UPPSALA UNIVERSITET
Matematiska institutionen
Erik Broman

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PhD course in Probability, Home Assignment 2: Simple Random Walks, Percolation and Fractal Percolation

This home assignment consists of 5 exercises. The deadline is on April 22 2015. On this day the solutions will be posted online, and anything handed in later will give 0 points.

1. SRW: Consider a random walk $(S_n)_{n\geq 1}$ on $\mathbb{Z}^d$, for which the step size X has distribution
$$\mathbb{P}(X = e_j) = \frac{p}{d},\ \mathbb{P}(X = -e_j) = \frac{1-p}{d}$$
where $p \in [0, 1]$ (when $p = 1/2$, this is what we referred to as a simple symmetric random walk). Prove that for any $p \neq 1/2$ and any $d \geq 1$, the random walk $(S_n)_{n\geq 1}$ is transient. Do this by using the combinatorial approach that we used to prove that ssrw is recurrent when $d = 1, 2$.

   Solution: As in demonstrated in class, we only need to consider the walk in $d = 1$ as transience there implies transience for $d \geq 2$.

   We have that
$$\mathbb{P}(S_n = 0) = \binom{n}{n/2} p^{n/2}(1-p)^{n/2} I(n \text{ is even}),$$
or
$$\mathbb{P}(S_{2n} = 0) = \binom{2n}{n} p^n(1-p)^n,$$
while $\mathbb{P}(S_{2n+1} = 0) = 0$ for any n. Furthermore, we have that
$$\binom{2n}{n} \leq \sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n},$$
so that
$$\mathbb{P}(S_{2n} = 0) \leq (4p(1-p))^n.$$
For any $p \neq 1/2$, we get that
$$\sum_{n=0}^{\infty} \mathbb{P}(S_n = 0) \leq \sum_{n=0}^{\infty} (4p(1-p))^n < \infty.$$
Non-recurrence follows by BC1.

2. SRW: Consider two random walks $(S_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ on $\mathbb{Z}^d$, for which the step size X has distribution
$$\mathbb{P}(X = e_j) = \frac{1}{2d},\ \mathbb{P}(X = -e_j) = \frac{1}{2d}.$$
Assume further that the two random walks are independent, and that $S_0 = 0$ and $T_0 = x$ for some $x \in \mathbb{Z}^d$. We say that $(S_n)_{n\geq 1}$ and $(T_n)_{n\geq 1}$ meets, if there exists $n \geq 1$ such that $S_n = T_n$. Prove the following

(a) If $||x|| := |x_1| + \cdots + |x_d|$ is odd, then for every $d \geq 1$, $\mathbb{P}(S_n = T_n$ for some $n \geq 1) = 0$.

(b) If $||x||$ is even, then for every $d \geq 1$, $\mathbb{P}(S_n = T_n$ for some $n \geq 1) > 0$. When is this probability 1?

Solution:

(a) The trick is to consider $W_n = S_n - T_n$ and note that $S_n = T_n$ iff $W_n = 0$. If $S_n = \sum_{k=1}^n X_k$ and $T_n = \sum_{k=1}^n Y_k$ (using obvious notation), then we have that $||W_n - W_{n-1}|| = ||X_n - Y_n|| \in \{0, 2\}$. Thus, if $||W_0|| = ||x||$ is odd, then $||W_n||$ is odd for every $n \geq 1$, and therefore will never hit 0.

(b) The fact that $\mathbb{P}(S_n = T_n$ for some $n \geq 1) > 0$ when $||x||$ is even is almost trivial given the solution in (a). If one really want to give an argument it might go along the following lines. Assume first that $d = 2$ and $x_1, x_2 > 0$ are both even. With positive probability, $X_1 - Y_1 = \cdots = X_{x_1/2} - Y_{x_1/2} = -2e_1$ and then $X_{x_1/2+1} - Y_{x_1/2+1} = \cdots = X_{(x_1+x_2)/2} - Y_{(x_1+x_2)/2} = -2e_2$ so that $W_{(x_1+x_2)/2} = o$. The other cases and dimensions are handed in the same way.

The issue is to determine when $\mathbb{P}(S_n = T_n$ for some $n \geq 1) = 1$. Define

$$Z_{2n} = \sum_{k=1}^n X_k - Y_k = S_n - T_n, \text{ and } Z_{2n+1} = X_{n+1} + \sum_{k=1}^n X_k - Y_k = S_{n+1} - T_n.$$

Since the random walk is symmetric, we see that $-Y_k$ has the same distribution as Y_k, and therefore $(Z_n)_{n \geq 1}$ is a ssrw. We know that this is recurrent iff $d = 1, 2$ and since

$$W_n = Z_{2n},$$

we see that also $(W_n)_{n \geq 1}$ is recurrent iff $d = 1, 2$. The last conclusion of course relies on the fact that $||x||$ is even.

