Effective Ratner equidistribution on $ASL_2(\mathbb{R})$

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$ASL_2(\mathbb{R})$

$$G = \mathrm{ASL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2 = \Big\{(g,v) \mid g \in \mathrm{SL}_2(\mathbb{R}),\ v \in \mathbb{R}^2\Big\}$$

$$(g_1, v_1) \cdot (g_2, v_2) = (g_1 g_2, g_1 v_2 + v_1).$$

Action by affine maps (preserving area & orientation) on $\mathbb{R}^2$:

$$\left(g, \begin{pmatrix}\alpha\\ \beta\end{pmatrix}\right) : \quad \begin{pmatrix}x\\ y\end{pmatrix} \mapsto g\begin{pmatrix}x\\ y\end{pmatrix} + \begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

Discrete group:

$$\Gamma = \mathrm{ASL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathbb{Z}^2.$$

$$\text{Let } \boxed{X = \Gamma \setminus G}$$

As a manifold, $X = \Big(\mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})\Big) \times (\mathbb{R}/\mathbb{Z})^2$.

$X = \Gamma \setminus G =$the space of shifted unimodular lattices;

$$\Gamma(g,v) \longleftrightarrow (g,v)^{-1}\mathbb{Z}^2 = (g^{-1}, -g^{-1}v)\mathbb{Z}^2 = g^{-1}\mathbb{Z}^2 - g^{-1}v.$$

Natural probability measure on X: $\quad \mu$
(from the Haar measure on G, hence G-invariant).

$SL_2(\mathbb{R})$

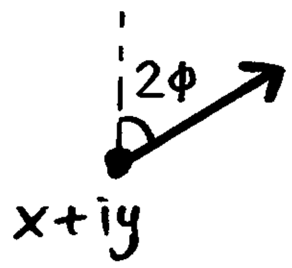
Iwasawa decomposition:

$$SL_2(\mathbb{R}) \ni g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$$

$$x \in \mathbb{R}, \quad y > 0, \quad \phi \in \mathbb{R}/2\pi\mathbb{Z}$$

Can view $SL_2(\mathbb{R})$ as double cover of $T_1\mathcal{H}$:

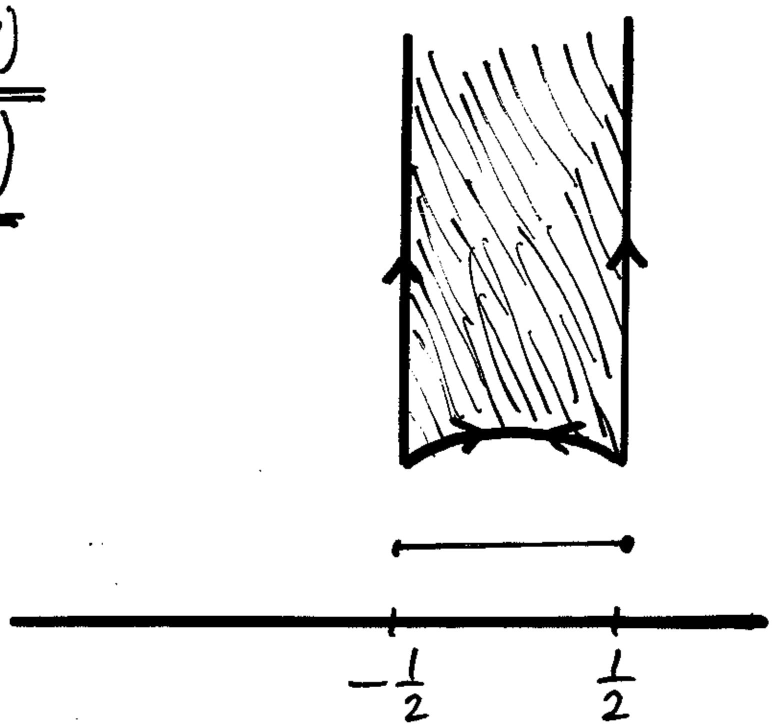
$$[g \text{ (above)}] \longleftrightarrow (\underbrace{x+iy}_{\mathcal{H}}, 2\phi)$$



Hence $\underline{\underline{SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}) \cong T_1(PSL_2(\mathbb{Z}) \backslash \mathcal{H})}}$

