Horocycles and asymptotic equidistribution

(or: An introduction to Ratner's Theorem on unipotent flows)

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Outline of talk

- Special case; horocycle flow on SM.

  $(G = PSL(2, \mathbb{R}))$

- Ratner's Theorem. ('90)
- Shah's Theorem ('96)
- Application in $\mathbb{H}^3$
- Application to horocycle "spaghetti".

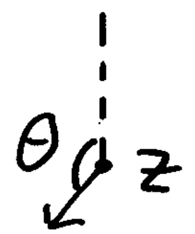
(Apologize for brief definitions!)

Hyperbolic plane, upper half plane model:

$$\mathcal{H} = \{ z = x+iy \mid y > 0 \}, \qquad d\mu(z) = \frac{dx\,dy}{y^2}$$

Sphere bundle (= unit tangent bundle):

$$S\mathcal{H} = \{ (z, \theta) \mid z \in \mathcal{H},\ \theta \in \mathbb{R}/2\pi\mathbb{Z} \}$$



Let $\boxed{G = PSL(2,\mathbb{R}) = \mathrm{Isom}^+(\mathcal{H})}$

$$= \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a,b,c,d \in \mathbb{R},\ ad-bc = 1 \right\} \Big/ \pm$$

The action of $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$ on $\mathcal{H}$ extends naturally to an action on $S\mathcal{H}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad (z, \theta) \longmapsto \left(\frac{az+b}{cz+d},\ \theta - 2\arg(cz+d) \right)$$

This action is <u>simply transitive</u>.

$\Rightarrow$ Can <u>IDENTIFY "$G = S\mathcal{H}$"</u> (as mfds)

by $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \longleftrightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}(i, 0)$

Facts about identification "$\underbrace{G}_{PSL(2,\mathbb{R})} = S\mathcal{H}$"

$\left[\begin{array}{l}\text{Liouville measure on } S\mathcal{H} \\ \text{(i.e } \mu \times \left[\begin{array}{l}\text{unif. measure} \\ \text{on circle}\end{array}\right]\text{)}\end{array}\right] \longleftrightarrow \left[\begin{array}{l}\text{Haar} \\ \text{measure} \\ \nu \text{ on } G\end{array}\right]$

Let $\Gamma \subset G$ be a Fuchsian group,

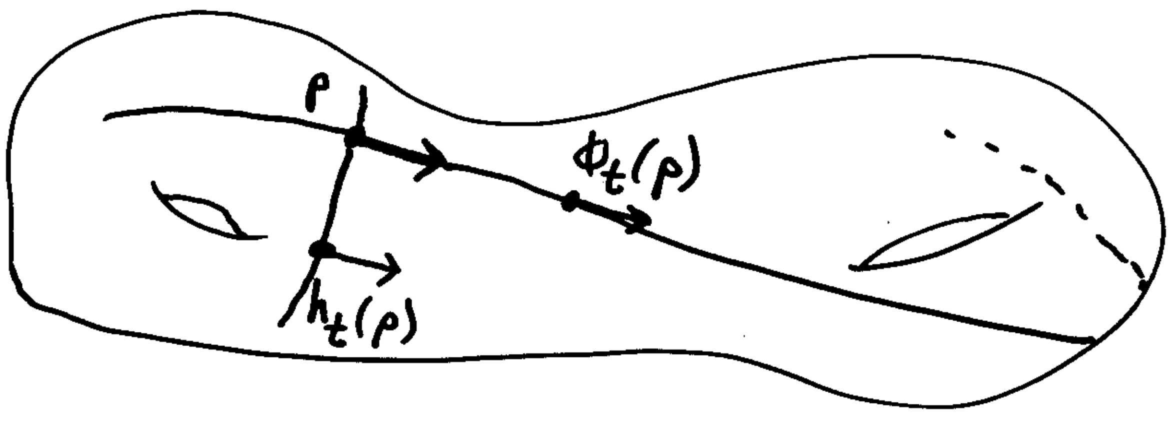
$\underline{M = \Gamma \backslash \mathcal{H}}$.