3. Percolation and Moment methods: We have that for any random variable X taking values in $\{1, 2, \ldots\}$

$$\mathbb{P}(X > 0) \leq \mathbb{E}[X], \tag{1}$$

and by an easy application of Cauchy-Schwarz (you should check this, but its not a part of the assignment)

$$\mathbb{P}(X > 0) \geq \frac{\mathbb{E}[X]^2}{\mathbb{E}[X^2]}. \tag{2}$$

These elementary upper and lower bounds on $\mathbb{P}(X > 0)$ can be surprisingly useful as we shall see in this exercise.

Consider a tree in which the root o has d children, each of which have d children etc. Denote this tree by $\mathbb{T}^d$. The case $d = 2$ is a binary tree which we encountered in the first home assignment. Obviously, $\mathbb{T}^d$ is a deterministic tree, but by performing percolation on it, we create a Galton-Watson tree (which we discussed in the beginning of the course) if we consider the open component of the root. You can choose to do either edge or site percolation, but they are basically equivalent on trees. For any $x, y \in \mathbb{T}^d$, let $\{x \leftrightarrow y\}$ denote the event that there exists an open path from x to y. Consider the event $\{o \leftrightarrow \infty\}$, which is simply the event that there exists an unbounded open path starting from o.

Let

$$Z_n = |\{x \in \mathbb{T}_n^d : o \leftrightarrow x\}|,$$

where $\mathbb{T}_n^d$ is the set of vertices of $\mathbb{T}^d$ at graph distance n from the root. Obviously, $\{o \leftrightarrow \infty\} = \cap_{n=1}^{\infty}\{Z_n > 0\}$.

(a) Prove that $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) = 0$, when $p < 1/d$ by using (1).

(b) Prove that $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) > 0$, when $p > 1/d$ by using (2).

It can be useful to observe that

$$Z_n = \sum_{x\in\mathbb{T}_n^d} I(o \leftrightarrow x),$$

and to think about what $\mathbb{P}(o \leftrightarrow x, o \leftrightarrow y)$ is.

The above is consistent with what we did at the start of the course, but we can do better! Consider a sequence $(a_n)_{n=0}^{\infty}$ such that $a_n \in \{1, 2, \ldots\}$ for every n, and let $A_n := \prod_{k=0}^{n-1} a_k$. Consider the tree T starting with a root, and let the root have a_0 children who in turn have a_1 children etc. Using analogous notation to above,

(c) Prove that $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) = 0$, when $p < 1/\liminf_n A_n^{1/n}$ by using (1).

(d) Prove that $\lim_{n\to\infty} \mathbb{P}(Z_n > 0) > 0$, when $p > 1/\liminf_n A_n^{1/n}$ by using (2).

Remark: The results (c) and (d) are generalizations of the results in (a) and (b). It is good to do (a) and (b) first to find the right idea for how to prove (c) and (d), but only solving (c) and (d) will of course give full credit.

Solution: I will do only (c) and (d).

(c) At level n, the tree has A_n vertices. Therefore, $\mathbb{E}[Z_n] = p^n A_n$. Since $p < 1/\liminf_n A_n^{1/n}$ there exists an $\alpha < 1$ and a subsequence $(n_k)_{k\geq 1}$ such that $pA_{n_k}^{1/n_k} \leq \alpha$ for every $k \geq 1$. We conclude that

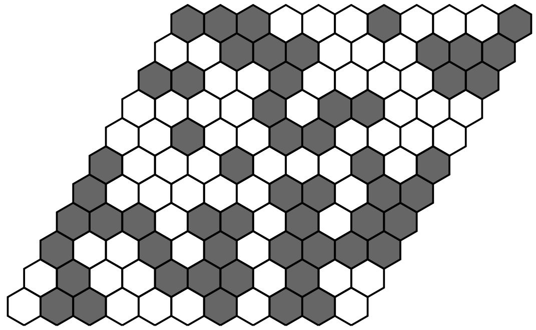
$$\mathbb{P}(Z_{n_k} > 0) \leq \mathbb{E}[Z_{n_k}] = \left(pA_{n_k}^{1/n_k}\right)^{n_k} \leq \alpha^{n_k}.$$

