# Numerical computations with the Selberg trace formula

Andreas Strömbergsson
(joint work with A. Booker and F. Strömberg)

## Overview

- Small eigenvalues on congruence groups.

- The Selberg Trace Formula with applications:
  - Verifying Selberg's conjecture
  - Computing eigenvalues

- Improvements using Hecke operators

- Statistics of $\lambda_1(p)$ as $p \to \infty$?

## Small eigenvalues on congruence groups

Let $\Gamma \subset \mathrm{SL}(2,\mathbb{Z})$ be a congruence subgroup, i.e.

$$\Gamma(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod N \right\} \subset \Gamma$$

for some $N \in \mathbb{Z}^+$.

The **Selberg Eigenvalue Conjecture** (1965): $\lambda_1(\Gamma) \geqq \frac{1}{4}$.

**Theorem** (Kim-Sarnak, 2003): $\lambda_1(\Gamma) \geqq \frac{1}{4} - \left(\frac{7}{64}\right)^2 = 0.238...$

**Theorem** (Huxley, 1985):

Selberg's conjecture is true for $N \leqq 18$.

---

Associated $L$-functions. Recent numerics by Farmer and Lemurell comparing with RMT $(O^+, O^-)$ statistics.

**Technicality:** Passing from $\Gamma(N)$ to $\Gamma_0(N^2),\chi$

$$\Gamma(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & 0\\ 0 & 1\end{pmatrix} \bmod N\right\}$$

$$\Gamma_1(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} 1 & *\\ 0 & 1\end{pmatrix} \bmod N\right\}$$

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in \mathrm{SL}(2,\mathbb{Z}) \mid \begin{pmatrix} a & b\\ c & d\end{pmatrix}\equiv\begin{pmatrix} * & *\\ 0 & *\end{pmatrix} \bmod N\right\}$$

Let $\chi$ — an even Dirichlet character modulo $N$:

We say $f:\mathcal{H}\to\mathbb{C}$ is $\Gamma_0(N),\chi$-*invariant* if

$$\boxed{f\Big(\frac{az+b}{cz+d}\Big)=\chi(d)f(z),\qquad \forall\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in\Gamma_0(N).}$$

Now:

$$L^2(\Gamma(N)\backslash\mathcal{H})\cong\bigoplus_{\substack{\chi\\ q(\chi)\mid N}} L^2\Big(\Gamma_0(N^2)\backslash\mathcal{H},\chi\Big).$$

## The Selberg Trace Formula

$\Gamma \subseteq \mathrm{SL}(2,\mathbb{Z})$ — a congruence subgroup.

$h(r)$ — any even, analytic function of decay $|r|^{-(2+\varepsilon)}$ in $|\mathrm{Im}\, r| < \frac{1}{2} + \varepsilon$.

$$\sum_{\lambda \text{ on } \Gamma\backslash\mathcal{H}} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right) = \begin{bmatrix}\text{Explicit} \\ \text{expression}\end{bmatrix}.$$

**Selberg Trace Formula or $\Gamma_0(N),\chi$ ($N$ squarefree), with split into even ($\varepsilon=0$) or odd ($\varepsilon=1$)**

$$\sum h\Big(\sqrt{\lambda-1/4}\Big)=\frac{\prod_{p|N}(p+1)}{24}\int_{-\infty}^{\infty}rh(r)\tanh(\pi r)\,dr+\Big[\text{Elliptic}\Big]$$

$$+\frac{1}{2}\sum_{n\in\{-1,1\}}n^{\varepsilon}\sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2-4n}\notin\mathbb{Q}}}\Big\langle\chi(\delta)\Big\rangle_{\delta^2-t\delta+n\equiv 0}\cdot\Bigg(\sum_{f|\ell}f\prod_{p|f}(1-(\tfrac{d}{p})p^{-1})$$

$$\cdot\prod_{p|N}\begin{Bmatrix}2 & \text{if } p\mid f\\ 1+\left(\frac{d}{p}\right) & \text{if } p\nmid f\end{Bmatrix}\Bigg)\cdot\frac{\mathbf{h}^{+}(d)\log\epsilon_d}{\sqrt{t^2-4n}}\cdot\widehat{h}\left(\log\frac{(|t|+\sqrt{t^2-4n})^2}{4}\right)$$

$$-\frac{d(N)}{4}\widehat{h}(0)\left(\log\Big(8N\Big)+C_{\chi,\varepsilon}\cdot\log\left(\frac{Nq(\chi)}{2\pi^2}\right)+\tfrac{1}{2}(C_{\chi,\varepsilon}-1)\log\left(N,2\right)\right)$$

$$-\frac{d(N)}{4\pi}\int_{-\infty}^{\infty}h(r)\left(\frac{\Gamma'(\frac{1}{2}+ir)}{\Gamma(\frac{1}{2}+ir)}+C_{\chi,\varepsilon}\frac{\Gamma'(1+ir)}{\Gamma(1+ir)}\right)dr+I_{\chi\neq 1}\frac{C_{\chi,\varepsilon}\,d(N)}{8}h(0)$$

$$+d(N)\sum_{n=1}^{\infty}\frac{\{\chi\}_{\varepsilon}(n)\Lambda(n)}{n}\widehat{h}(2\log n)-\frac{d(N)}{2}\sum_{p|\frac{N}{q(\chi)}}\sum_{m=1}^{\infty}\frac{\{\chi\}_{\varepsilon}(p^m)\log p}{p^m}\widehat{h}(2\log p^m).$$

**Notation:**

$d, \ell$ implicit variables; $t^2 - 4n = d\ell^2$, $\ell \in \mathbb{Z}^+$, $d$ a fund. discr.

$\mathbf{h}^+(d)$ =narrow class number of $\mathbb{Q}(\sqrt{d})$.
$\epsilon_d$=proper fundamental unit in $\mathbb{Q}(\sqrt{d})$.

$$\Big\langle \chi(\delta) \Big\rangle_{\delta^2 - t\delta + n \equiv 0} = \text{average of } \chi(\delta) \text{ over all } \delta \bmod q(\chi) \text{ with } \delta^2 - t\delta + n \equiv 0 \bmod q(\chi).$$

$$C_{\chi,\varepsilon} := \begin{cases} 1 & \text{if } \chi \text{ not pure} \\ 2 & \text{if } \chi \text{ pure, } \varepsilon = 0\,; \\ 0 & \text{if } \chi \text{ pure, } \varepsilon = 1 \end{cases}$$

$$\{\chi\}_\varepsilon(\delta) := \frac{2}{d(q(\chi))} \sum_{\substack{A | q(\chi) \\ \chi_A(-1) = (-1)^\varepsilon}} \chi^{(A)}(\delta).$$

$$\sum h\Big(\sqrt{\lambda - 1/4}\Big) = \frac{\prod_{p|N}(p+1)}{24}\int_{-\infty}^{\infty} rh(r)\tanh(\pi r)\,dr$$

$$+\frac{1}{2}\sum_{n\in\{-1,1\}} n^{\varepsilon} \sum_{\substack{t\in\mathbb{Z}\\ \sqrt{t^2-4n}\notin\mathbb{Q}}} \Big\langle \chi(\delta)\Big\rangle_{\delta^2 - t\delta + n\equiv 0}$$

$$\cdot\left(\sum_{f|\ell} f\prod_{p|f}(1-(\tfrac{d}{p})p^{-1})\cdot\prod_{p|N}\begin{Bmatrix} 2 & \text{if } p\mid f\\ 1+\left(\frac{d}{p}\right) & \text{if } p\nmid f\end{Bmatrix}\right)$$

$$\cdot\frac{\mathbf{h}^{+}(d)\log\epsilon_d}{\sqrt{t^2-4n}}\cdot\hat{h}\left(\log\frac{(|t|+\sqrt{t^2-4n})^2}{4}\right)$$

$$+\Big[\text{Elliptic}\Big]+\Big[\text{Parabolic}\Big]+\Big[\text{Eisenstein part}\Big].$$

$\Big(d,\ell$ implicit variables; $t^2-4n = d\ell^2$, $\ell\in\mathbb{Z}^+$, $d$ a fund. discr.$\Big)$

We tabulate class numbers $\mathbf{h}^{+}(d)$ for all $d = t^2 \pm 4 \lesssim e^{36}$. Thus we can evaluate STF for all $h$ with $supp(\hat{h}) \subset [-36, 36]$.