Then $SM = \Gamma \backslash S\mathcal{H} = \Gamma \backslash G$

$= \{ \Gamma g \mid g \in G \}$

Geodesic flow: $\quad \Phi_t(\Gamma g) = \Gamma g \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2} \end{pmatrix}$.

Horocycle flow
(along stable orbit): $\quad h_t(\Gamma g) = \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$.



A <u>very</u> special case of Ratner's Theorem: <u>ASSUME $\mu(M) < \infty$</u>.

For all $\Gamma g \in \Gamma \backslash G = SM$:

EITHER

$\left\{ \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$ is a closed orbit,

OR

$\left\{ \Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid t \in \mathbb{R} \right\}$ goes asymptotically equidistributed wrt ν, i.e. for every bounded continuous function $f : \Gamma \backslash G \to \mathbb{C}$ we have

$$\frac{1}{T} \int_0^T f\left(\Gamma g \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\right) dt \longrightarrow \int_{\Gamma \backslash G} f(\Gamma g)\, d\nu(g)$$

as $T \to \infty$.

(This special case was actually proved earlier by Dani & Smillie 84.)

Assume

<u>$\nu(\Gamma \backslash G) = 1$</u>

General result

$\underline{G}$ — a connected Lie group.

$\underline{U = \{u(t) \mid t \in \mathbb{R}\}}$ — an Ad-unipotent subgroup. (For G s.s. matrix group: Every $u(t)$ should have all eigenvalues $= 1$)

$\underline{\Gamma}$ — A lattice in G, i.e. a discrete subgroup in G s.t. $\nu(\Gamma \backslash G) < \infty$ for $\nu =$ Haar measure.

Our space: $\underline{\Gamma \backslash G}$. Flow: $\underline{\Gamma g \longmapsto \Gamma g \cdot u(t)}$.

Ratner's Theorem (1990): (still in special case)

For every $\Gamma g \in \Gamma \backslash G$, there exists a connected closed subgroup $H \subset G$ s.t.:

- $U \subseteq H$
- $\Gamma g H = $ Closure of $\{\Gamma g \cdot u(t)\}$ in $\Gamma \backslash G$.
- $H \cap g^{-1} \Gamma g$ is a lattice in H.
- For every bounded continuous $f : \Gamma \backslash G \to \mathbb{C}$:

$$\frac{1}{T} \int_0^T f(\Gamma g \cdot u(t))\, dt \longrightarrow \int_{\Gamma \backslash \Gamma g H} f \, d\nu_H \quad \text{as } T \to \infty$$

normalized!

Applications

Eskin, Margulis, Mozes 98 (2002)

$m^2 + \beta^2 n^2 \quad (m, n \in \mathbb{Z})$

- Poisson pair correlation for **explicit** $\beta \in \mathbb{R}$!

(Applies Ratner with $G = SL(4,\mathbb{R})$, $\Gamma = SL(4,\mathbb{Z})$, $U = SO(2,2)$.)

