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PhD course in Probability, Home Assignment 1: Probabilities and Convergence

This home assignment consists of 7 exercises. The deadline is on Wednesday 4/3. You can hand in the assignment later, but if you hand it in k days too late, then your score will be multiplied by $1 - 0.15 * k$.

1. Consider a branching process with offspring distribution X such that $\mathbb{P}(X = k) = p^k(1-p)$ for $k = 0, 1, \ldots$ (this is essentially a geometric distribution) where $p \in (0,1)$ is the parameter. Let Z_n denote the number of individuals at level/generation n. In class, we proved that $\mathbb{P}(\lim Z_n \neq 0) > 0$ iff $\mathbb{E}[X] > 1$, which in our case translates into $p > 1/2$. We can call the induced graph the family tree of a Galton-Watson process with geometric offspring distribution, or just a geometric tree for short. We always start the process with one individual in generation 0, i.e. $Z_0 = 1$, and we call the corresponding vertex the root.

   (a) It can be of interest to know the value of the survival probability $\theta(p) = \mathbb{P}(\lim Z_n \neq 0)$. Determine the function $\theta(p)$ explicitly.

   (b) A binary tree is a tree in which every individual has exactly two descendants. We want to determine the probability that, starting at the root, the geometric tree contains a binary tree. Let $\theta_2(p)$ denote this probability and prove that $\theta_2(p)$ satisfies the equation

   $$(1-\theta_2(p))(1-p(1-\theta_2(p)))^2 - (1-p)(1-p+2p\theta_2(p)) = 0. \tag{1}$$

   Remark: Solving (1) gives the solution

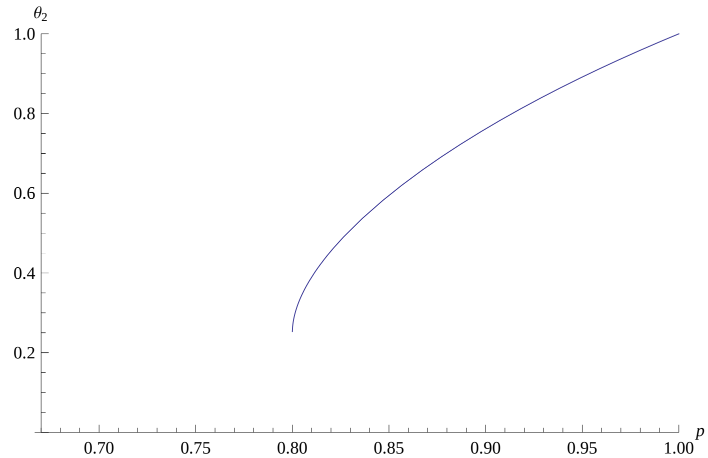
   $$\theta_2(p) = \begin{cases} \frac{3p-2+\sqrt{p(5p-4)}}{2p} & \text{if } 4/5 \leq p \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

   As can be seen from a plot (see Figure 1) and indeed from the expression itself, this has a real solution only when $4/5 \leq p (\leq 1)$. Therefore, we conclude that the geometric tree can contain a binary tree iff $4/5 \leq p \leq 1$. However, much more interesting is that *at* the critical value $4/5$, the existence of a binary subtree is strictly positive!

   Hint: It can be useful to condition on the number of individuals in the first generation.

   Solution:

   (a) We simply condition on the number k of individuals in generation 1. Given k, we then have k independent branching processes with exactly the same distribution as the original.
   Therefore, $1-\theta(p) = \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k)(1-\theta(p))^k$, since the original process



Figur 1: A plot of $\theta_2(p)$. Note the discontinuity.

dies out iff all the trees belonging to the first generation dies out. Thus,

$$\begin{aligned}
1-\theta(p) &= \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k)(1-\theta(p))^k \\
&= (1-p)\sum_{k=0}^{\infty}(p(1-\theta(p)))^k = \frac{1-p}{1-p(1-\theta(p))}.
\end{aligned}$$