STF: $$\sum_{\lambda \text{ on } \Gamma\backslash\mathcal{H}} h\left(\sqrt{\lambda - \tfrac{1}{4}}\right) = \begin{bmatrix}\text{Explicit} \\ \text{expression}\end{bmatrix}.$$

**Application to the Selberg conjecture**

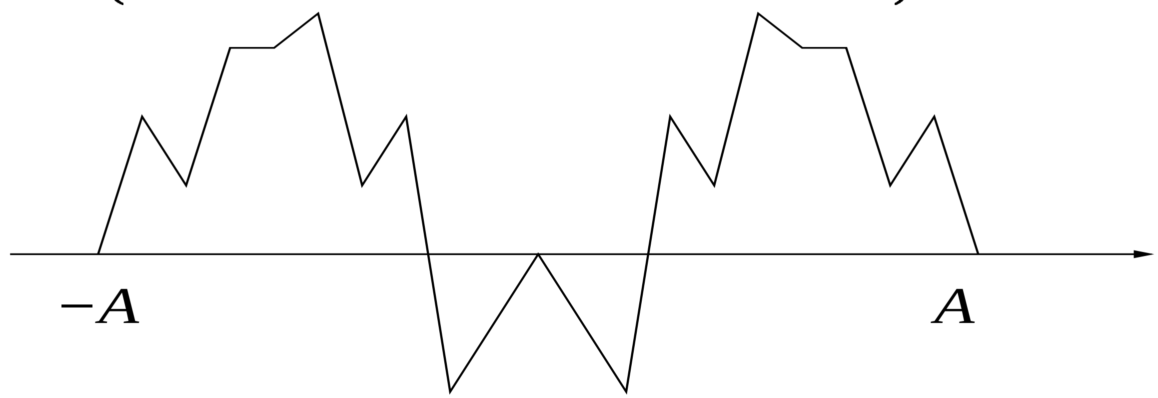
• Choose $h$: $h(r) \geqq 0$ on $[0, \infty)$, $h(r) \geqq 1$ on $i[0, \frac{7}{64}]$.
• Calculate $\sum h\left(\sqrt{\lambda - \frac{1}{4}}\right)$. If $< 1$ then there are no exceptional eigenvalues.
• Must remove contribution from Galois representations ($\lambda = \frac{1}{4}$).
(Also: Remove CM forms, sieve for *newforms*.)

$\Longrightarrow$ (Booker, S, 2005)
Selberg's conjecture holds for all $\Gamma_0(N), \chi$ with $N$ squarefree $< 857$.

Also get: Classification of all even 2-dim Galois representations with squarefree Artin conductor $N < 3000$. (For the icosahedral reprs, at $N = 1951$ and $2141$, this is modulo Artin's conjecture.)

## Application: computing eigenvalues

- Let $S(A) = \Big\{ \widehat{h} \mid \widehat{h} \text{ even}, \ supp(\widehat{h}) \subset [-A, A] \Big\}$.



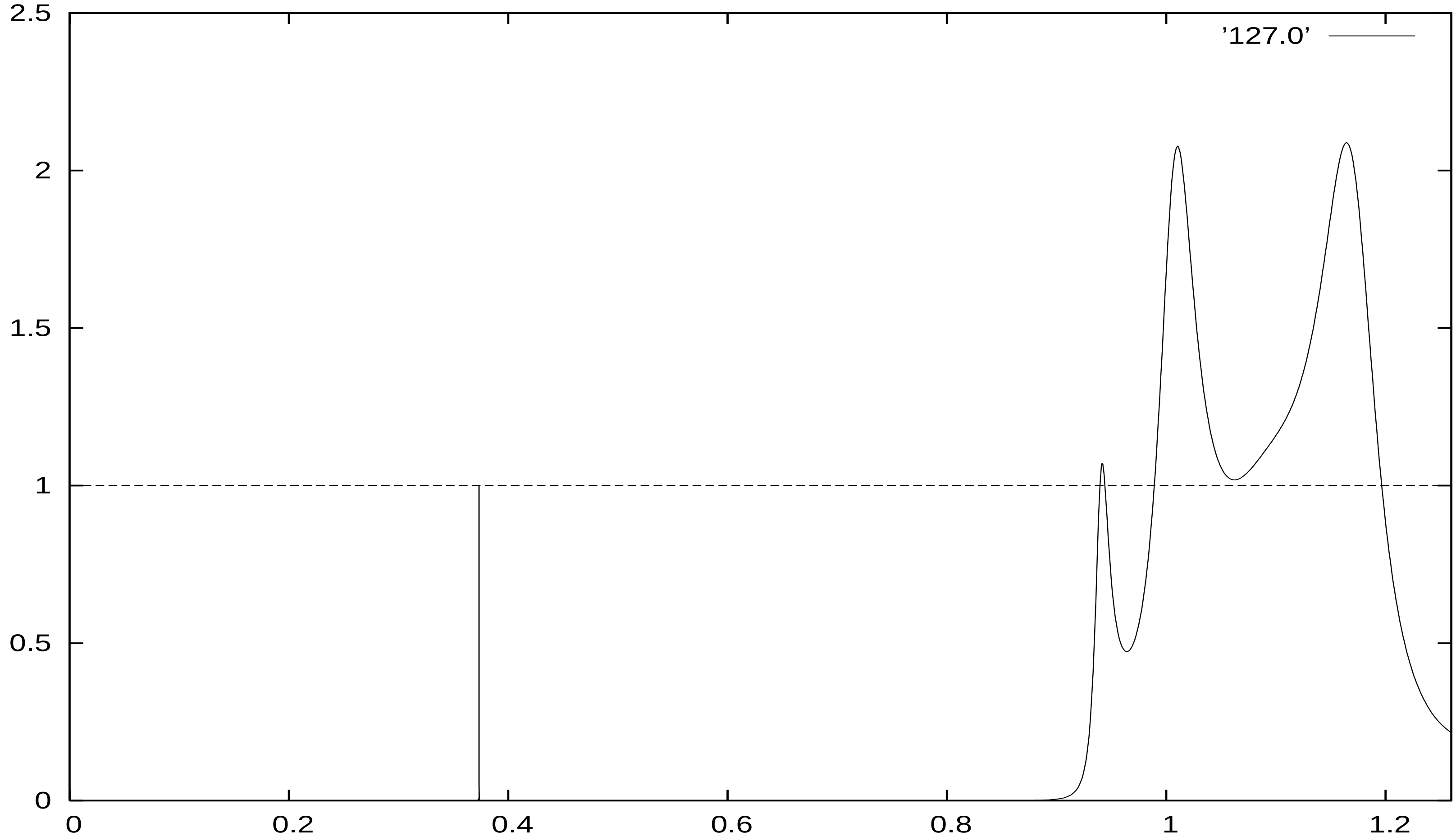

- For each $\rho > 0$, set:

$$F_A(\rho) = \inf \Big\{ \sum h\Big(\sqrt{\lambda - \tfrac{1}{4}}\Big)^2 \mid \widehat{h} \in S(A), \ h(\rho) = 1 \Big\}.$$

Then

$$\lim_{A \to \infty} F_A(\rho) = \begin{cases} 1 & \text{if } \rho \text{ (single) eigenvalue} \\ 0 & \text{otherwise.} \end{cases}$$