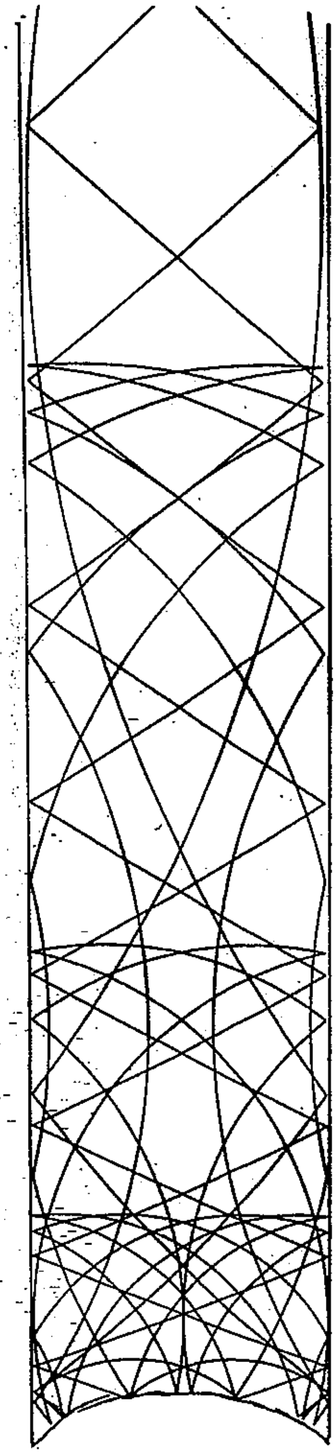
Marklof 2001,

same for $(m-\alpha)^2 + (n-\beta)^2 \quad (m, n \in \mathbb{Z})$

(Applies Ratner with
$G = SL(2,\mathbb{R}) \ltimes \mathbb{R}^4$, $\Gamma = SL(2,\mathbb{Z}) \ltimes \mathbb{Z}^4$,
$U = $ "$\begin{pmatrix} 1 & \mathbb{R} \\ 0 & 1 \end{pmatrix} \ltimes 0$")

LONG CLOSED HOROCYCLES

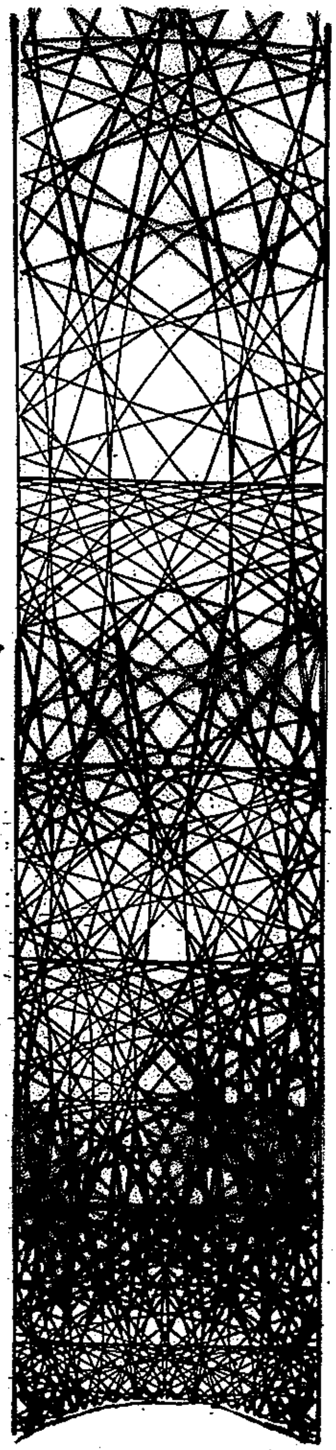
$\Gamma \subset PSL(2,\mathbb{R})$
standard cusp
at ∞



$\Gamma = PSL(2,\mathbb{Z})$

$\leftarrow y = 0.025$

$y = 0.005 \rightarrow$



$$\int_0^1 f(x+iy)\,dx = \langle f \rangle + [\text{corrections}] + o(y^{\frac{1}{2}})$$

as $y \to 0$. (Sarnak, 1981)

In fact, $\displaystyle\frac{1}{b-a}\int_a^b f(x+iy)\,dx \to \langle f \rangle$

as long as $b-a > y^{\frac{1}{2}-\varepsilon}$, $y \to 0$ (S, 2001)

A theorem of Shah (96)

Here: A very special case!

G – a connected Lie group.

Γ – a lattice in G,

ν – Haar measure on G, normalized by $\nu(\Gamma\backslash G) = 1$.

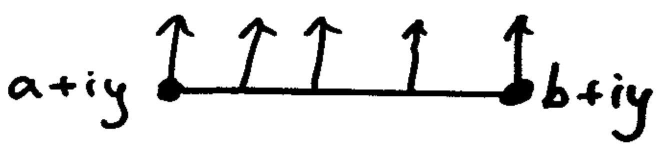
Suppose $G_0 = PSL(2,\mathbb{R})$ is realized as a Lie subgroup of G, and that ΓG_0 is dense in $\Gamma\backslash G$.