Solving this gives the answer

$$\theta(p) = \begin{cases} \frac{2p-1}{p} & \text{if } p \geq 1/2 \\ 0 & \text{otherwise.} \end{cases}$$

(b) Again, we condition on the number k of individuals in generation 1. Then, at least two of these individuals must themselves contain a binary subtree among their descendants. As in part (a) we have that

$$\begin{aligned}
1-\theta_2 &= \sum_{k=0}^{\infty} \mathbb{P}(Z_1 = k)\left((1-\theta_2)^k + k\theta_2(1-\theta_2)^{k-1}\right) \\
&= \sum_{k=0}^{\infty}(1-p)p^k\left((1-\theta_2)^k + k\theta_2(1-\theta_2)^{k-1}\right) \\
&= (1-p)\sum_{k=0}^{\infty}(p(1-\theta_2))^k + \frac{(1-p)\theta_2}{1-\theta_2}\sum_{k=0}^{\infty} k(p(1-\theta_2))^k \\
&= \frac{1-p}{1-p(1-\theta_2)} + \frac{(1-p)\theta_2}{1-\theta_2}\frac{p(1-\theta_2)}{(1-p(1-\theta_2))^2} \\
&= \frac{1-p}{1-p(1-\theta_2)}\left(1 + \frac{p\theta_2}{1-p(1-\theta_2)}\right) = \frac{(1-p)(1-p+2p\theta_2)}{(1-p(1-\theta_2))^2}
\end{aligned}$$

2. Let $f:[0,1]\to\mathbb{R}$ be a continuous function (i.e. $f \in C([0,1])$), and let $X_n \sim \mathrm{Bin}(n,x)$, i.e. such that

$$\mathbb{P}(X_n = k) = \binom{n}{k}x^k(1-x)^{n-k}.$$

Let $A = \{|X_n/n - x| > \delta\}$, and let $Z_n = f(x) - f(X_n/n)$. Using that $\mathbb{E}[Z_n] = \mathbb{E}[Z_n I(A)] + \mathbb{E}[Z_n I(A^c)]$, prove that

$$\lim_{n\to\infty} \sup_{0\le x\le 1} \left| f(x) - \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \right| = 0.$$

Remark: This is Weierstrass' approximation theorem.

Solution: We first note that

$$\mathbb{P}(A) = \mathbb{P}(|X_n/n - x| > \delta) \le \frac{Var(X_n/n)}{\delta^2} = \frac{x(1-x)}{n\delta^2}.$$

Furthermore, f is in fact bounded and uniformly continuous since $[0,1]$ is compact. By using that f is bounded by some constant $C < \infty$, we get that

$$\mathbb{E}[Z_n I(A)] \le 2C\mathbb{P}(A) \le 2C\frac{x(1-x)}{n\delta^2}.$$

By using that f is uniformly continuous, we see that for any $\epsilon > 0$, there exists a $\delta > 0$ such that $|f(x) - f(X_n/n)| \le \epsilon$ if $|X_n/n - x| \le \delta$. Therefore,

$$\mathbb{E}[Z_n I(A^c)] \le \epsilon.$$

Thus, for any $\epsilon > 0$, by first taking $\delta > 0$ small enough, and then n large enough (uniformly for $x \in [0,1]$) we have that

$$\left| f(x) - \sum_{k=0}^{n} f(k/n) \binom{n}{k} x^k (1-x)^{n-k} \right| = |\mathbb{E}[Z_n]| \le 2\epsilon.$$

3. In many applications, one is interested in the maxima of i.i.d. random variables. For example, let X_k denote the (random) height of the largest wave to hit the coastal area of the Netherlands during some fixed month. Then the distribution of $M_n := \max_{k\le n} X_k$ will give some information about how high you should build the walls protecting the people living there. This is also relevant when pricing insurances. How bad will the worst storm be that occurs over a hundred year period? Prove the following:

   (a) **Thin tail distribution:** Assume that X_k have distribution function $F(x)$ such that
   $$\lim_{x\to\infty} \frac{1-F(x)}{e^{-x}} = 1.$$
   Prove that $(M_n - \log n) \xrightarrow{w} X$ where X has distribution function $F_X(x) = \exp(-e^{-x})$ for every $x \in \mathbb{R}$.