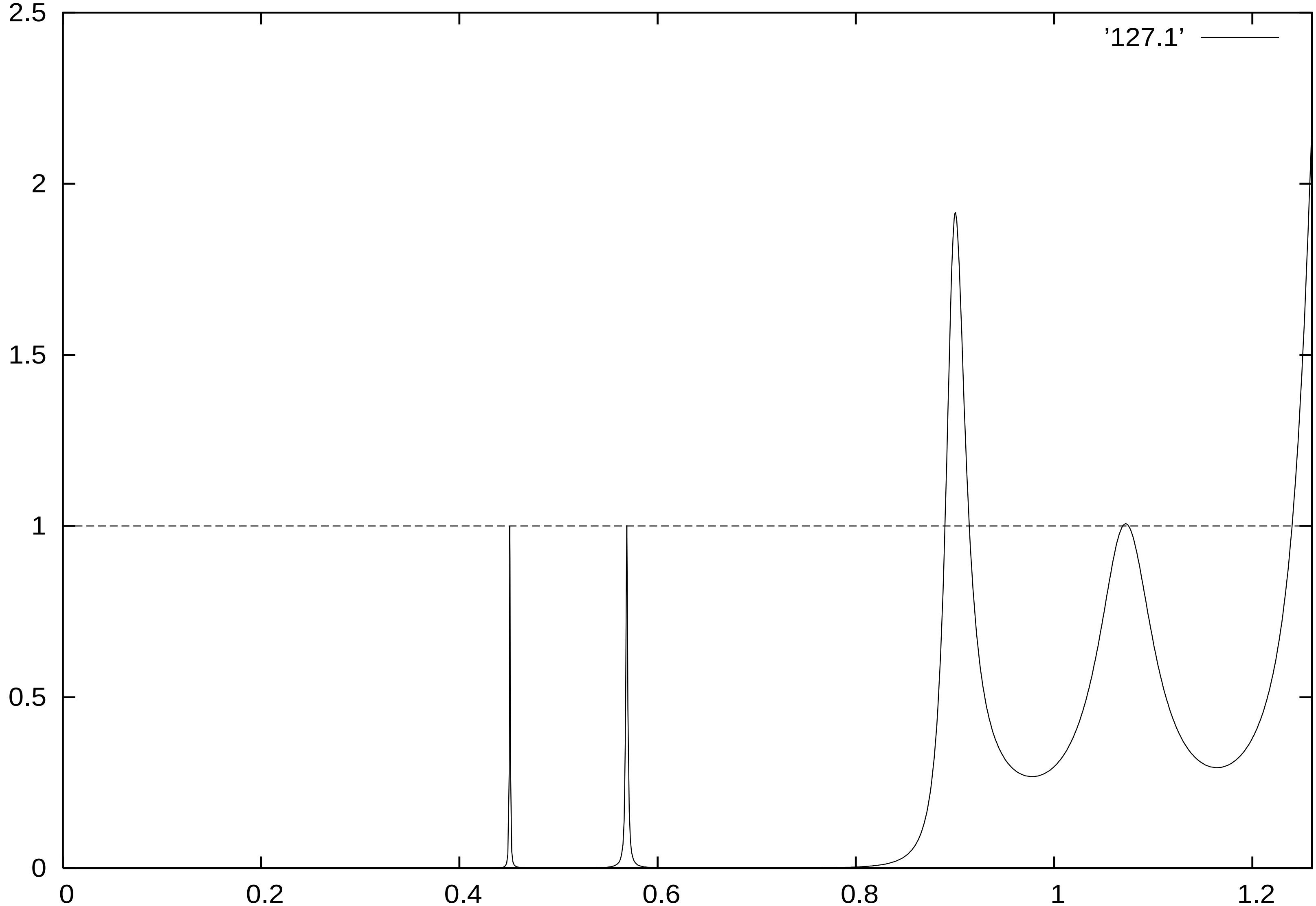
## Maxima for $N = 127, \chi = 1, even$

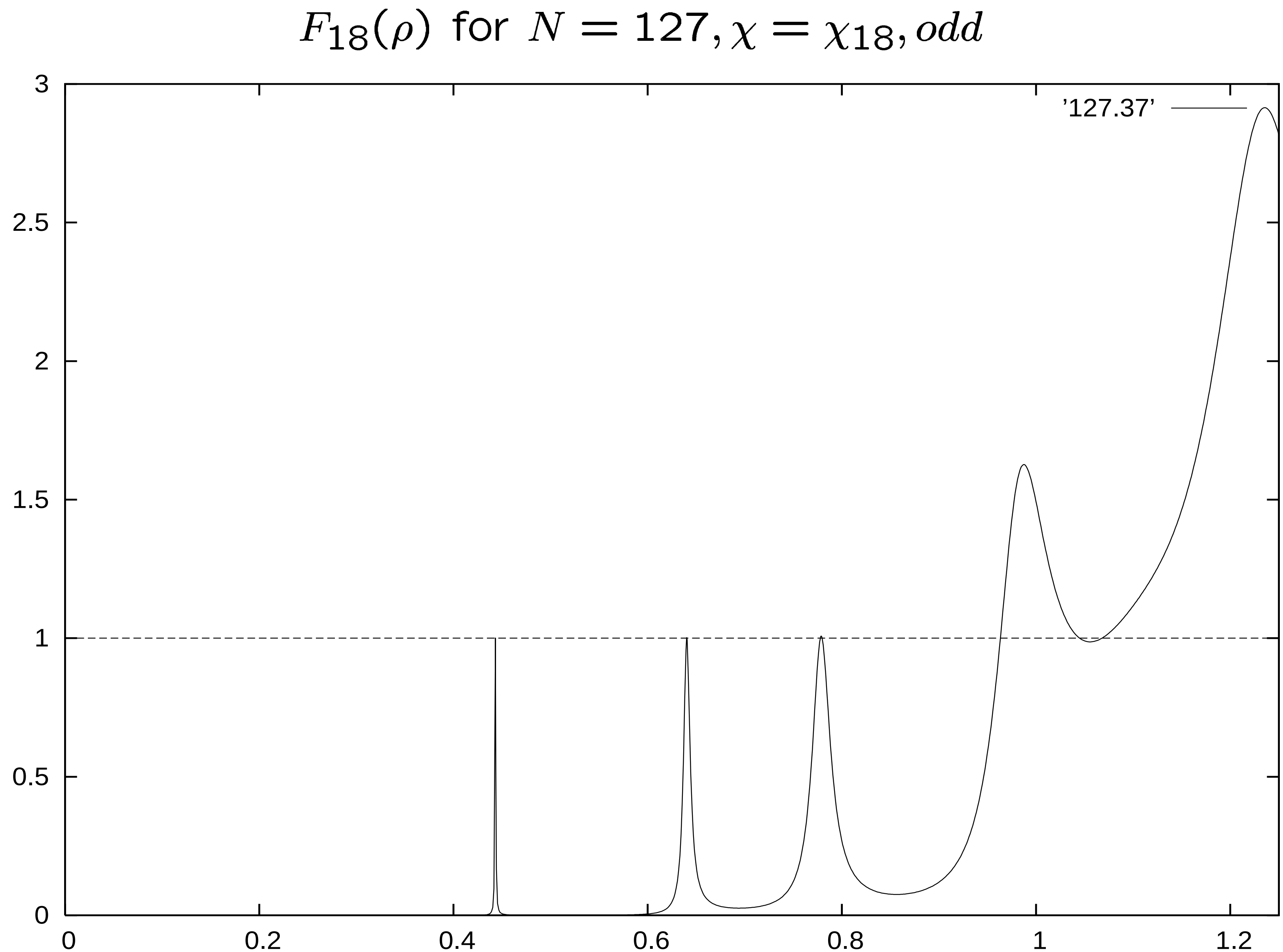



This gives: first eigenvalue $r = 0.3733851150 \pm 2 \cdot 10^{10}$.

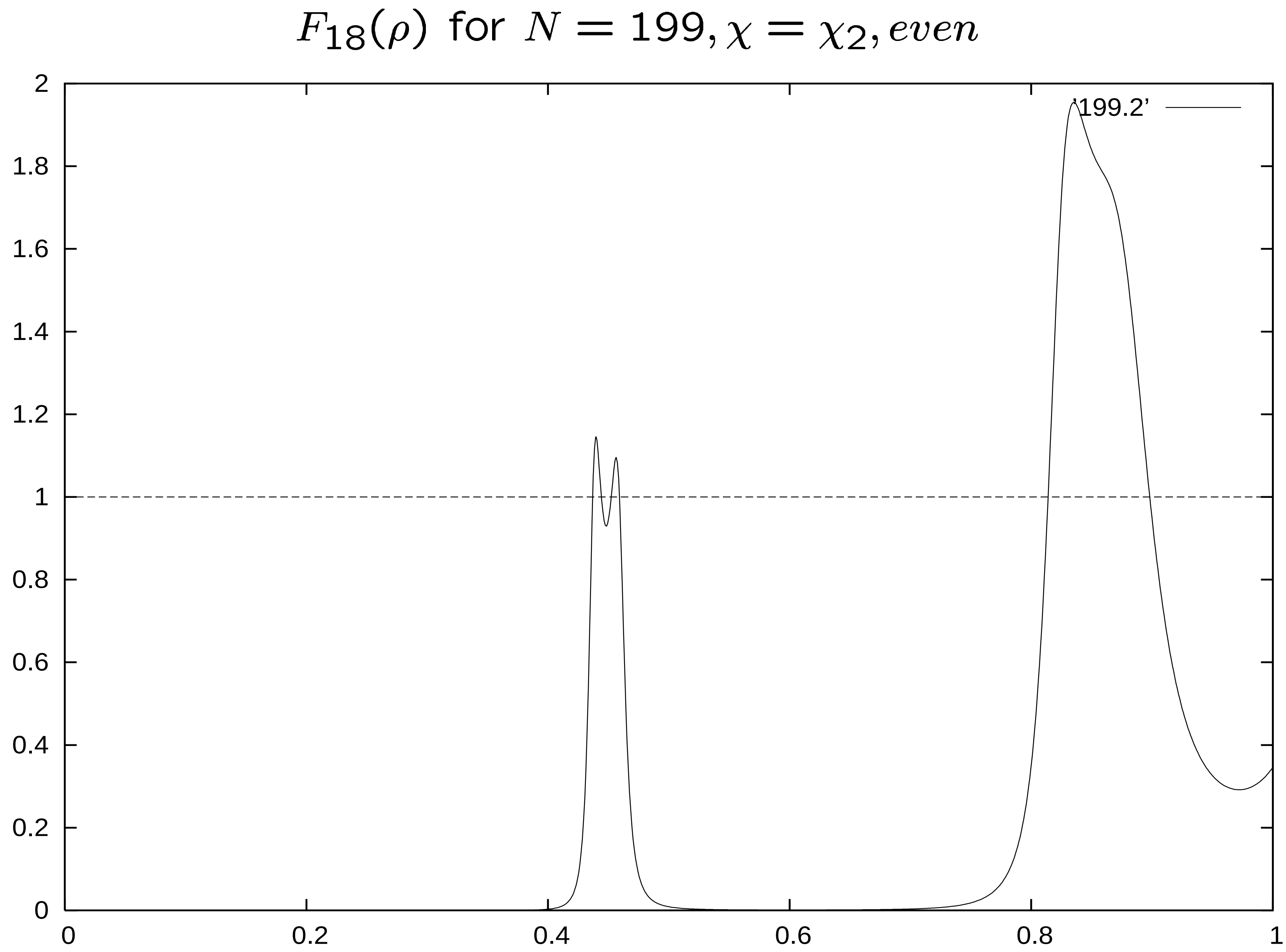
Check with Hejhal's algorithm (in pari); 0.37338511491202197757.