We then have, for any bounded continuous $f : \Gamma\backslash G \to \mathbb{C}$ and any fixed $a<b$:

$$\frac{1}{b-a}\int_a^b f\Big(\Gamma \underbrace{\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{y} & 0 \\ 0 & 1/\sqrt{y} \end{pmatrix}}_{\text{in } G_0 \subseteq G}\Big)\,dt \to \int_{\Gamma\backslash G} f\,d\nu \quad \text{as } y \to 0^+.$$

- Proof uses Ratner classification of inv. measures (90).

- Under $G_0 = S\mathcal{H}$:



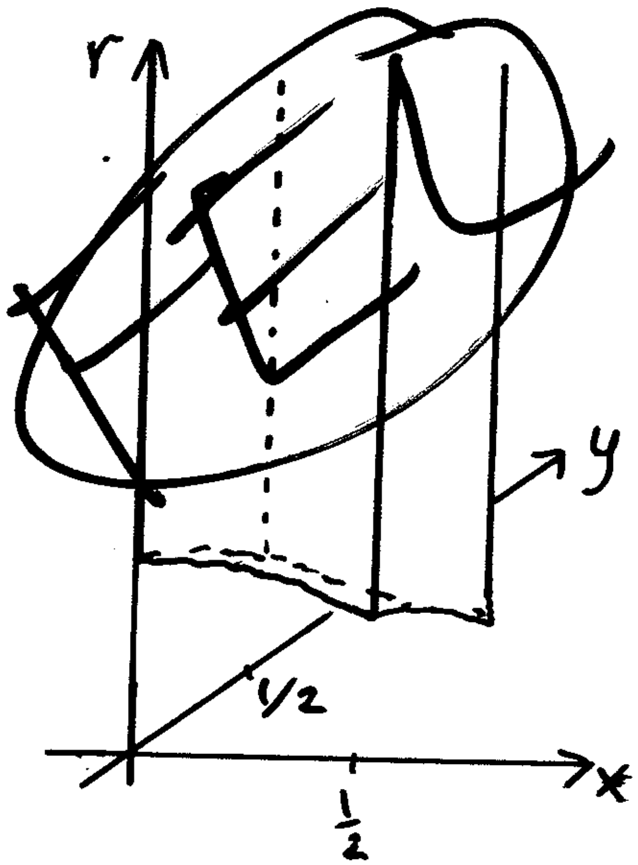
Hyperbolic 3-space $\mathbb{H}^3$

Upper half space model:

$$\mathcal{H} = \{ (x, y, r) \mid x, y \in \mathbb{R},\ r \in \mathbb{R}^+ \}$$

$(x + iy = z \in \mathbb{C})$

$$d\mu = \frac{dx\,dy\,dr}{r^3}, \qquad ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}.$$



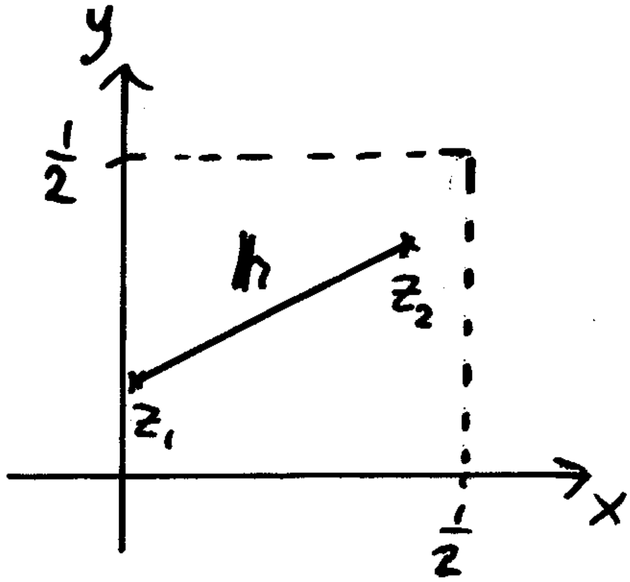
Let $M = \Gamma \backslash \mathbb{H}^3$,

$\Gamma = PSL(2, \mathbb{Z}[i])$.

