j = 1, \ldots, m$, and $f_{|W_j}$ is a diffeomorphism of W_j onto V_k (since $f_{|U_j''}$ is a diffeomorphism and $V_k \subset f(U_j'')$). The fact that every point $p \in N_2$ has such an open neighborhood V_k proves that f is a covering map. □

The lemma applies to our situation, and gives that $\Psi : S \to M$ is a covering map. Hence since we are assuming that M is simply connected, Ψ must be a homeomorphism, hence an isometry of S onto M. □ □

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