Thus, since $\lim_{n\to\infty} \mathbb{P}(Z_n > 0)$ exists by monotonicity ($\mathbb{P}(Z_n > 0) \geq \mathbb{P}(Z_{n+1} > 0)$), we get that

$$\lim_{n\to\infty} \mathbb{P}(Z_n > 0) = \lim_{k\to\infty} \mathbb{P}(Z_{n_k} > 0) \leq \lim_{k\to\infty} \alpha^{n_k} = 0.$$

(d) We want to bound $\mathbb{E}[Z_n^2]$. Let x, y be two vertices at level n, and let $k_{x,y}$ be the generation of their latest common ancestor. That is, if we consider the two paths from o to x and from o to y, then $k_{x,y}$ is the generation at which these paths split. Thus, $\mathbb{P}(o \leftrightarrow x, o \leftrightarrow y) = p^{2n-k_{x,y}}$. For a given x, the number of y such that $k_{x,y} = k$ is bounded by A_n/A_k. Therefore we get that

$$\mathbb{E}[Z_n^2] \leq \sum_{x\in T_n}\sum_{k=0}^{n} p^{2n-k} A_n/A_k = p^{2n} A_n^2 \sum_{k=0}^{n} \frac{1}{A_k p^k} \leq C\mathbb{E}[Z_n]^2,$$



Figur 1: This is the box B_{11} in the notation from class. We also see a configuration of open (black) and closed (white) vertices in this box. The origin is that white guy in the middle.

where $C < \infty$ can be made independent of n as we now argue. To see this, observe that since $p > 1/\liminf_n A_n^{1/n}$ we have that there exists a $\beta > 1$ and an $N < \infty$ such that $pA_n^{1/n} \geq \beta$ for every $n \geq N$. Thus,

$$C = \sum_{k=0}^{n} \frac{1}{A_k p^k} \leq \sum_{k=0}^{N} \frac{1}{A_k p^k} + \sum_{k=N+1}^{\infty} \frac{1}{\beta^k} < \infty.$$

4. Percolation: Consider percolation on the vertices of the triangular lattice (this is the model that we have studied in class) with density $p < p_c$. Let B_n be the box of side length n centered at the origin (see Figure 1). In order to avoid degenerate situations, we assume that n is odd.

   Let $o \leftrightarrow \partial B_n$ denote the event that there exists an open path from the origin to the boundary of the "box" B_n Furthermore, let $\theta_n(p) := \mathbb{P}(o \leftrightarrow \partial B_n)$.

   (a) Prove that there exists a constant $C < \infty$ such that $\theta_n(1/2) \geq 1/(Cn)$ for every (odd) $n \geq 1$.

   (b) Prove that there exist some constants $\nu > 0$ and $C < \infty$ such that for every (odd) $n \geq 1$, and $p \leq 1/2$,

$$\theta_n(p) \leq Cn^{-\nu}.$$

   Hints: For (a): draw a picture and compare to the crossing events we used in class. For (b): circuits!

   Solution:

   (a) We know that $\mathbb{P}(H^o_{n,n}) = 1/2$. Let $v_1, \dots, v_n$ denote the vertices on the LHS of B_n, and let $H^{o,k}_{n,n}$ denote the event that we can find an open path connecting the vertex v_k to the RHS of B_n. Obviously, $H^{o,k}_{n,n}$ implies that there exists an open path connecting the vertex v_k to the boundary of $v_k + B_{2(n-1)+1}$. Thus

$$1/2 = \mathbb{P}(H^o_{n,n}) = \mathbb{P}(\bigcup_{k=1}^{n} H^{o,k}_{n,n}) \leq \sum_{k=1}^{n} \mathbb{P}(v_k \leftrightarrow (v_k + \partial B_{2n-1})) = n\theta_{2n-1}(1/2).$$

Thus

$$\theta_{2n-1}(1/2) \geq 1/(2n) \geq 1/(2(2n-1)),$$

proving the statement for $C = 2$.