(Picard's group)

$M = \Gamma \backslash \mathbb{H}^3$

$\Gamma = PSL(2, \mathbb{Z}[i])$



FIX $z_1 \neq z_2 \in \mathbb{C}$.

At height $r > 0$:

Slope: $k = \dfrac{\operatorname{Im}(z_2 - z_1)}{\operatorname{Re}(z_2 - z_1)}$

Shah's thm (for $G = PSL(2, \mathbb{C})$) implies:

If $k \notin \mathbb{Q}$: h goes equidistr on M as $r \to 0$.

If $k = 0$, $\operatorname{Im} z_1 = \operatorname{Im} z_2 \notin \mathbb{Q}$ $\Big\}$: SAME!

But if $k = 0$, $\operatorname{Im} z_1 = \operatorname{Im} z_2 \in \mathbb{Q}$: NO equidistr

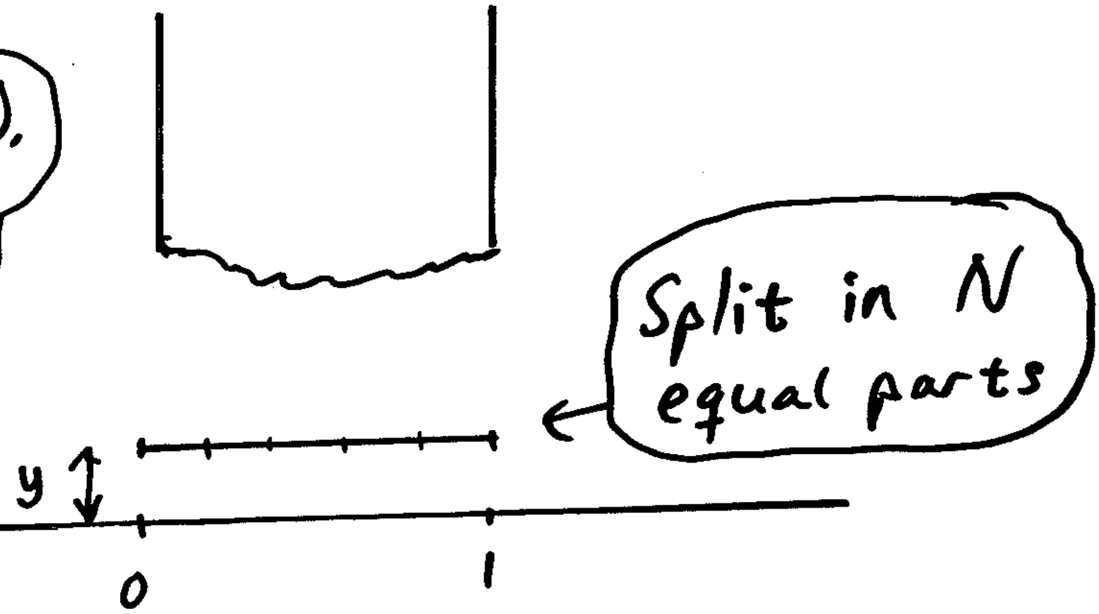
(h stays in imbedded 2-dim mfd $\subset M$)

Horocycle "spaghetti"

$\Gamma \subset PSL(2,\mathbb{R})$, standard cusp at ∞.

$f \in C(\Gamma \backslash \mathcal{H})$, fixed.

$\langle f \rangle = 0$



$$S_{y,N}(x) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f\left(\frac{x+j}{N} + iy\right)$$

Let $y \to 0$, $N \to \infty$, keeping $Ny \to 0$.

$S_{y,N} \to$ Gaussian ??