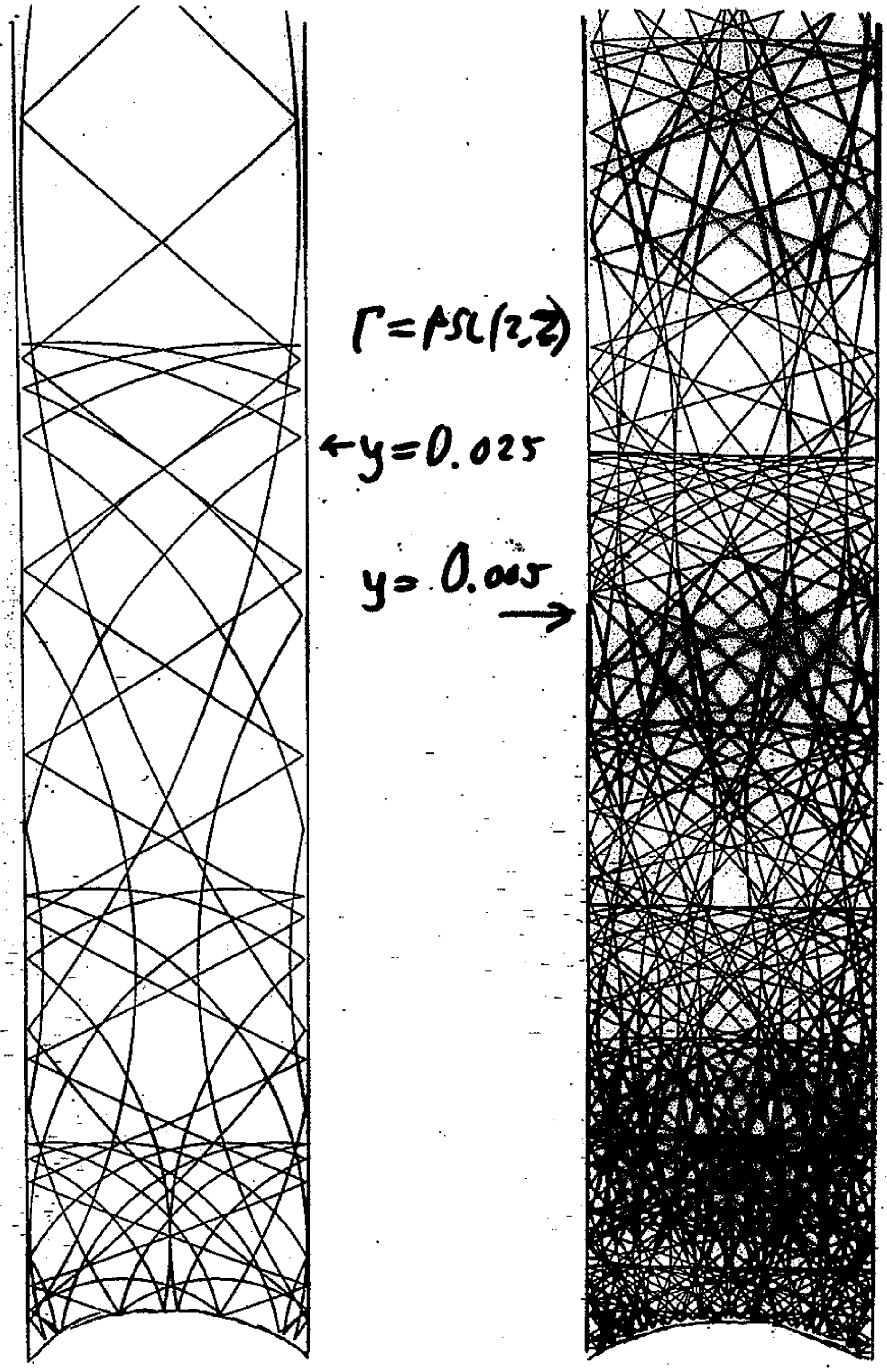
closed horocycle

(length $\frac{1}{y}$)



LONG CLOSED HOROCYCLES

$\Gamma \subset PSL(2,\mathbb{R})$
standard cusp
at ∞



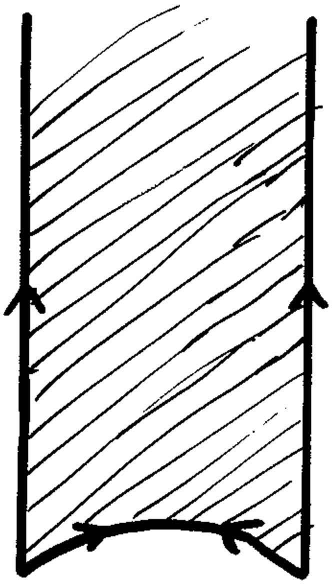
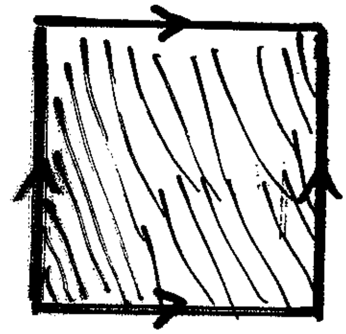
In fact, for any $f \in C^4(SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R}))$, any $0 < y < 1$ and $\beta - \alpha \geq y$:

$$\underline{\frac{1}{\beta-\alpha} \cdot \int_\alpha^\beta f(x+iy, 0)\, dx = \langle f \rangle +} \\ \underline{+ O\left(\|f\|_{W_4} \cdot \frac{y^{\frac{1}{2}-\varepsilon}}{\beta - \alpha} \right)}$$

(impl. const. absolute)

X as manifold

(Recall $\underline{X = \Gamma \backslash G = ASL_2(\mathbb{Z}) \backslash ASL_2(\mathbb{R})}$

T_1 of  $\times$ 

$SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$ $\qquad$ $(\mathbb{R}/\mathbb{Z})^2$

Horocycle lifts to $X = ASL_2(\mathbb{Z}) \backslash ASL_2(\mathbb{R})$

$$\left\{ \left(\begin{pmatrix} \sqrt{y} & \alpha/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right) \;\middle|\; \alpha \in \mathbb{R}/\mathbb{Z} \right\} \qquad (\xi \text{ fixed})$$

closed curve!

--→ statistics of $n\alpha$ mod 1.

(Marklof 2000)

$$\left\{ \left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} \begin{pmatrix} 0 \\ 0 \end{pmatrix} \\ \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \end{pmatrix} \right) \;\middle|\; x \in \mathbb{R}/\mathbb{Z} \right\} \qquad \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \text{ fixed, Diophantine}$$

non-closed curve!

in $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^4 \backslash SL_2(\mathbb{R}) \ltimes \mathbb{R}^4$

--→ pair corr. for $(m-\xi_1)^2 + (n-\xi_2)^2$.

(Marklof 2003)

$$\left\{ \left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} x^2/4 \\ -x/2 \end{pmatrix} \right) \;\middle|\; x \in \mathbb{R}/2\mathbb{Z} \right\}$$

--→ statistics of $\sqrt{n}$ mod 1.

(Elkies & McMullen '04)

Application: $n\alpha$ mod 1, for α random
Take α random in $\mathbb{R}/\mathbb{Z}$, study:

$$\alpha, 2\alpha, 3\alpha, 4\alpha, \cdots \in \mathbb{R}/\mathbb{Z}.$$

Given $\xi \in \mathbb{R}/\mathbb{Z}$, let

$$S_\xi(N) = \sharp\Big\{1 \leqq n \leqq N \mid n\alpha \in \big[\xi - \tfrac{1}{2N}, \xi + \tfrac{1}{2N}\big] \mod 1\Big\}$$

(Also $S_{\ell,\xi}(N) = \sharp\Big\{1 \leqq n \leqq N \mid n\alpha \in \big[\xi - \tfrac{\ell}{2N}, \xi + \tfrac{\ell}{2N}\big] \mod 1\Big\}$.

Theorem: For each $k \in \mathbb{Z}_{\geqq 0}$ there is a number $E(k) > 0$ such that
$\forall \xi \in \mathbb{R}/\mathbb{Z}$ *irrational:* $\displaystyle\lim_{N\to\infty} Prob\{S_\xi(N) = k\} = E(k)$

In fact, we get *same* limit $E(k)$ for ξ *random*.
(However, ξ *rational* gives *different* limit.)

