THE EULER SERIES CONVERGES TO $\pi^2/6$

Consider the function

$$f(x) = \frac{sin(nx)}{sin(x)^n}$$

Let let x belong to the open interval $(0, \pi/2)$ and n be an odd number. We can then with some rewritings use Moivre to get the following.

$$f(x) = \frac{sin(nx)}{sin(x)^n} = Im\frac{\cos(nx) + isin(nx)}{sin(x)^n} =$$

$$Im\frac{e^{inx}}{sin(x)^n} = Im\left(\frac{\cos(x) + isin(x)}{sin(x)}\right)^n = Im(\cot(x) + i)^n$$

This expression can be expanded with binomial coefficients to get the following;

$$f(x) = Im\sum_{k=0}^{n}\binom{n}{k} i^k \cot^{n-k} x$$

By putting in n=2m+1 (since n was odd) we get the following expression:

$$f(x) = Im\sum_{k=0}^{2m+1}\binom{2m+1}{k} i^k \cot^{2m+1-k} x =$$

$$\sum_{k=oddnumber\geq 1}^{2m+1}\binom{2m+1}{k}(-1)^{\frac{k-1}{2}}\cot^{2m+1-k} x$$

We now substitute $\cot^2 x = t$, noting that $\cot^2 x$ is an injective function on our interval. This will give us a polynomial of degree m in t.

$$f(x) = p(t) = \binom{2m+1}{1}t^m - \binom{2m+1}{3}t^{m-1} + ...$$

It is known that the sum of the roots of a polynomial of degree d is the coefficient before the term of order $d-1$ divided by the coefficient before the term of order d with switched sign. This gives us

$$(sumofroots) = \frac{\binom{2m+1}{3}}{\binom{2m+1}{1}} = \frac{2m(2m-1)}{6}$$

To find the roots we look at the original function $f(x)$. It is zero iff $sin(nx) = sin((2m+1)x) = 0$ so it must be true for a solution to $f(x) = 0$ that $(2m+1)x = r\pi$ where r is an integer. This gives us $x = \frac{r\pi}{2m+1}$. Since x must be the open interval $(0, \pi/2)$ this gives us the m solutions $x_r = \frac{r\pi}{2m+1}$ for $r = 1...m$.

Due to $\cot(x)^2$ being injective this gives us m unique roots to p(t), so we have all roots. That gives us the following equality from the root sum formula:

$$\sum_{r=1}^{m}\cot^2\left(\frac{r\pi}{2m+1}\right) = \frac{2m(2m-1)}{6}$$

A similar equality can be gained for $\csc^2(x)$ by using the known formula $\cot^2(x) = \csc^2(x) - 1$. Putting this into our sum gives

$$\sum_{r=1}^{m}(\csc^2\left(\frac{r\pi}{2m+1}\right) - 1) = \frac{2m(2m-1)}{6}$$

$$\sum_{r=1}^{m}\csc^2\left(\frac{r\pi}{2m+1}\right) = \frac{2m(2m+2)}{6}$$

The inequality $sin(x) \leq x \leq tan(x)$ holds for our interval and gives rise to the inequality $\csc^2(x) \geq 1/x^2 \geq \cot^2(x)$. By using this inequality with our sums we get the following:

$$\sum_{r=1}^{m}\csc^2\left(\frac{r\pi}{2m+1}\right) = \frac{2m(2m+2)}{6} \geq \sum_{r=1}^{m}(\frac{2m+1}{r\pi})^2 \geq \sum_{r=1}^{m}\cot^2\left(\frac{r\pi}{2m+1}\right) = \frac{2m(2m-1)}{6}$$

Somewhat simplified as

$$\frac{2m(2m+2)}{6} \geq \frac{(2m+1)^2}{\pi^2}\sum_{r=1}^{m}\frac{1}{r^2} \geq \frac{2m(2m-1)}{6}$$

This gives us the Euler series partial sums as bounded from above and below by the following:

$$\frac{\pi^2}{6}\frac{2m(2m+2)}{(2m+1)^2} \geq \sum_{r=1}^{m}\frac{1}{r^2} \geq \frac{\pi^2}{6}\frac{2m(2m-1)}{(2m+1)^2}$$

A simple limit process then gives us:

$$\frac{\pi^2}{6} \geq \sum_{r=1}^{\infty}\frac{1}{r^2} \geq \frac{\pi^2}{6}$$

and we have proved convergence to $\frac{\pi^2}{6}$.