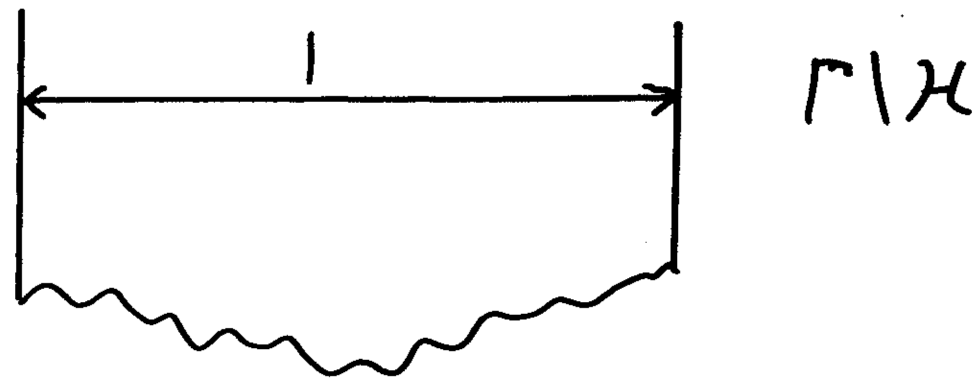
mean 0
variance $\langle f^2 \rangle$.

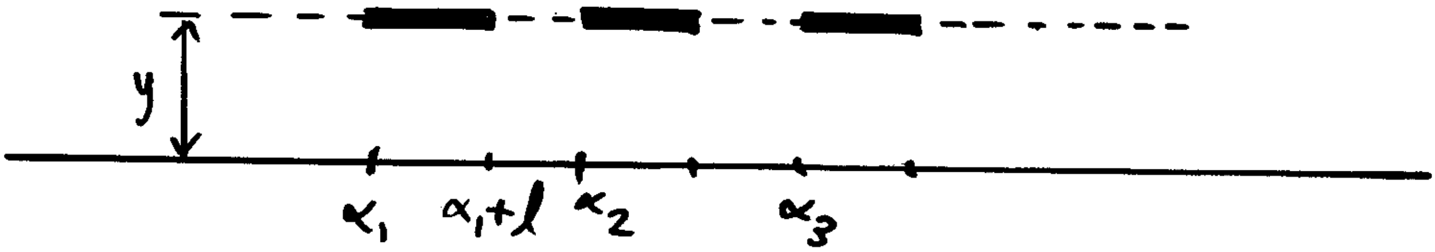
On $\Gamma = G_5, G_7$: YES!

On $\Gamma = G_3 = PSL(2,\mathbb{Z})$: NO!

(Hecke operators)

Joint equidistribution of subsegments of closed horocycles





Fix $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$, $\ell > 0$, and

ASSUME $\underline{\begin{pmatrix} 1 & \alpha_j - \alpha_k \\ 0 & 1 \end{pmatrix} \notin \mathrm{Comm}(\Gamma), \quad \forall j \neq k.}$

We then have, for every bounded continuous function $f : (\Gamma\backslash\mathcal{H})^n \to \mathbb{C}$:

$$\lim_{y \to 0} \frac{1}{\ell} \int_0^\ell f(\alpha_1 + x + iy, \ldots, \alpha_n + x + iy)\, dx =$$

$$= \frac{1}{\mu(\Gamma\backslash\mathcal{H})^n} \int_{(\Gamma\backslash\mathcal{H})^n} f(z_1, z_2, \ldots, z_n)\, d\mu(z_1) \ldots d\mu(z_n)$$

Corollaries for $S_{y,N}$

(Recall $S_{y,N}(x) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} f\left(\frac{x+j}{N} + iy\right)$)

On G_L, $L \neq 3, 4, 6$:

If $y \to 0$, and $N = N(y) \to \infty$ SUFFICIENTLY SLOWLY, then

$S_{y,N} \to$ Gaussian!