Thus for example

$$\lim_{N\to\infty} Prob\{S_{\sqrt{2}}(N) = 42\} = E(42) = \frac{5291}{913801686\pi^2}$$
$$(\text{and} = \lim_{N\to\infty} Prob\{S_\pi(N) = 42\}).$$

(à la Marklof, 2000)

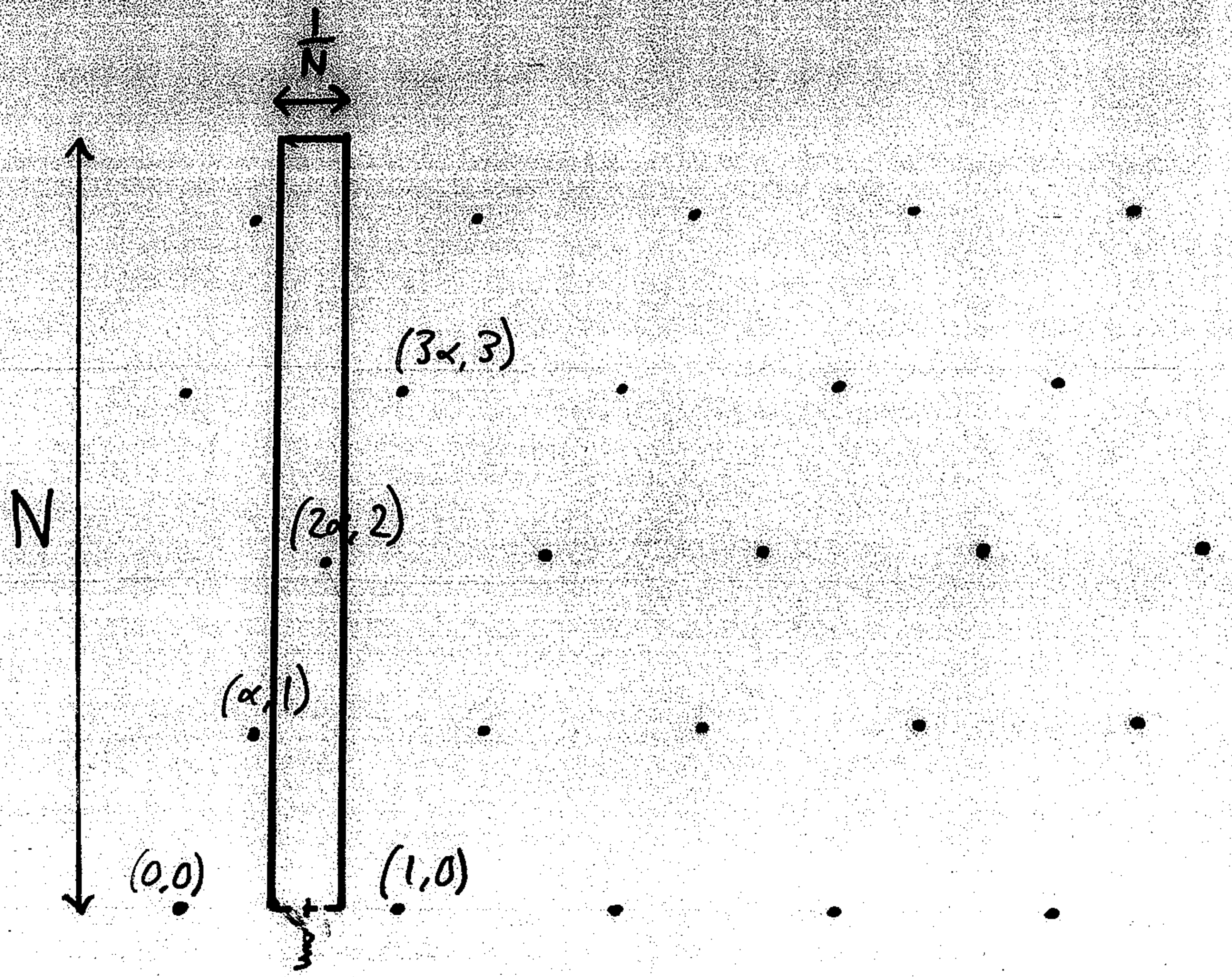
Connection with $\Gamma \setminus G = \mathbf{ASL}_2(\mathbb{Z})\backslash\mathbf{ASL}_2(\mathbb{R})$

$$S_\xi(N) = \sharp\left\{1 \leqq n \leqq N \mid n\alpha \in \left[\xi - \tfrac{1}{2N}, \xi + \tfrac{1}{2N}\right] \mod 1\right\}$$

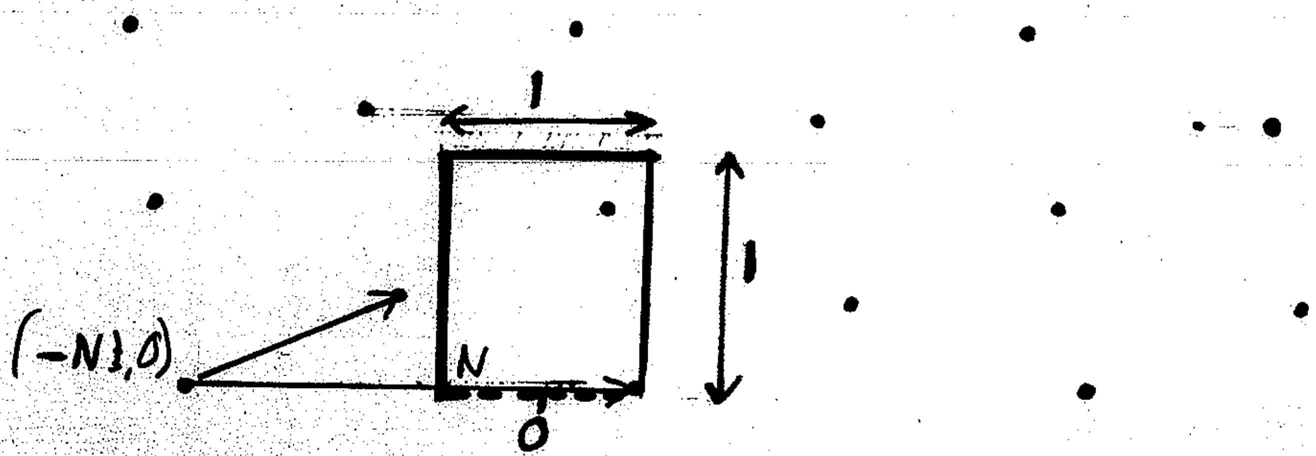
Lattice spanned by $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}\alpha\\1\end{pmatrix}$:

Scale, then shift $N\xi$ to the left!