$F_{18}(\rho)$ for $N = 127, \chi = 1, odd$

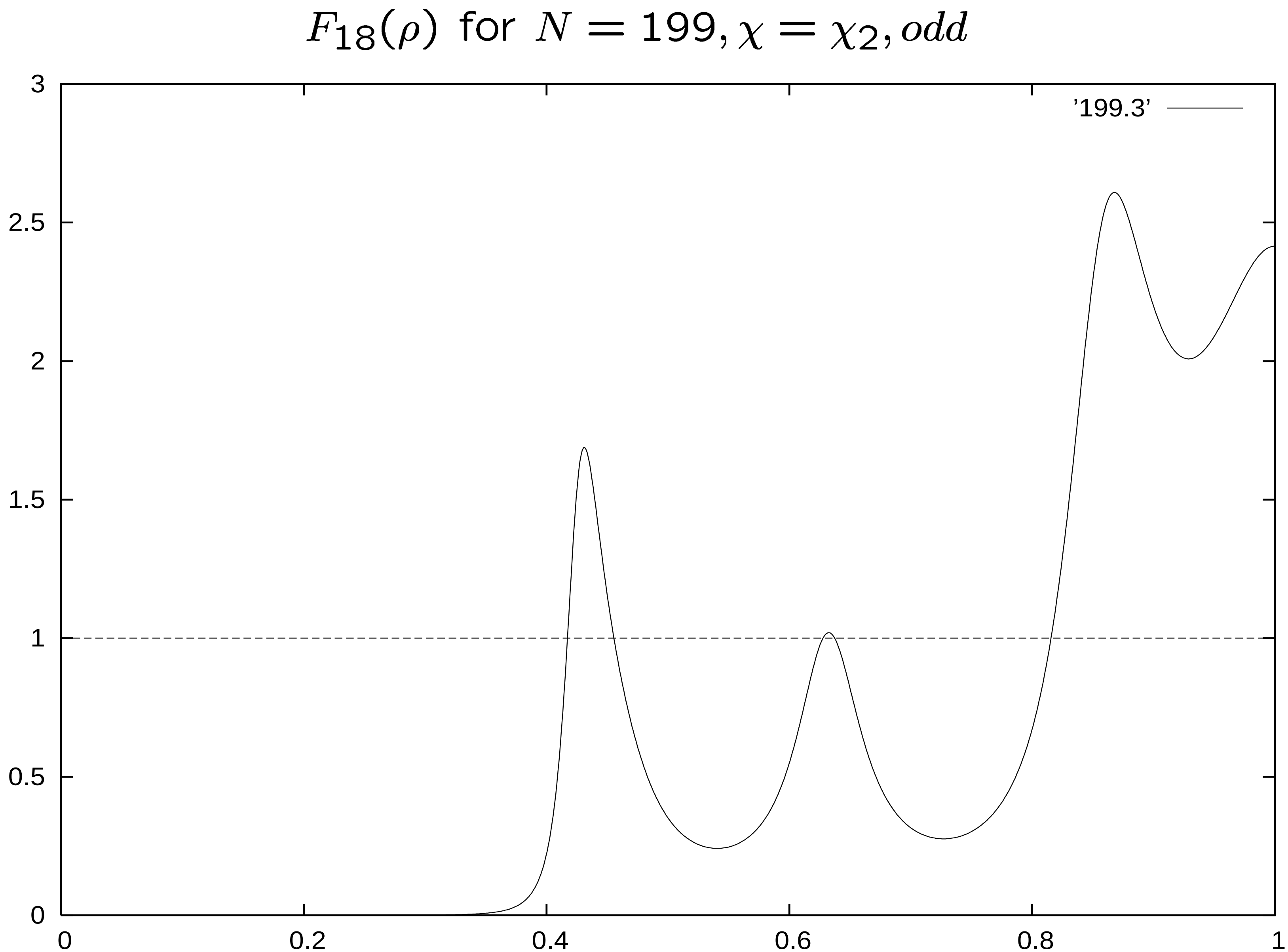




$F_{18}(\rho)$ for $N = 127, \chi = \chi_{18}, odd$
'127.37'
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1
1.2



$F_{18}(\rho)$ for $N = 199, \chi = \chi_2, even$
'199.2'
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


$F_{18}(\rho)$ for $N = 199, \chi = \chi_2, odd$
'199.3'
3
2.5
2
1.5
1
0.5
0
0
0.2
0.4
0.6
0.8
1

## Trace formula for the Hecke operator $T_n$ on $\Gamma_0(N)$ ($N, n \in \mathbb{Z}^+$, $(N,n)=1$, and $N$ square free):

$$\sum_{\lambda \geq 0} h(r_\lambda) \cdot \mathrm{Tr}\ T_n = \frac{\chi(\sqrt{n}) \prod_{p|N}(p+1)}{12\sqrt{n}} \int_{-\infty}^{\infty} r h(r) \tanh(\pi r)\, dr$$

$$+ \sum_{\substack{t \in \mathbb{Z} \\ \sqrt{t^2-4n} \notin \mathbb{Q}}} \Big\langle \chi(\delta) \Big\rangle_{\delta^2 - t\delta + n \equiv 0}$$

$$\cdot \left( \sum_{f|\ell} f \prod_{p|f} (1 - (\tfrac{d}{p}) p^{-1}) \cdot \prod_{p|N} \begin{Bmatrix} 2 & \text{if } p \mid f \\ 1 + \left(\frac{d}{p}\right) & \text{if } p \nmid f \end{Bmatrix} \right)$$

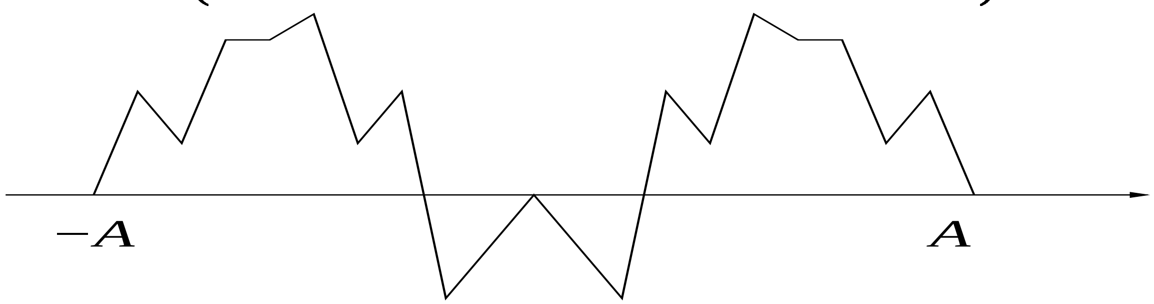
$$\cdot \frac{\mathbf{h}(d) \log \epsilon'_d}{\sqrt{t^2-4n}} \cdot \widehat{h}\left( \log \frac{(|t| + \sqrt{t^2-4n})^2}{4} \right)$$

$$+ \Big[\text{Elliptic}\Big] + \Big[\text{Parabolic}\Big] + \Big[\text{Eisenstein part}\Big].$$

$\Big(d, \ell$ implicit variables; $t^2 - 4n = d\ell^2$, $\ell \in \mathbb{Z}^+$, $d$ a fund. discr.$\Big)$

## Computing eigenvalues using STF for $T_{2^0}, T_{2^1}, T_{2^2}, T_{2^3}, \ldots, T_{2^{20}}$

- Again: $S(A) = \left\{ \widehat{h} \mid \widehat{h} \text{ even}, \; supp(\widehat{h}) \subset [-A, A] \right\}.$