Definition of $\mathrm{Comm}(\Gamma) \subseteq PSL(2,\mathbb{R})$:

$g \in \mathrm{Comm}(\Gamma)$

$\iff |\Gamma : \Gamma \cap g\Gamma g^{-1}| < \infty$

$\iff \Gamma g \Gamma = \bigsqcup_{j=1}^{d} \Gamma g_j \qquad (d < \infty)$

$\iff g : \mathcal{H} \to \mathcal{H}$ gives a "1-to-d" map $z \mapsto \{z_1, z_2, \ldots, z_d\}$ on $\Gamma \backslash \mathcal{H}$.

(May here take $z_j = g_j z$)

THM (Margulis): $[\mathrm{Comm}(\Gamma):\Gamma] = \infty \iff \Gamma$ arithm

THM (Leutbecher): $[\mathrm{Comm}(G_N) : G_N] = 1$ for $N \neq 3, 4, 6$.

Every $g \in \mathrm{Comm}(\Gamma)$ gives a Hecke operator:

For $f : \Gamma \backslash \mathcal{H} \to \mathbb{C}$, let $Tf(z) = \sum_{j=1}^{d} f(g_j z)$.

(Ex: For $\Gamma = PSL(2,\mathbb{Z})$, in this way construct

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{ad=n} \sum_{b=1}^{d} f\left(\frac{az+b}{d}\right)$$

Proof (sketch)

Realize $G_0 = PSL(2,\mathbb{R})$ as a subgroup of $G = \underbrace{G_0 \times G_0 \times \dots \times G_0}_{n}$ by imbedding

$\underline{g_0 \in G_0}$

as

$$\underline{\left(\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} g_0 \begin{pmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix} g_0 \begin{pmatrix} 1 & -\alpha_2 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} g_0 \begin{pmatrix} 1 & -\alpha_n \\ 0 & 1 \end{pmatrix} \right)} \in G.$$

Let $\underline{\Lambda = \Gamma \times \Gamma \times \dots \times \Gamma}$, a lattice in G.

<u>IF</u> ΛG_0 DENSE IN $\Lambda \backslash G$:

Shah's thm gives, $\forall f \in C_b(\Lambda \backslash G)$:

$$\lim_{y \to 0} \frac{1}{\ell} \int_0^\ell f \left(\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \underline{\underline{\begin{pmatrix} \sqrt{y} & t/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}}} \begin{pmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & t/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \begin{pmatrix} 1 & -\alpha_n \\ 0 & 1 \end{pmatrix} \right) dt =$$

$$= \int_{\Lambda \backslash G} f \, d\nu.$$

Set

$f(g) =$ [redacted]

$$= h\left[g \cdot \left(\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} \right) \right],$$

with arbitrary $h \in C_b(\Lambda \backslash G)$.

Gives:

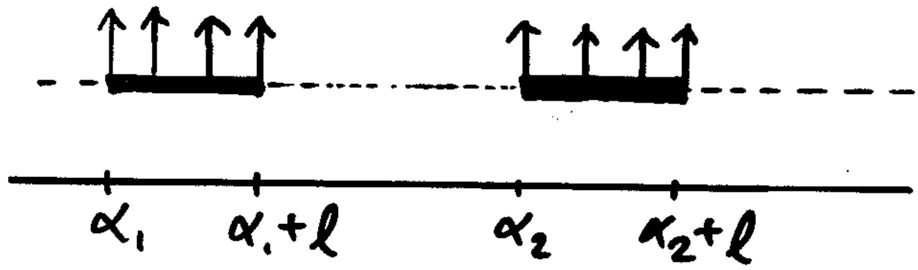
[Integral] =

$$= \frac{1}{\ell} \int_0^\ell h\left(\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & t/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} 1 & \alpha_2 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & t/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \ldots, \right.$$

$$\left. \ldots, \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & t/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dt$$

$$= \frac{1}{\ell} \int_0^\ell h\left(\begin{pmatrix} \sqrt{y} & (t+\alpha_1)/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \begin{pmatrix} \sqrt{y} & (t+\alpha_2)/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix}, \ldots \right.$$

$$\left. \ldots, \begin{pmatrix} \sqrt{y} & (t+\alpha_n)/\sqrt{y} \\ 0 & 1/\sqrt{y} \end{pmatrix} \right) dt$$



$M = \Gamma \backslash \mathcal{H}$

$\Longrightarrow$ EQUIDISTRIBUTION ON $\underbrace{SM \times SM \times \ldots \times SM}$!