(b) By monotonicity, it suffices to prove this for $p = 1/2$. Consider the annulus $A_k = B_{3\cdot 4^k} \setminus B_{4^k}$, and let $O(A_k)$ denote the event that there exists a closed circuit surrounding the origin, contained in A_k. By RSW+result in class, we know that there exists a constant $c > 0$ such that

$$\mathbb{P}(O(A_k)) \geq c,$$

uniformly in $k \geq 1$. For any n, let k_n be the largest integer such that $3 \cdot 4^k \leq n$, so that $k_n = \lfloor \frac{\log n - \log 3}{\log 4} \rfloor$. We have that

$$\begin{aligned}\theta_n(p) &\leq \mathbb{P}_p(O(A_k) \text{ does not occur for } k = 1, 2, \ldots, k_n) \\ &= \prod_{k=1}^{k_n} (1 - \mathbb{P}_p(O(A_k))) \leq c^{k_n} \leq c^{\log n/\log 4 - 1} = \frac{1}{c} n^{\log c/\log 4},\end{aligned}$$

and so we have proven the statement with $C = 1/c$ and $\nu = -\log c/\log 4$.

5. Fractals: Consider the fractal percolation model discussed in class, and assume for convenience that $d = 2$ and $N = 3$. Recall the notation $\mathcal{C}^1(p) \supset \mathcal{C}^2(p) \supset \cdots$, and as usual define

$$\mathcal{C}(p) := \cap_{k=1}^{n} \mathcal{C}^k(p).$$

(a) Consider the set F of "corners" , i.e.

$$F := \bigcup_{k=1}^{\infty} \bigcup_{l_1=0}^{3^k} \bigcup_{l_2=0}^{3^k} \left\{ \left(\frac{l_1}{3^k}, \frac{l_2}{3^k} \right) \right\}.$$

The set F consists of all corners of any box on any scale in the fractal construction. Prove that

$$\mathbb{P}(F \cap \mathcal{C}(p) \neq \emptyset) = 0.$$

(b) Consider the lattice $G = (\mathbb{Z}^2, \mathbb{E}^2)$ (i.e. not the hexagonal lattice from class). Perform *site* percolation on G, i.e. pick a random configuration $\omega \in \{0,1\}^{\mathbb{Z}^2}$ (in the first lecture on percolation, we considered edge percolation on this graph). Let

$$B_n := \mathbb{Z}^2 \cap [0, n]^2,$$

so that B_n is a box of side length n. We let $H^o(B_n)$ denote the event that there exists a left-right open crossing of B_n. Define

$$p_{c,site} := \sup\{p > 0 : \limsup_{n\to\infty} \mathbb{P}_p(H^o(B_n)) = 0\}.$$

Define also

$$HF(p) := \{\mathcal{C}(p) \text{ contains a L-R crossing of } [0,1]^2\}$$

(H=Horizontal, F=Fractal), so that with notation from class, $\bar{\varphi}(p) = \mathbb{P}(HF(p))$. Let

$$\bar{p}_c := \inf\{p > 0 : \bar{\varphi}(p) > 0\}.$$

Use part (a), i.e. the fact that with probability one, no corners are in the fractal set, to prove that

$$\bar{p}_c \geq p_{c,site}.$$

Hint: Scaling invariance!

Remark: With $\theta_{site}(p)$ defined as in class, but considering site-percolation on $\mathbb{Z}^2$, it is known that

$$p_{c,site} := \inf\{p > 0 : \theta_{site}(p) > 0\}.$$

This is a non-trivial statement, which is not needed for the exercise.

Solution:

(a) Fix a level k-cube and one of its corners x. The probability that $x \in \mathcal{C}^{k+l}(p)$ is bounded by $(1-(1-p)^4)^l$, and by considering further smaller scales we conclude that $\mathbb{P}(x \in \mathcal{C}(p)) = 0$. The result follows since F is clearly countable.

(b) By the same type of scaling invariance that we utilized during class, we get that for any $n \geq 1$,

$$\begin{aligned}\bar{\varphi}(p) &\leq \mathbb{P}(\mathcal{C}(p) \text{ contains a L-R crossing of } [0,3^n]^2) \\ &= \mathbb{P}(\mathcal{C}(p) \setminus F \text{ contains a L-R crossing of } [0,3^n]^2),\end{aligned}$$

where the last equality follows from part (a).
In order for $\mathcal{C}(p) \setminus F$ to contain a L-R crossing of $[0,3^n]^2$, there must be a path of level-2 squares connected by line-segments (not through corners!) and which are all retained in $\mathcal{C}^1(p)$. This is equivalent to looking for a crossing of $\mathbb{Z}^2 \cap [0,3^{n+1}]^2$ in the site percolation model. Thus, for any $p < p_{c,site}$

$$\bar{\varphi}(p) \leq \mathbb{P}_p(H^o(B_{3^{n+1}})) \to 0.$$

This concludes the proof.