Lattice: $\begin{pmatrix}N & N\alpha\\0 & 1/N\end{pmatrix}\mathbb{Z}^2 + \begin{pmatrix}-N\xi\\0\end{pmatrix} \longleftrightarrow \boxed{\Gamma\left(\begin{pmatrix}N^{-1} & -N\alpha\\0 & N\end{pmatrix}, \begin{pmatrix}\xi\\0\end{pmatrix}\right) \in \mathbf{\Gamma} \setminus \boldsymbol{G}}$



$\frac{1}{N}$
N
$(3\alpha, 3)$
$(2\alpha, 2)$
$(\alpha, 1)$
$(0,0)$
$(1,0)$


1
1
(-N3,0)
N
O

$S_\xi(N)$ extends to a function on $\Gamma \setminus G$:

$$S(\Gamma g) = \sharp\Big(g^{-1}\mathbb{Z}^2 \cap \big([-\tfrac{1}{2}, \tfrac{1}{2}] \times (0,1]\big)\Big).$$

We get:

$$Prob\Big(S_\xi(N) = k\Big)$$

$$= meas\Big\{\alpha \in \mathbb{R}/\mathbb{Z} \mid S\left(\Gamma\left(\begin{pmatrix} N^{-1} & -N\alpha \\ 0 & N \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix}\right)\right) = k\Big\}$$

Ratner (1991) $\Longrightarrow$ Our curve becomes *equidistributed* (wrt. Haar measure μ) in $\Gamma \setminus G$ as $N \to \infty$.

Hence

$$\lim_{M\to\infty} Prob\Big(S_\xi(N) = k\Big) = \mu\Big\{\Gamma g \in \Gamma \setminus G \mid S(\Gamma g) = k\Big\}.$$

However, if $\xi = \frac{a}{b} \in \mathbb{Q}$ (say $(a,b) = 1$) then

$$\forall \alpha : \quad \Gamma\left(\begin{pmatrix} N^{-1} & -N\alpha \\ 0 & N \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix}\right) \in \Gamma \backslash \mathrm{SL}_2(\mathbb{R}) \ltimes \begin{pmatrix} b^{-1}\mathbb{Z} \\ b^{-1}\mathbb{Z} \end{pmatrix},$$

a 3-dim *imbedded submanifold* in $\Gamma \setminus G$.
Hence the curve does *not* go equidistributed in $\Gamma \setminus G$!

(S, '06)

THEOREM

Let $f \in C_c^8(\Gamma \setminus G)$ with

$$\forall g \in \mathrm{SL}(2,\mathbb{R}) : \mathcal{Y}_\Gamma(g) \geqq Y_0 \Longrightarrow f\left(g; \begin{pmatrix} * \\ * \end{pmatrix}\right) = 0;$$

$$g(i) \in \mathcal{F} \Longrightarrow \left|\partial_{x_j}^m X_1 \cdots X_r f\left(g; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right)\right| \leqq C_m^{(r)}.$$

Let $\xi \in \mathbb{R}$ (irrational) be Diophantine with

$$\forall j \in \mathbb{Z}^+ : \quad ||j\xi|| > cj^{1-K}.$$

$(c > 0,\ K > 2)$

Then, for all $y \leqq Y_0^{-1}$ and all $\alpha < \beta < \alpha + y$:

$$\left|\frac{1}{\beta-\alpha}\int_\alpha^\beta f\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix}\right) dx - \langle f \rangle\right|$$

$$\ll_{\varepsilon,K} \quad C_3^{(1)} c^{-\frac{2}{K}} Y_0^{1-\frac{1}{K}} y^{\frac{1}{K}} + \frac{1}{\beta-\alpha}\Big(C_6^{(2)} Y_0^{\frac{11}{4}+\frac{\varepsilon}{2}} y^{\frac{1}{4}-\varepsilon}$$

$$+ C_6 Y_0^{\frac{1}{2}+\frac{\varepsilon}{2}} y^{\frac{1}{2}-\varepsilon} + (C_0 + C_0^{(4)}) y^{\frac{1}{2}-\varepsilon}\Big).$$

(The implied constant is effective.)

(S. '06)

THEOREM (for *all* ξ!)
Let $f \in C_c^8(\Gamma \setminus G)$ with

$$\forall g \in \mathrm{SL}(2,\mathbb{R}) : \mathcal{Y}_\Gamma(g) \geqq Y_0 \Longrightarrow f\Big(g; \begin{pmatrix} * \\ * \end{pmatrix}\Big) = 0;$$

$$g(i) \in \mathcal{F} \Longrightarrow \Big| \partial_{x_j}^m X_1 \cdots X_r f\Big(g; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\Big)\Big| \leqq C_m^{(r)}.$$

Let $\xi \in \mathbb{R}$ (either rational or irrational!), and for each $Y \geqq 1$ let

$$k_0(Y) = \inf\Big\{ k \in \mathbb{Z}^+ \ \Big| \ k^{-2} \inf_{1 \leqq j \leqq k} j||j\xi|| \leqq Y^{-1} \Big\}.$$

Then, for all $y \leqq Y_0^{-1}$ and all $\alpha < \beta < \alpha + y$:

$$\left| \frac{1}{\beta - \alpha} \int_\alpha^\beta f\left(\left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} \xi \\ 0 \end{pmatrix} \right) \right) dx - \langle f \rangle \right|$$

$$\ll_{\varepsilon} \quad C_3^{(1)} Y_0 \cdot k_0\big(\sqrt{Y_0/y}\big)^{-1} + \frac{1}{\beta - \alpha} \Big(C_6^{(2)} Y_0^{\frac{11}{4} + \frac{\varepsilon}{2}} y^{\frac{1}{4} - \varepsilon}$$

$$+ C_6 Y_0^{\frac{1}{2} + \frac{\varepsilon}{2}} y^{\frac{1}{2} - \varepsilon} + (C_0 + C_0^{(4)}) y^{\frac{1}{2} - \varepsilon} \Big).$$

(The implied constant is effective.)