   (b) **Heavy tail distribution:** Assume that X_k have distribution function $F(x)$ such that
   $$\lim_{x\to\infty} \frac{1-F(x)}{x^{-\alpha}} = 1.$$
   Prove that $M_n/n^{1/\alpha} \xrightarrow{w} X$ where X has distribution function $F_X(x) = \exp(-x^{-\alpha})$ for every $x > 0$.

Solution:

(a) **Thin tail distribution:** First we observe that

$$\mathbb{P}(M_n - \log n \le y) = \prod_{k=1}^{n} \mathbb{P}(X_k \le y + \log n) = F(y + \log n)^n.$$

The condition gives that for any $c > 1$, and x large enough, we have that $F(x) \le 1 - ce^{-x}$ and so for any y and n large enough

$$F(y + \log n)^n \le \left(1 - ce^{-y-\log n}\right)^n = \left(1 - ce^{-y}/n\right)^n \to e^{-ce^{-y}}.$$

Since $c > 1$ was arbitrary, we have that

$$\limsup_n \mathbb{P}(M_n - \log n \le y) = \limsup_n F(y + \log n)^n \le e^{-e^{-y}}.$$

The statement follows by a similar argument for the other direction.

(b) **Heavy tail distribution:** We have that $\mathbb{P}(M_n \le x) = \mathbb{P}(X_1 \le x)^n = F(x)^n$ and so for any $c > 1$,

$$\begin{aligned}
&\limsup_n \mathbb{P}(M_n/n^{1/\alpha} \le x) \\
&\quad = \limsup_n F(n^{1/\alpha}x)^n \le \limsup_n \left(1 - c(n^{1/\alpha}x)^{-\alpha}\right)^n \\
&\quad = \limsup_n \left(1 - cn^{-1}x^{-\alpha}\right)^n = e^{-cx^{-\alpha}}.
\end{aligned}$$

As above, the statement follows.

4. Let $(X_n)_{n\ge1}$ be a sequence of random variables such that for every $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(|X_n - X| \ge \epsilon) = 0.$$

(a) Prove that

$$\sup_{n_1,n_2\ge N} \mathbb{P}(|X_{n_1} - X_{n_2}| \ge \epsilon) \to 0 \text{ as } N \to \infty.$$

(b) Prove that if $(X_n)_{n\ge1}$ are independent, then X is almost surely constant.

Solution:

(a) This is short and mainly meant as a hint for (b). Obviously,

$$\{|X_{n_1} - X_{n_2}| \ge \epsilon\} \subset \{|X_{n_1} - X| \ge \epsilon/2\} \cup \{|X_{n_2} - X| \ge \epsilon/2\},$$

and so for any $\delta > 0$, by letting N be so large that $\mathbb{P}(|X_n - X| \ge \epsilon/2) \le \delta/2$ for every $n \ge N$, we see that $\mathbb{P}(|X_{n_1} - X_{n_2}| \ge \epsilon) \le \delta$ for all $n_1, n_2 \ge N$.

(b) Assume that X is not a.s. constant. Then, there exists c and $\delta, \epsilon > 0$ such that $\mathbb{P}(X < c) \ge \epsilon$ and $\mathbb{P}(X > c + \delta) \ge \epsilon$. (This fact can be argued in a number of ways, for example, let I, S be the essential infimum/supremum of X respectively. Then, let $\delta = (S - I)/3$ and $c = (I + S)/2 - \delta/2$, the claim follows with these choices of c, δ.)
Using this, and by taking M large enough, $\mathbb{P}(X_n < c) \ge \epsilon/2$ and $\mathbb{P}(X_n > c + \delta) \ge \epsilon/2\ \forall\ n \ge M$ (this follows since $\xrightarrow{P}$ implies $\xrightarrow{d}$ by theorem during lecture). Thus, for $n_1, n_2 \ge M$,

$$\begin{aligned}
&\mathbb{P}(|X_{n_1} - X_{n_2}| \ge \delta) \\
&\quad \ge \mathbb{P}(X_{n_1} < c, X_{n_2} > c + \delta) = \mathbb{P}(X_{n_2} > c + \delta)\mathbb{P}(X_{n_1} < c) \ge \epsilon^2/4.
\end{aligned}$$