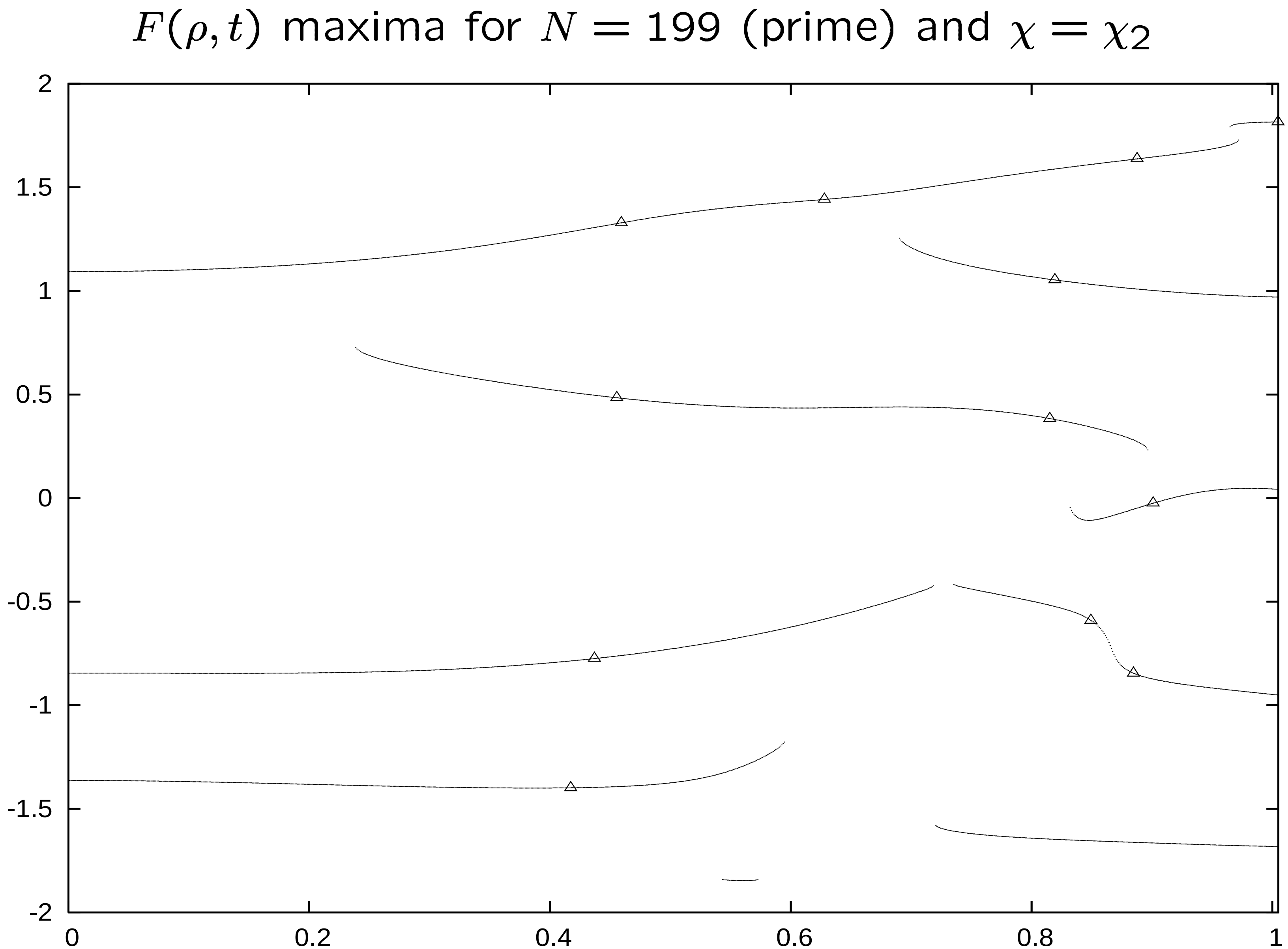

- For each $\rho > 0$, $t \in \mathbb{R}$ set:

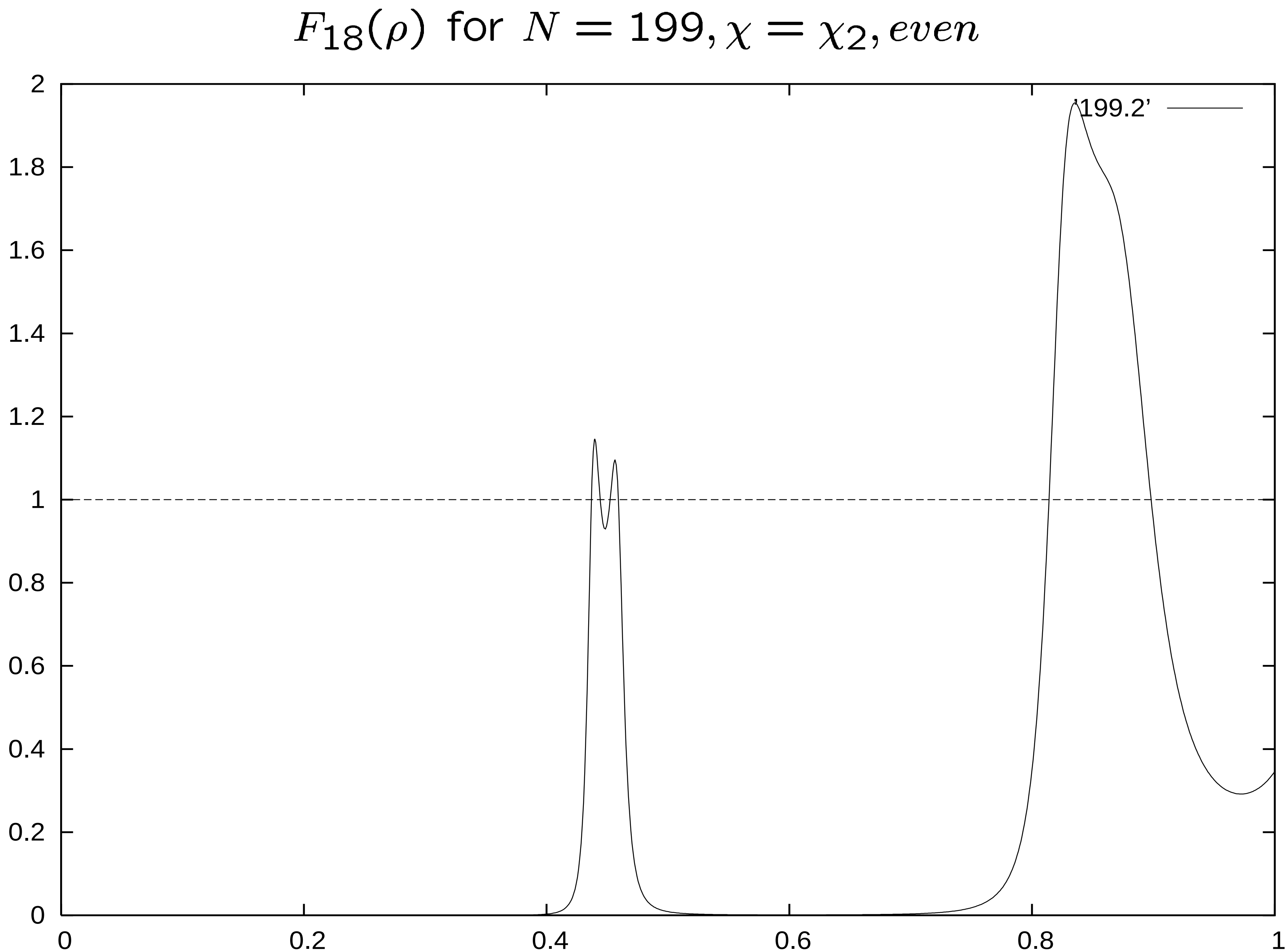
$$F(\rho,t) = \inf \left\{ \sum_{\lambda} \Big( \sum_{k=0}^{10} h_k(r_\lambda) t_{2,\lambda}^k \Big)^2 \mid \widehat{h}_k \in S(A_k), \; \sum_{k=0}^{10} h_k(\rho) t^k = 1 \right\}.$$

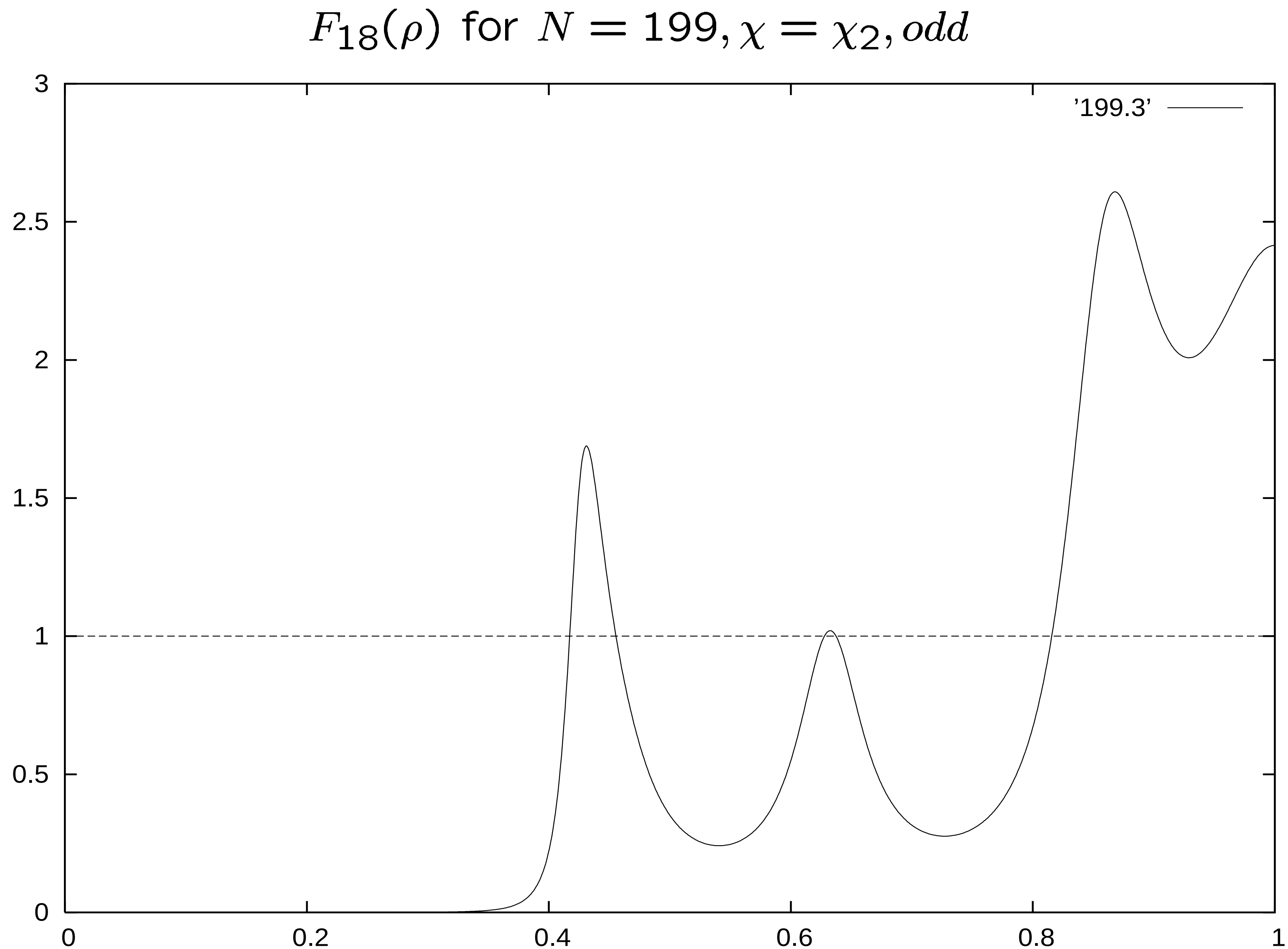
($A_0 = 14 > A_1 > \ldots > A_{10}$, uses $\sim$ 25.000.000 $\mathbf{h}(d)$'s.)