$G_0 = PSL(2,\mathbb{R})$, $\quad G = \underbrace{G_0 \times G_0 \times \ldots \times G_0}_{n}$.

$\Lambda = \Gamma \times \Gamma \times \ldots \times \Gamma$,

$[G_0 \to G]$ - imbedding:

$$g_0 \mapsto \left(\begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} g_0 \begin{pmatrix} 1 & -\alpha_1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & \alpha_n \\ 0 & 1 \end{pmatrix} g_0 \begin{pmatrix} 1 & -\alpha_n \\ 0 & 1 \end{pmatrix} \right).$$

$$\boxed{\Lambda G_0 \text{ DENSE IN } \Lambda \backslash G \text{ ???}}$$

Change notation to: $S^{\alpha} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix}$

$$\underline{\Lambda = (S^{-\alpha_1} \Gamma S^{\alpha_1}) \times (S^{-\alpha_2} \Gamma S^{\alpha_2}) \times \ldots \times (S^{-\alpha_n} \Gamma S^{\alpha_n})},$$

$[G_0 \to G]$ - imbedding:

$$\underline{g_0 \mapsto (g_0, g_0, \ldots, g_0)}.$$

$$\boxed{\Lambda G_0 \text{ DENSE IN } \Lambda \backslash G \text{ ???}}$$

ΛG_0 DENSE IN $\Lambda \backslash G$???

(Recall $G_0 = PSL(2,\mathbb{R})$, $G = G_0^n$,

$[G_0 \to G]$ - imbedding: $g_0 \mapsto (g_0, g_0, \dots, g_0)$.

$\Lambda = (S^{-\alpha_1} \Gamma S^{\alpha_1}) \times (S^{-\alpha_2} \Gamma S^{\alpha_2}) \times \dots \times (S^{-\alpha_n} \Gamma S^{\alpha_n})$.)

G_0 is generated by Ad-unipotent subgroups. Hence **Ratner applies!**

$\therefore$ $\exists$ closed connected $H \subset G$ s.t.:

(I) $G_0 \subset H$

(II) ΛH = closure of ΛG_0 in $\Lambda \backslash G$.

(III) $H \cap \Lambda$ = is a lattice in H

I) $\Rightarrow$ $H = \{ (g_1, g_2, \dots, g_n) \in G_0^n \mid i \sim j \Rightarrow g_i = g_j \}$

for some equivalence relation $\sim$ on $\{1, 2, \dots, n\}$.

IF $H = G_0^n = G$: DONE, by (II) !!

(ie ΛG_0 DENSE IN $\Lambda \backslash G$!)

Otherwise, example:

$$\underline{H = \{(g_1, g_1) \mid g_1 \in G_0\}}$$

Then (III) says that

$$\underline{H \cap (S^{-\alpha_1}\Gamma S^{\alpha_1} \times S^{-\alpha_2}\Gamma S^{\alpha_2})}$$

is a lattice in H!

$\therefore$ $\underline{S^{-\alpha_1}\Gamma S^{\alpha_1} \cap S^{-\alpha_2}\Gamma S^{\alpha_2}}$ lattice in $G_0 = PSL(2,\mathbb{R})$

$\therefore$ $\Gamma \cap (S^{\alpha_1-\alpha_2}\Gamma S^{\alpha_2-\alpha_1})$ lattice in $G_0 = PSL(2,\mathbb{R})$.

$\therefore$ $\underline{\underline{S^{\alpha_1-\alpha_2} \in \mathrm{Comm}(\Gamma)}}$,

contrary to our assumptions.

□ □ □