Since this is uniform over $n_1, n_2 \geq M$ we cannot have that

$$\sup_{n_1, n_2 \geq N} \mathbb{P}(|X_{n_1} - X_{n_2}| \geq \epsilon) \to 0 \text{ as } N \to \infty,$$

which contradicts (a).

5. In many applications (e.g financial mathematics), one is interested in some function of normal random variables. In particular taking the exponential. The purpose of this exercise is to explore some convergence properties of this. Thus, let $\alpha, \beta \in \mathbb{R}$ and $Z_1, Z_2, \ldots$ be an i.i.d. sequence of $N(0,1)$ random variables. Define

$$X_n = e^{\alpha S_n - \beta n},$$

where $S_n = \sum_{i=1}^n Z_i$.

Recall that:

- $X_n \xrightarrow{P} X$ if $\forall\ \epsilon > 0$, $\mathbb{P}(|X_n - X| \geq \epsilon) \to 0$.
- $X_n \xrightarrow{r} X$ if $\mathbb{E}[|X_n - X|^r] \to 0$ (defined for $r > 0$).

(a) Prove that $X_n \xrightarrow{P} 0$ iff $\beta > 0$. (This should be done by direct methods, i.e. it does not suffice to do (c) and then use that a.s. convergence implies convergence in probability).

(b) Prove that $X_n \xrightarrow{r} 0$ iff $r < 2\beta/\alpha^2$.

(c) Prove that $X_n \xrightarrow{a.s.} 0$ iff $\beta > 0$.

Solution:

(a) We may assume that $\alpha > 0$ by symmetry. Thus, for any $\epsilon > 0$,

$$\begin{aligned}\mathbb{P}(X_n > \epsilon) &= \mathbb{P}\left(e^{\alpha S_n - \beta n} > \epsilon\right) \\ &= \mathbb{P}\left(\alpha S_n - \beta n > \log \epsilon\right) = \mathbb{P}\left(\frac{S_n}{\sqrt{n}} > \frac{\log \epsilon + \beta n}{\sqrt{n}}\right) \to 0,\end{aligned}$$

by using that $S_n/\sqrt{n} \sim N(0,1)$ and that $\beta > 0$.
On the other hand if $\beta < 0$,

$$\mathbb{P}(X_n > 1) = \mathbb{P}(\alpha S_n - \beta n > 0) = \mathbb{P}\left(\frac{S_n}{\sqrt{n}} \geq \frac{\beta}{\alpha}\sqrt{n}\right) \to 1,$$

as above. When $\beta = 0$, then in fact $\mathbb{P}(X_n > 1) = 1/2$ for every n.

(b) We have that

$$\mathbb{E}[X_n^r] = \mathbb{E}[e^{\alpha r S_n - \beta r n}] = e^{-\beta r n} \prod_{i=1}^n \mathbb{E}[e^{\alpha r Z_i}] = e^{-\beta r n} e^{n \frac{(\alpha r)^2}{2}}.$$

This goes to 0 iff $(\alpha r)^2 < 2\beta r$ from which the proof follows. The fact that

$$\mathbb{E}[e^{\alpha r Z_i}] = e^{n \frac{(\alpha r)^2}{2}},$$

can be explicitly calculated, or looked up in a table.

(c) For this we use good-old Borel-Cantelli as follows. If $\beta > 0$, let $r < 2\beta/\alpha^2$ and conclude by using Markovs inequality that

$$\mathbb{P}(X_n > 1/n) = \mathbb{P}(X_n^r > 1/n^r) \leq n^r \mathbb{E}[X_n^r].$$

By part (b) and since we picked $0 < r < 2\beta/\alpha^2$, we have that for some $c > 0$, $\mathbb{E}[X_n^r] \leq e^{-cn}$. Therefore

$$\sum_{i=1}^{\infty} \mathbb{P}(X_n > 1/n) \leq \sum_{i=1}^{\infty} n^r e^{-cn} < \infty,$$

and so by BC

$$\mathbb{P}(X_n > 1/n \text{ i.o.}) = 0.$$

This means that indeed $X_n \xrightarrow{a.s.} 0$.