Application to $n\alpha$ mod 1

Recall $$S_\xi(N) = \#\left\{1 \leq n \leq N \;\middle|\; n\alpha \in \left[\xi - \frac{1}{2N}, \xi + \frac{1}{2N}\right] \bmod 1\right\}$$

Fix $\xi \in \mathbb{R}$ irrational, type $K > 2$.

(Thus $\|j\xi\| \gg j^{1-K}$, $\forall j \in \mathbb{Z}^+$)

Then
$$\left|\mathrm{Prob}(S_\xi(N) = k) - E(k)\right| \ll_{k,\xi} \begin{cases} N^{-\frac{2}{59}+\varepsilon} & K \leq 8 \\ N^{-\frac{4}{15K-2}} & K > 8 \end{cases}$$

as $N \to \infty$

(S, '06)

Fourier expansion on $\Gamma \setminus G = \mathbf{ASL}_2(\mathbb{Z}) \setminus \mathbf{ASL}_2(\mathbb{R})$

Given $f \in C_c^\infty(\Gamma \setminus G)$ (i.e. $f : G \to \mathbb{R}$, Γ-left invariant):

$$f\left(g, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \sum_{m \in \mathbb{Z}^2} \hat{f}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; m\right) \cdot e\left(m^t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right),$$

where

$$\hat{f}(g; m) := \int_{(\mathbb{R}/\mathbb{Z})^2} f\left(g; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) \cdot e\left(-m^t \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) dx_1 dx_2,$$

$(m \in \mathbb{Z}^2,\ g \in \mathrm{SL}_2(\mathbb{R}))$.

Now $f(\gamma g) = f(g)$, $\forall \gamma \in \mathrm{ASL}_2(\mathbb{Z})$ leads to:

$$\boxed{\hat{f}(\gamma g; m) = \hat{f}(g; \gamma^t m), \qquad \gamma \in \mathrm{SL}_2(\mathbb{Z}),\ g \in \mathrm{SL}_2(\mathbb{R}),\ m \in \mathbb{Z}^2.}$$

We set:

$$\boxed{F_n(g) := \hat{f}\left(g, \begin{pmatrix} 0 \\ n \end{pmatrix}\right), \qquad (g \in \mathrm{SL}_2(\mathbb{R}),\ n \in \mathbb{Z}_{\geqq 0}).}$$

For $n = 0$: $F_0(\gamma g) = F_0(g)$, $\forall \gamma \in \mathrm{SL}_2(\mathbb{Z})$.

For $n > 0$: $F_n(\gamma g) = F_n(g)$, $\forall \gamma \in \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix}$.

f in terms of the F_n's:

$$\boxed{\begin{aligned} f\left(g; \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) &= F_0(g) \\ &+ \sum_{n \geqq 1} \sum_{\begin{pmatrix} * & * \\ c & d \end{pmatrix} \in \begin{pmatrix} 1 & \mathbb{Z} \\ 0 & 1 \end{pmatrix} \setminus \mathrm{SL}_2(\mathbb{Z})} F_n\left(\begin{pmatrix} * & * \\ c & d \end{pmatrix} g\right) \cdot e((cx_1 + dx_2)n) \end{aligned}}$$

Asymptotic equidistribution of horocycle lift:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta f\left(\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}, \begin{pmatrix}\xi\\ 0\end{pmatrix}\right)dx \qquad \approx \langle f\rangle? \quad (\xi\notin\mathbb{Q})$$

Using the Fourier expansion we get:

$$=\frac{1}{\beta-\alpha}\int_\alpha^\beta F_0\left(\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dx+\sum_{n=1}^\infty \mathcal{J}_n,$$

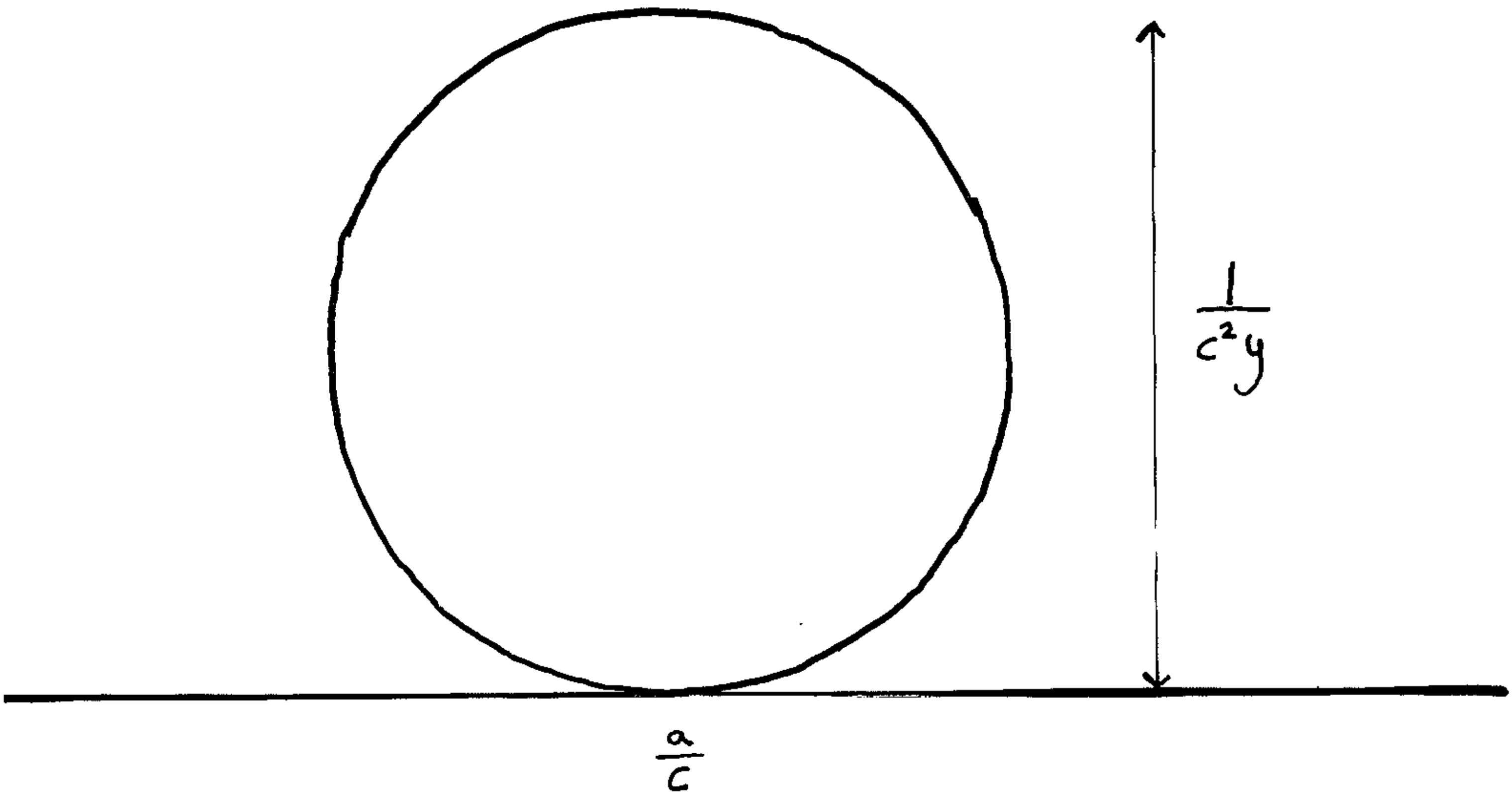
Here, via spectral analysis on $\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})$ one proves:

$$\frac{1}{\beta-\alpha}\int_\alpha^\beta F_0\left(\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dx$$
$$=\Big\langle F_0\Big\rangle_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathrm{SL}_2(\mathbb{R})}+O\Big(\|F_0\|_{W_4}y^{\frac12-\varepsilon}(\beta-\alpha)^{-1}\Big)$$
$$=\Big\langle f\Big\rangle_{\mathrm{ASL}_2(\mathbb{Z})\backslash\mathrm{ASL}_2(\mathbb{R})}+O\Big(\|F_0\|_{W_4}y^{\frac12-\varepsilon}(\beta-\alpha)^{-1}\Big)$$

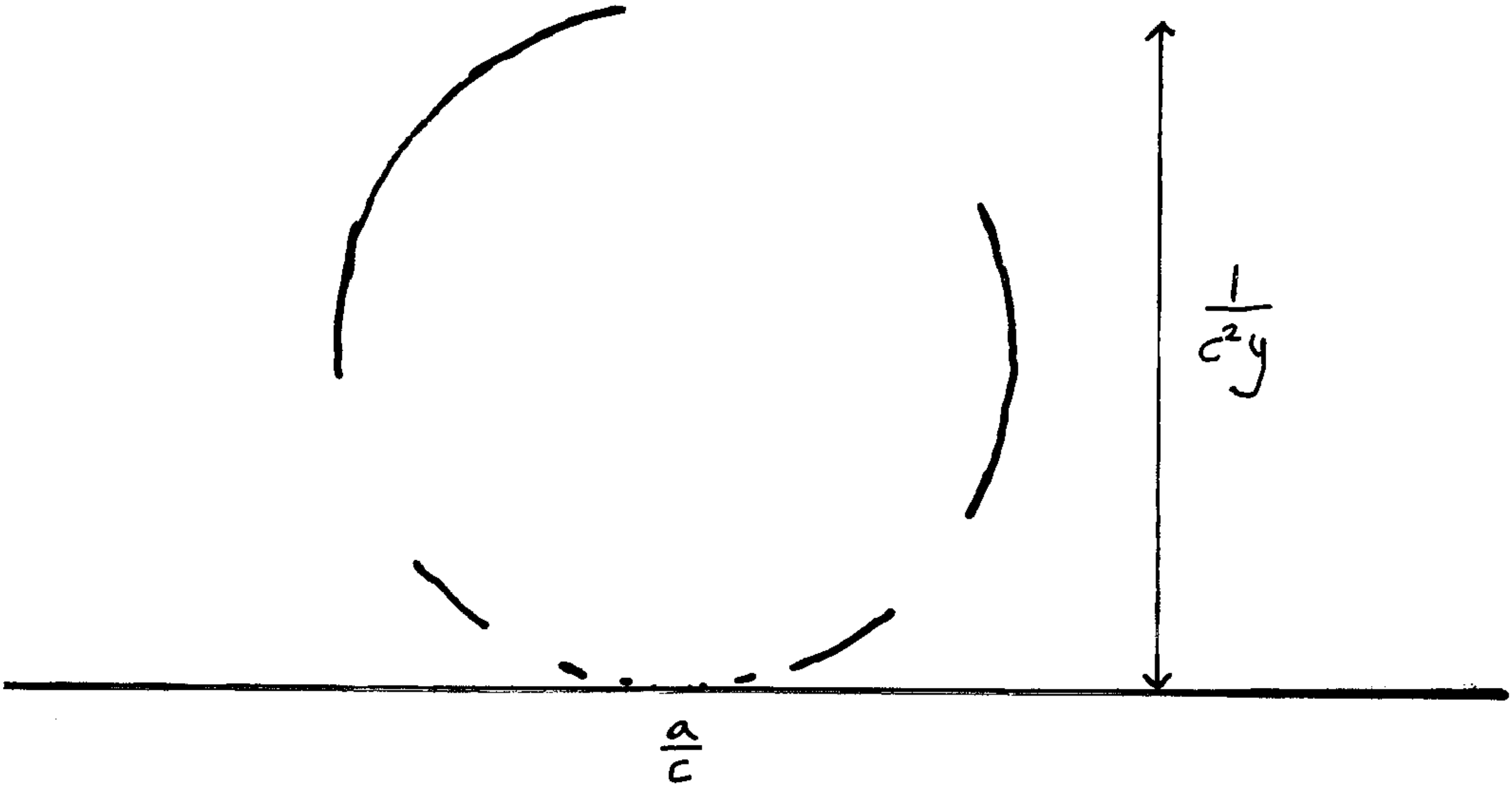
Also:

$$\mathcal{J}_n=\frac{1}{\beta-\alpha}\sum_{k=0}^{1}\int_\alpha^\beta F_n\left((-1)^k\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dx$$
$$+\frac{1}{\beta-\alpha}\sum_{c\in\mathbb{Z}-\{0\}}e(cn\xi)\sum_{\substack{d\in\mathbb{Z}\\(d,c)=1}}\int_\alpha^\beta F_n\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dx$$
$$=O\Big(C_6 n^{-6}Y_0^3y^3\Big)$$
$$+\frac{1}{\beta-\alpha}\sum_{c\neq0}e(cn\xi)\sum_{\substack{d \bmod c\\(d,c)=1}}\int_{(\alpha,\beta)+\mathbb{Z}}F_n\left(\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}\sqrt{y} & x/\sqrt{y}\\ 0 & 1/\sqrt{y}\end{pmatrix}\right)dx.$$

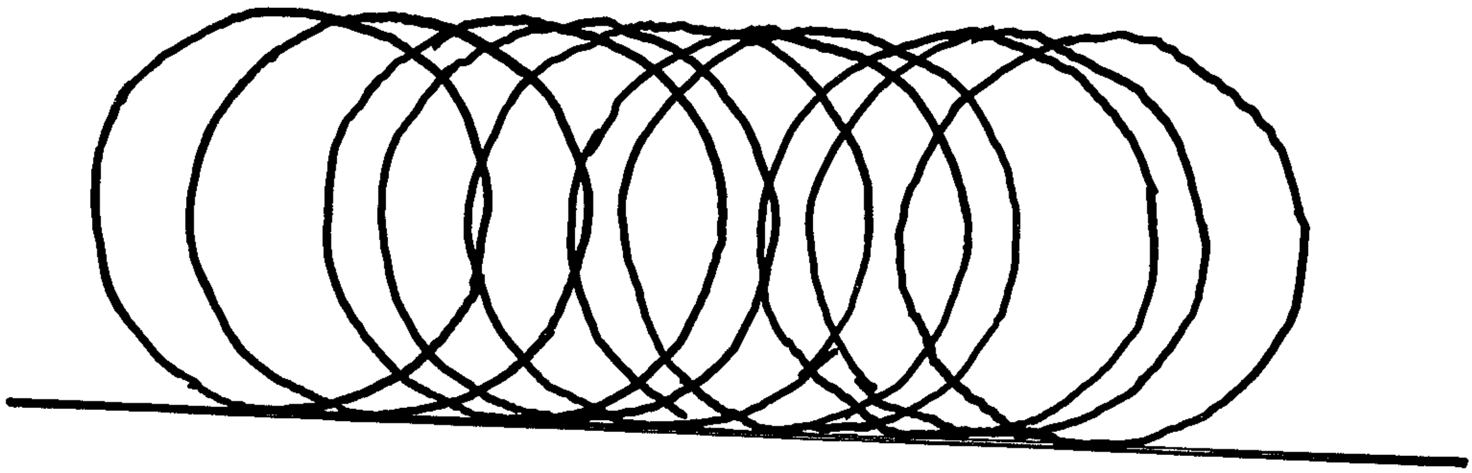
$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \;\middle|\; x \in \mathbb{R} \right\}$$



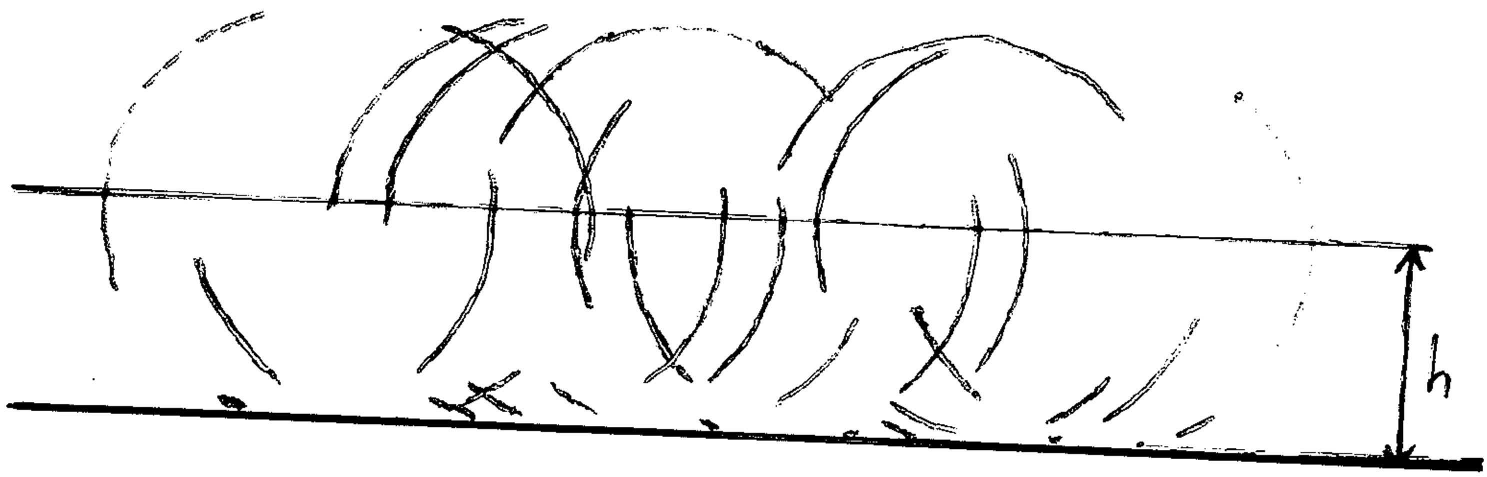
$$\left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \;\middle|\; x \in (\alpha, \beta) + \mathbb{Z} \right\}$$



$$a \equiv d^{-1} \, (c)$$



<u>At height h:</u>



Change variables:

$$h = \operatorname{Im} \begin{pmatrix} a & b \\ c & d \end{pmatrix} (x+iy) = \frac{y}{(cx+d)^2 + (cy)^2},$$