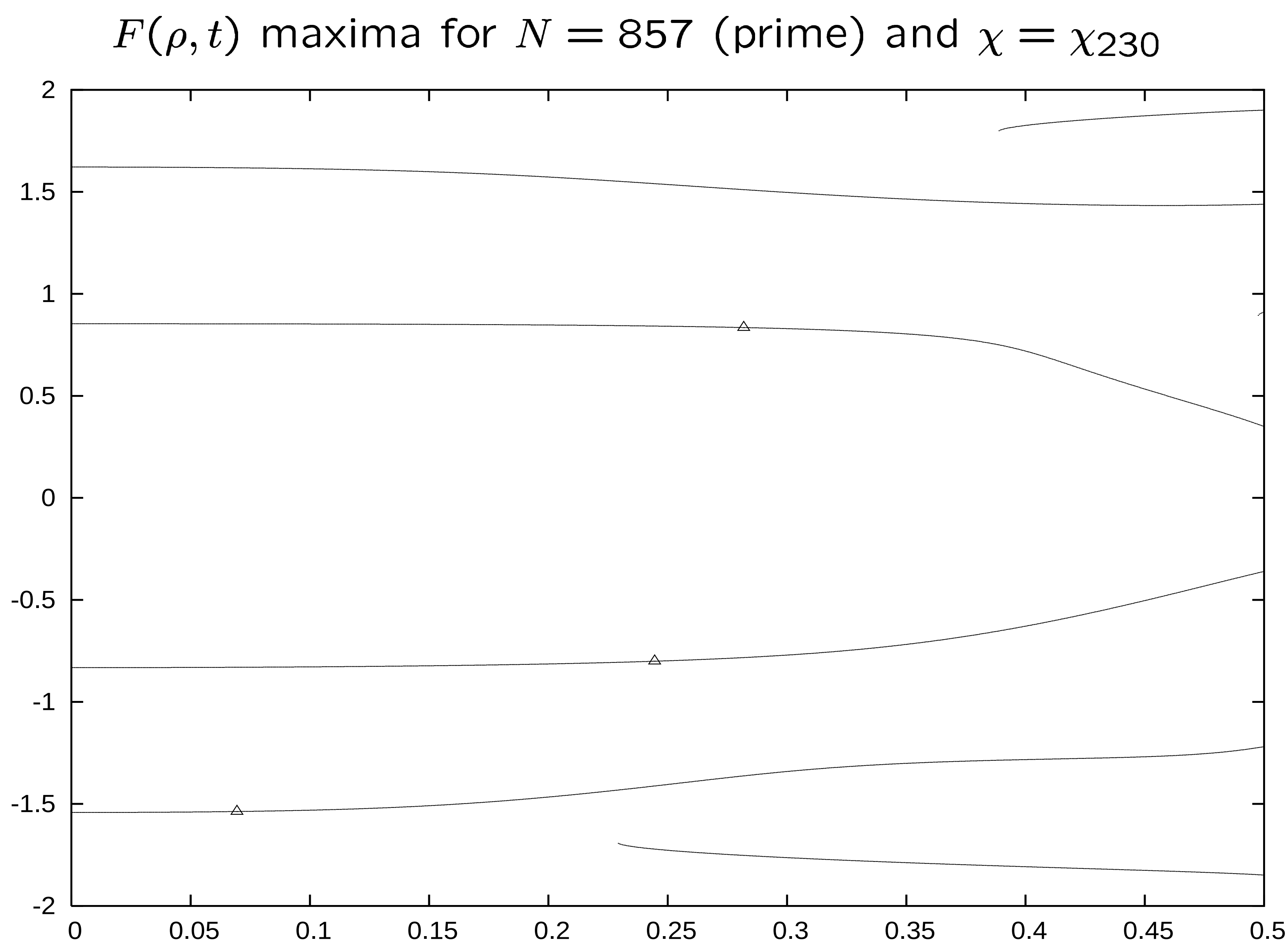
Then

$$\lim_{A\to\infty} F(\rho,t) \geqq \begin{cases} 1 & \text{if } \exists \; f \text{ with } \Delta f + (\rho^2 + \frac{1}{4}) f = 0 \text{ and } T_2 f = tf \\ 0 & \text{otherwise.} \end{cases}$$


$F(\rho,t)$ maxima for $N = 199$ (prime) and $\chi = \chi_2$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.2
0.4
0.6
0.8
1


$F_{18}(\rho)$ for $N = 199, \chi = \chi_2, even$
'199.2'
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0
0.2
0.4
0.6
0.8
1


$F_{18}(\rho)$ for $N = 199, \chi = \chi_2, odd$
'199.3'
0
0.5
1
1.5
2
2.5
3
0
0.2
0.4
0.6
0.8
1


$F(\rho, t)$ maxima for $N = 857$ (prime) and $\chi = \chi_{230}$
2
1.5
1
0.5
0
-0.5
-1
-1.5
-2
0
0.05
0.1
0.15
0.2
0.25
0.3
0.35
0.4
0.45
0.5

**Statistics of $\lambda_1(p)$ as $p \to \infty$?** $\qquad (\lambda_1 = r_1^2 + \frac{1}{4},\ r_1 \geqq 0)$.

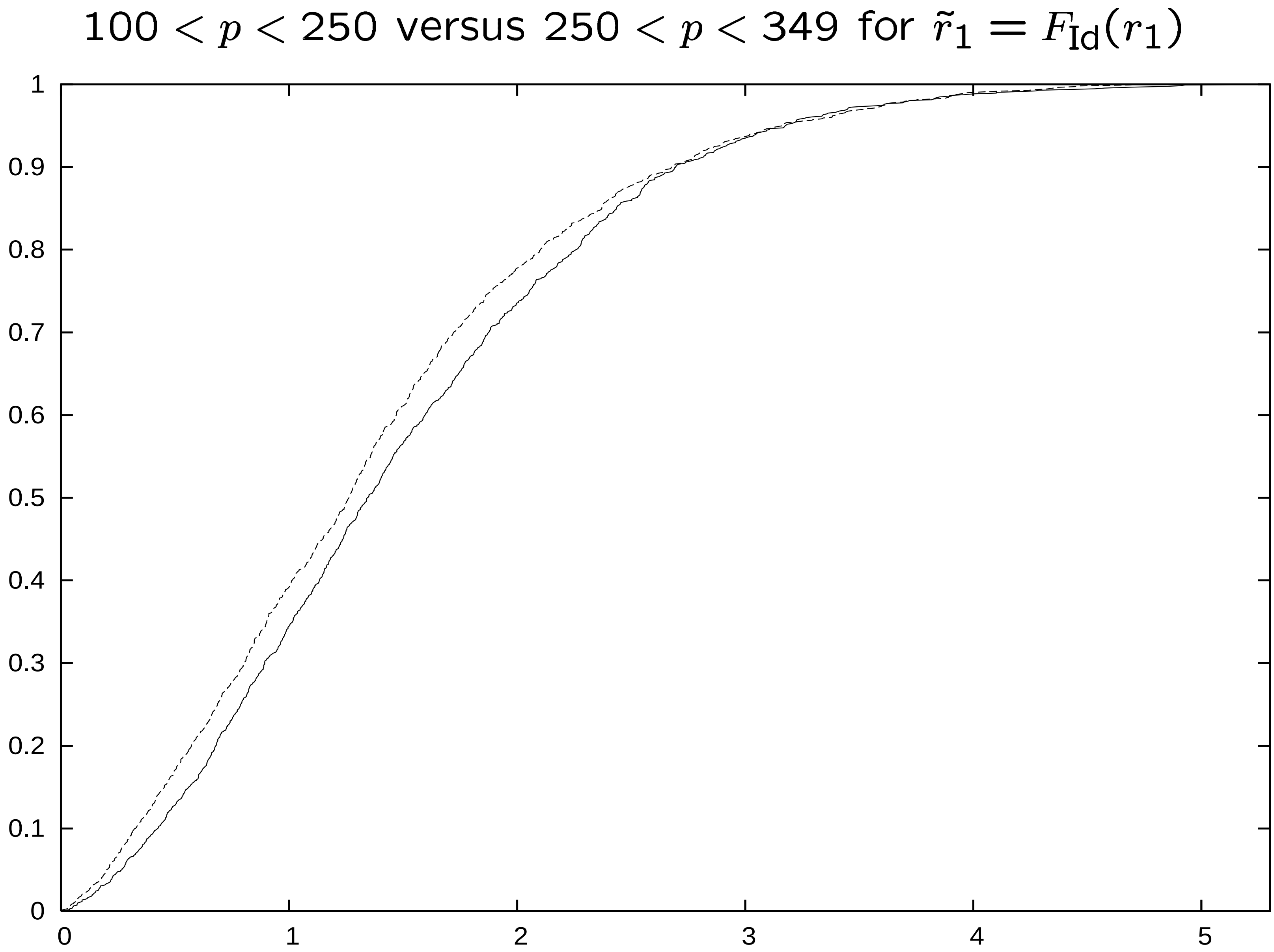
Taking $h = h_x := \chi_{[-x,x]}$ (formally!) in the trace formula, skipping all except identity term:

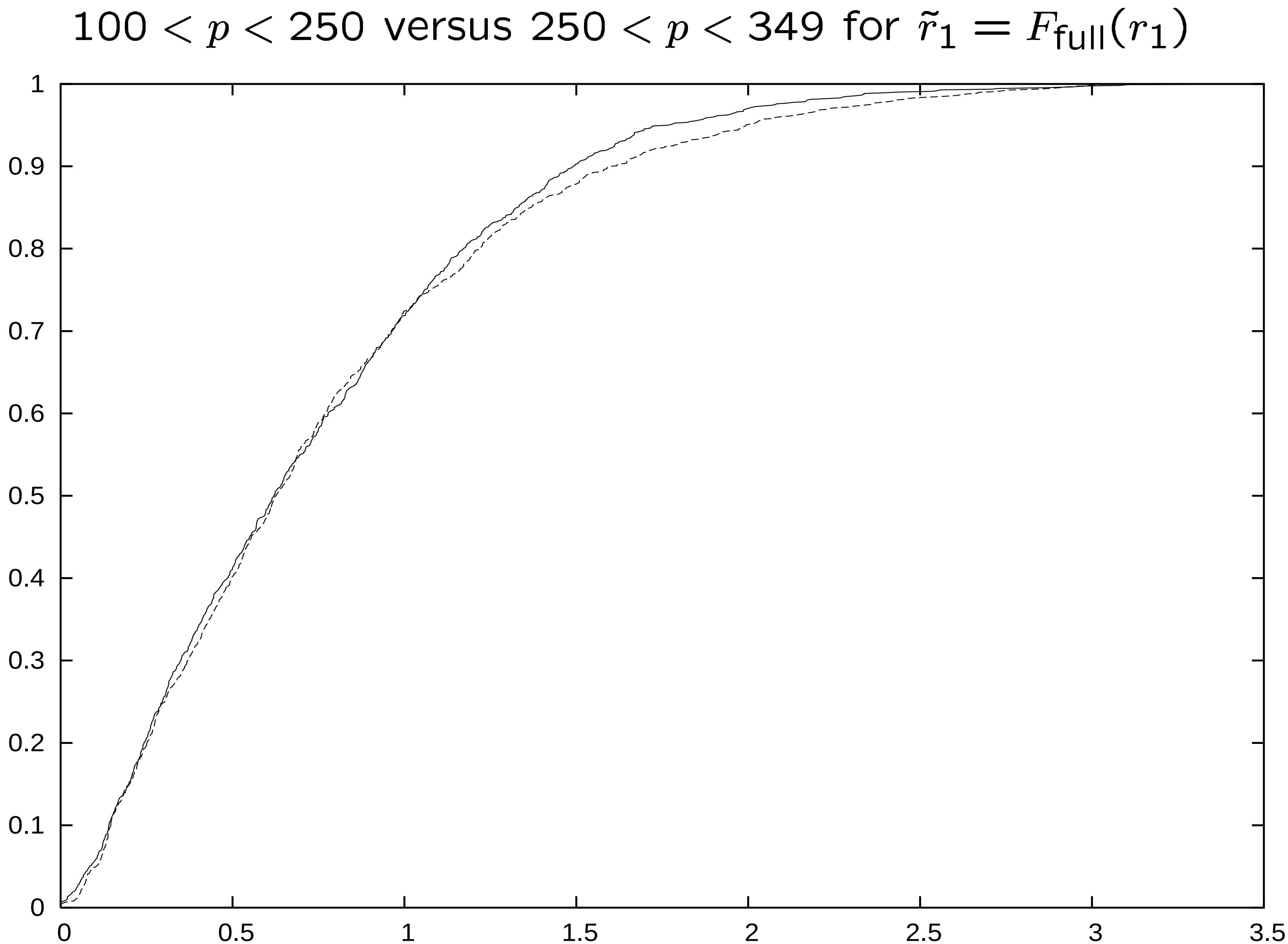
$$'' \sum_{0 \leqq r_n \leqq x} 1'' \qquad \approx F_{\mathrm{Id}}(x) := \frac{p+1}{6} \int_0^x r \tanh(\pi r)\, dr.$$

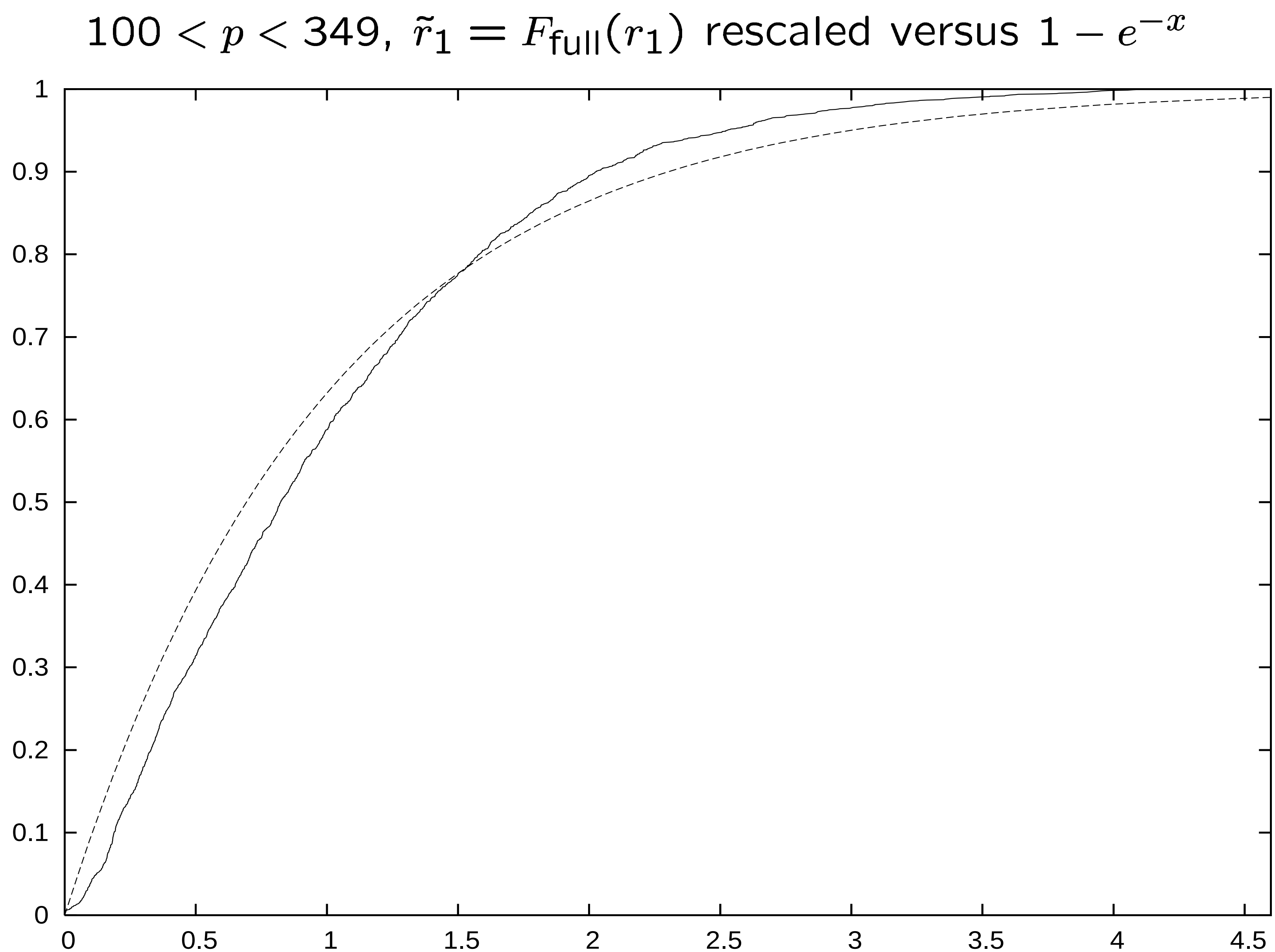
Including all except hyperbolic and $L$-function terms:

$$F_{\mathrm{full}}(x) := \frac{p+1}{6} \int_0^x r \tanh(\pi r)\, dr + \frac{1}{2} - \frac{x}{\pi} \log\left(\frac{4p^3}{\pi^2}\right)$$
$$+ \left[\text{If } p \equiv_4 1 : \quad \frac{(-1)^{ord(\chi)/2}}{2} \int_0^x \frac{dr}{\cosh(\pi r)}\right]$$
$$+ \left[\text{If } p \equiv_3 1 : \quad \frac{4 \mathrm{Re}\, e(ord(\chi)/3)}{3\sqrt{3}} \int_0^x \frac{\cosh(\pi r/3)}{\cosh(\pi r)}\, dr\right]$$