You can also use the SLLN proved during the lectures, but this is somewhat of an overkill...

If $\beta \leq 0$, we do not have a.s. convergence by (a) and theorem from class.

6. Consider $C[0,1]$, the set of continuous functions from $[0,1]$ to $\mathbb{R}$. Let

$$f_n(x) = nxI(x \in [0, 1/n]) + (2 - nx)I(x \in (1/n, 2/n]),$$

which is simply the function that increases linearly from 0 to 1 for $x \in [0, 1/n]$, and then decreases linearly from 1 to 0 for $x \in (1/n, 2/n]$.

Let μ_n be the measure on $C[0,1]$ which assigns unit mass to f_n, and let μ be the measure which assigns unit mass to the function $f(x) = 0 \; \forall \; x \in [0,1]$.

Prove that $\mu_n \overset{w}{\not\to} \mu$ but that the finite dimensional marginals of μ_n converges to those of μ.

Remark: This proves that while $\mathcal{C}_f$ is a separating class, it is not a convergence determining class.

Solution: Let $A := \{f \in C[0,1] : \sup_{x\in[0,1]} f(x) \geq 1/2\}$. Obviously, A is a continuity set since $\mu(A) = 0$. However $\mu_n(A) = 1 \; \forall n \geq 1$, so by the Portmanteau theorem, $\mu_n \overset{w}{\not\to} \mu$.

We need to prove that $\mu_n \pi^{-1}_{x_1 \cdots x_k} \xrightarrow{w} \mu \pi^{-1}_{x_1 \cdots x_k}$. For all n large enough we have that $\pi_{x_1 \cdots x_k}(f_n) = (0, 0, \ldots, 0)$ so that for any $H \subset \mathbb{R}^k$ (or rather in $\mathcal{B}(\mathbb{R}^k)$) $\mu_n \pi^{-1}_{x_1 \cdots x_k}(H) = I((0,0,\ldots,0) \in H) = \mu \pi^{-1}_{x_1 \cdots x_k}(H)$.

7. Let S be a metric space with metric d. Assume that (X_n, Y_n) is a random variable taking values in $S \times S$ (so that X_n and Y_n are random variables taking values in S).

Define the random variable $Z_n(\omega) := d(X_n(\omega), Y_n(\omega))$, which is then a random variable taking values in $\mathbb{R}^+$.

Prove that if $X_n \xrightarrow{w} X$ and $Z_n \xrightarrow{w} 0$, then $Y_n \xrightarrow{w} X$.

Solution: Let F be a closed set, and define $F_\epsilon := \{s \in S : d(s, F) \leq \epsilon\}$ which is then also a closed set. Then,

$$\mathbb{P}(Y_n \in F) \leq \mathbb{P}(Z_n \geq \epsilon) + \mathbb{P}(X_n \in F_\epsilon).$$

By the Portmanteau theorem

$$\begin{aligned}
&\limsup \mathbb{P}(Y_n \in F) \\
&\quad \leq \limsup \mathbb{P}(Z_n \geq \epsilon) + \limsup \mathbb{P}(X_n \in F_\epsilon) \\
&\quad \leq \mathbb{P}(0 \geq \epsilon) + \mathbb{P}(X \in F_\epsilon) = \mathbb{P}(X \in F_\epsilon),
\end{aligned}$$

since $[\epsilon, \infty)$ is a closed set. Since F is closed we have that $F = \cap_{\epsilon>0} F_\epsilon$, so that by letting $\epsilon \downarrow 0$ we get that $\limsup \mathbb{P}(Y_n \in F) \leq \mathbb{P}(X \in F)$ which proves the statement by again using the Portmanteau theorem.