Condition of integration (if $c > 0$):

$$x = -\frac{d}{c} + \nu\sqrt{\frac{y}{c^2 h} - y^2} \quad \in (\alpha, \beta) + \mathbb{Z} \qquad\qquad (\nu = \pm 1)$$

$$\iff d \in (-c\beta, -c\alpha) + c\nu\sqrt{\frac{y}{c^2 h} - y^2} + c\mathbb{Z}$$

$$\iff d \in J = J_{c,\alpha,\beta,\nu,y,h}, \qquad \text{an interval mod } c \text{ (length } |c|(\beta-\alpha)).$$

Get:

$$\mathcal{J}_n \approx \frac{1}{\beta-\alpha} \sum_{c \neq 0} e(cn\xi) \sum_{\substack{d \bmod c \\ (d,c)=1}} \int_{(\alpha,\beta)+\mathbb{Z}} F_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx$$

$$= \frac{1}{\beta-\alpha} \sum_{\nu=\pm 1} \sum_{c \neq 0} e(cn\xi) \int_0^{\min(1/c^2 y, Y_0)} \frac{-\nu y}{2c^2h^2\sqrt{\frac{y}{c^2h} - y^2}}$$

$$\times \sum_{\substack{d \in J \bmod c \\ (d,c)=1}} F_n\left(\frac{a}{c} - \nu\sqrt{\frac{h}{c^2 y} - h^2}; h; \pi_{c\nu} + \nu \arcsin(c\sqrt{hy}) \right) dh$$

$$= \sum_{\nu=\pm 1} \sum_{c \neq 0} e(cn\xi) \int_0^{\min(1/c^2 y, Y_0)} \frac{-\nu y}{2c^2h^2\sqrt{\frac{y}{c^2h} - y^2}}$$

$$\times \left\{ \varphi(|c|) \int_{\mathbb{R}/\mathbb{Z}} F_n(x; h; *)\, dx + O\left(n^{-6} h |c|^{\frac{1}{2}+\varepsilon}\right) \right\} dh$$

$$= \sum_{\nu=\pm 1} \sum_{c\neq 0} \varphi(|c|) e(cn\xi) \int_0^{\min(1/c^2y, Y_0)} \frac{-\nu y\, dh}{2c^2h^2\sqrt{\frac{y}{c^2h} - y^2}} \int_{\mathbb{R}/\mathbb{Z}} F_n\, dx$$
$$+ O\Big(n^{-6} y^{\frac{1}{4}-\frac{\varepsilon}{2}}\Big).$$

The sum is roughly:

$$\ll \text{``}n^{-6} y \sum_{1\leqq c\ll y^{-\frac{1}{2}}} \varphi(c) e(cn\xi).\text{''}$$

Now:

$$\sum_{m\leqq X} \varphi(m) e(mn\xi) = \sum_{m\leqq X} e(mn\xi) \sum_{d|m} \mu\Big(\frac{m}{d}\Big) d$$
$$= \sum_{k\leqq X} \mu(k) \sum_{d\leqq X/k} d\, e(kd\xi) \ll \sum_{k\leqq X} \min\Big((X/k)^2, \frac{X/k}{||k\xi||}\Big)$$

$$\Big\{\text{Assume } \xi \text{ Diophantine; } ||m\xi|| \gg m^{1-K}, \forall m \in \mathbb{Z}^+ \quad (K>2)\Big\}$$

$$\ll n^{\frac{2(K-1)}{K}} X^{\frac{2(K-1)}{K}+\varepsilon}.$$

Hence in total:

$$\mathcal{J}_n \ll n^{\frac{2(K-1)}{K}-6} y^{\frac{1}{K}+\varepsilon} + n^{-6} y^{\frac{1}{4}-\frac{\varepsilon}{2}}.$$

Alteration for $\left\{ \left(\begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} 0 \\ \xi \end{pmatrix} \right) \;\middle|\; \alpha \le x \le \beta \right\}$

Get instead

$$J_n = \frac{1}{\beta - \alpha} \sum_{c \neq 0} \underbrace{\sum_{\substack{d \in \mathbb{Z} \\ (d,c)=1}} e(dn\xi) \int_\alpha^\beta F_n\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \sqrt{y} & x/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dx}_{\text{cancelation already here!}}$$

$\rightsquigarrow$

need, for any $f \in C^\infty(\mathbb{R}/\mathbb{Z})$

$$\sum_{d \in (\delta_1 c,\, \delta_2 c)} f\left(\frac{a}{c}\right) \cdot e(d\xi) \ll c^{1-\eta}$$

(some $\eta > 0$)

as $c \to \infty$

Alteration; f on $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^4 \backslash SL_2(\mathbb{R}) \ltimes \mathbb{R}^4$

$$\left(g_1, \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix}\right) \cdot \left(g_2, \begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix}\right) = \left(g_1 g_2, \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix} + g_1 \begin{pmatrix} \underline{w}_1 \\ \underline{w}_2 \end{pmatrix}\right)$$

$SL_2(\mathbb{R})$ acts on $\mathbb{R}^4$ by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix} = \begin{pmatrix} a\underline{v}_1 + b\underline{v}_2 \\ c\underline{v}_1 + d\underline{v}_2 \end{pmatrix}$

Fourier expansion of f:

$$f\left(g, \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix}\right) = \sum_{\underline{m} \in \mathbb{Z}^4} \hat{f}(g; m) \cdot e\left(m^t \begin{pmatrix} \underline{v}_1 \\ \underline{v}_2 \end{pmatrix}\right),$$

again $\hat{f}(\gamma g; m) = \hat{f}(g; \gamma^t m)$, $\gamma \in SL_2(\mathbb{Z})$

$\underline{3}$ types of $m \in \mathbb{Z}^4$:

- $m = 0 \Rightarrow \hat{f}(\cdot; m)$ on $SL_2(\mathbb{Z}) \backslash SL_2(\mathbb{R})$
- $m = \begin{pmatrix} 0 \\ 0 \\ * \\ * \end{pmatrix} \Rightarrow \hat{f}(\cdot, m)$ on $\begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \backslash SL_2(\mathbb{R})$
- $m = \begin{pmatrix} 0 \\ * \\ * \\ * \end{pmatrix} \Rightarrow \hat{f}(\cdot; m)$ on $SL_2(\mathbb{R})$