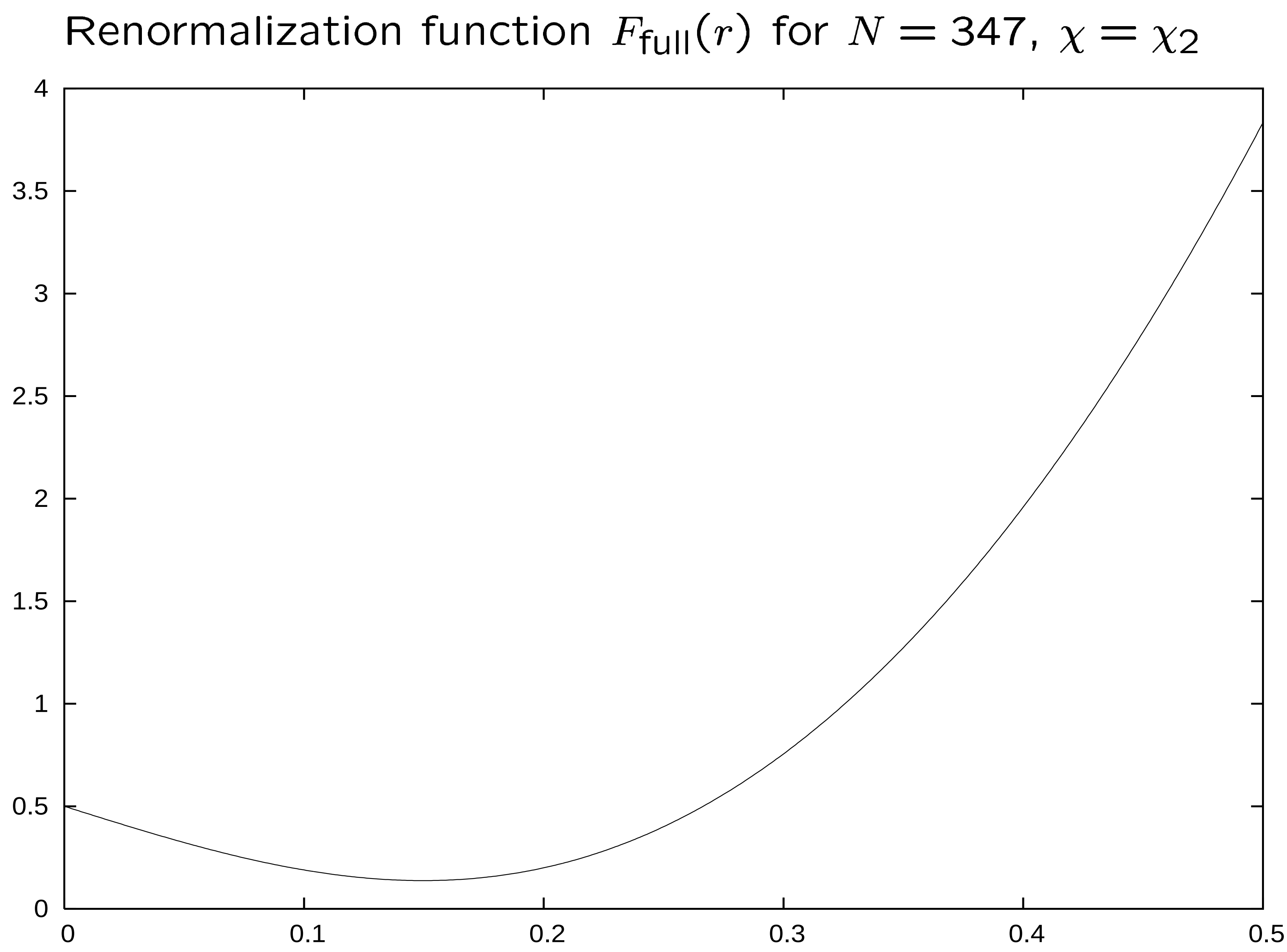
$$- \frac{2}{\pi} \int_0^x \mathrm{Re}\left(\frac{\Gamma'(\frac{1}{2} + ir)}{\Gamma(\frac{1}{2} + ir)} + \frac{\Gamma'(1 + ir)}{\Gamma(1 + ir)}\right) dr$$

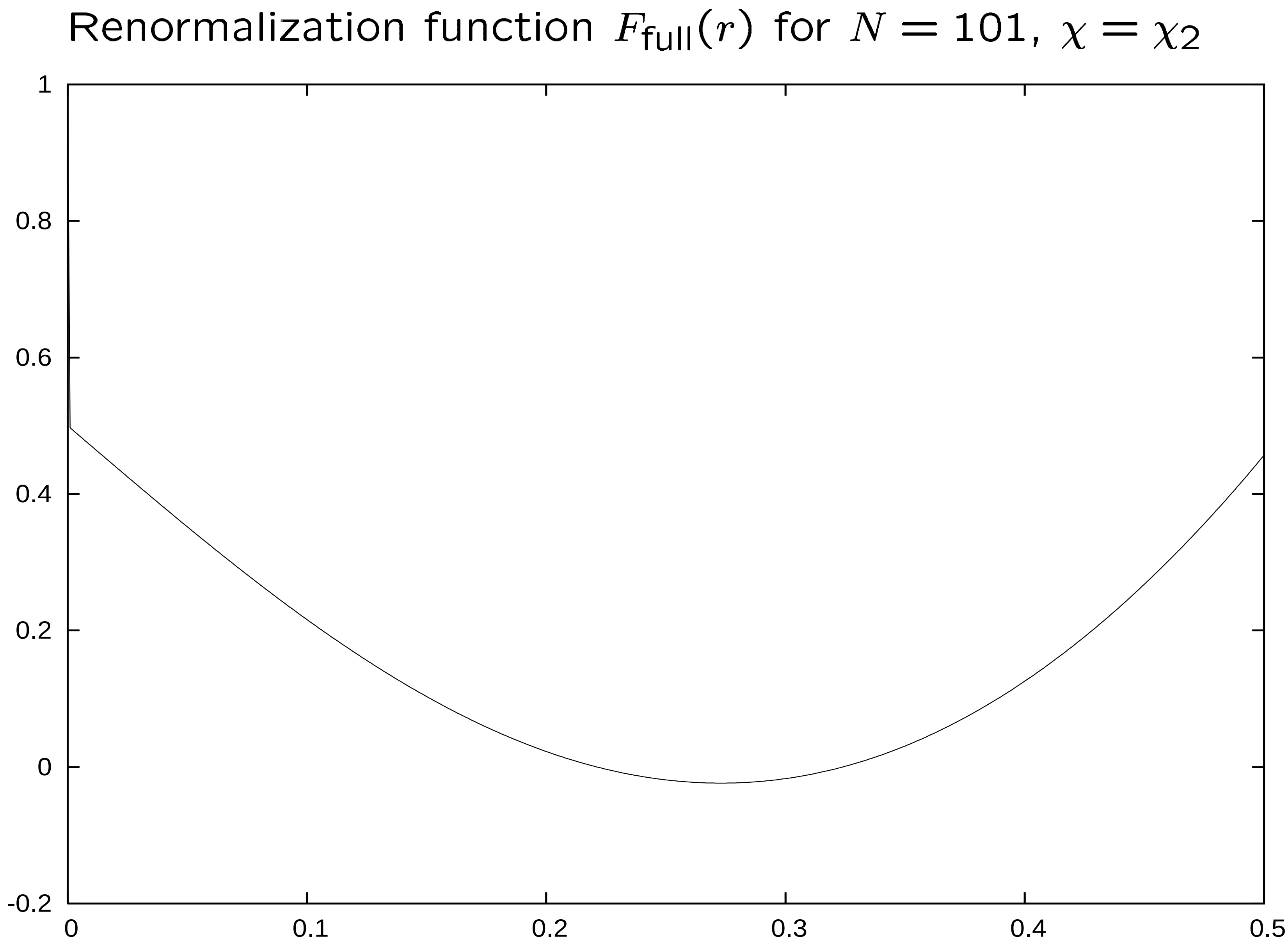

$100 < p < 250$ versus $250 < p < 349$ for $\tilde{r}_1 = F_{\mathrm{Id}}(r_1)$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
1
2
3
4
5


$100 < p < 250$ versus $250 < p < 349$ for $\tilde{r}_1 = F_{\text{full}}(r_1)$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5



$100 < p < 349$, $\tilde{r}_1 = F_{\text{full}}(r_1)$ rescaled versus $1 - e^{-x}$
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5


Renormalization function $F_{\mathsf{full}}(r)$ for $N = 347$, $\chi = \chi_2$
4
3.5
3
2.5
2
1.5
1
0.5
0
0
0.1
0.2
0.3
0.4
0.5



Renormalization function $F_{\mathrm{full}}(r)$ for $N = 101$, $\chi = \chi_2$
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.1
0.2
0.3
0.4
0.5

Renormalization function $F_{\text{full}}(r)$ for $N = 101$, $\chi = \chi_4$