# Survival of inhomogeneous Galton-Watson processes

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**Abstract**

We study survival properties of inhomogeneous Galton-Watson processes. We determine the so-called branching number (which is the reciprocal of the critical value for percolation) for these random trees (conditioned on being infinite), which turns out to be an a.s. constant. We also shed some light on the way the survival probability varies between the generations. When we perform independent percolation on the family tree of an inhomogeneous Galton-Watson process, the result is essentially a family of inhomogeneous Galton-Watson processes, parametrized by the retention probability $p$. We provide growth rates, uniformly in $p$, of the percolation clusters, and also show uniform convergence of the survival probability from the $n$-th level along subsequences. These results also establish, as a corollary, the supercritical continuity of the percolation function. Some of our results are generalisations of results by Lyons (1992).

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## 1 Introduction and main results

We start by defining the main object of study in this paper, namely inhomogeneous Galton-Watson processes. Start with a root $o$ and let $L_1$ be the

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distribution of the (random) number of offspring of the root. Proceed by letting each child (if any) of the root have an i.i.d. number of offspring with distribution $L_2$, and also let these offsprings be independent of the number of children of the root. Given a sequence $\{L_n\}_{n=1}^{\infty}$, we let $L_n$ be the offspring distribution of every individual of generation $n-1$. Sometimes we will treat $L_n$ as a random variable rather than as a distribution, this is standard abuse of notation. The root is considered to be generation 0. Observe that if the distributions $\{L_n\}_{n=1}^{\infty}$ all are the same we get a regular Galton-Watson process. Observe also that if $\mathbb{P}(L_n = l_n) = 1$ for every $n$ and some sequence of numbers $\{l_n\}_{n=1}^{\infty}$, then we a.s. get a (deterministic) spherically symmetric tree, that is, a rooted tree in which any two vertices in the same generation have the same degree.

We denote the random family tree of such an inhomogeneous Galton-Watson process by $T$. We will let $\overline{T}$ be a tree with distribution equal to $T$ conditioned on survival, and we will also let $I \subset \overline{T}$ be the tree that consists of those vertices $x \in \overline{T}$ (and the edges between them) that have infinitely many descendents in $\overline{T}$. We will denote by $T_n$, $\overline{T}_n$ and $I_n$ the number of points in the $n$th generation of $T$, $\overline{T}$ and $I$ respectively.

It is well known (see e.g. [7]) and not hard to see that $I$ is itself the family tree of an inhomogeneous Galton-Watson process; we will use this fact later on.

For inhomogeneous Galton-Watson processes we define the survival probability $\theta_n$ from the $n$th generation, that is,

$$\theta_n := \lim_{m\to\infty} \mathbb{P}(T_m > 0 \big| T_n = 1).$$

For an infinite tree, a *cutset* $\pi$ is defined to be a finite set of edges such that every infinite path starting at the origin must contain at least one edge of the cutset. We denote by $\Pi$ the set of all such cutsets. Any infinite tree $S$ has a so-called *branching number* which is defined as follows.

**Definition 1.1** *The* branching number *of an infinite tree $S$ with root $o$ is denoted by* $\mathrm{br}S$ *and defined by*

$$\mathrm{br}S := \sup\left\{\lambda; \inf_{\pi\in\Pi}\sum_{e\in\pi}\lambda^{-|e|} > 0\right\}.$$

The branching number is a very important property for trees (see [8]). For instance it is known that (see [6]) the critical density $p_c(S)$ for independent percolation (we are assuming that the reader is familiar with the concept of percolation, otherwise please see [4]) on $S$ is the reciprocal of the branching number, that is,

$$p_c(S) = 1/\mathrm{br}S.$$

Closely related to the branching number is the *lower growth number* $\underline{\mathrm{gr}}S$ which is defined by

$$\underline{\mathrm{gr}}S := \liminf_{n\to\infty} S_n^{1/n},$$

where $S_n$ denotes the number of vertices in the $n$th generation of $S$. It is not hard to see that we always have $\mathrm{br}S \leq \underline{\mathrm{gr}}S$, while equality is not always true. It is however well-known that if $S$ is spherically symmetric, then $\mathrm{br}S = \underline{\mathrm{gr}}S$.

We start with the following simple survival criterion. This result is essentially contained in Proposition 4.15 of [7], but we do give a different proof based on even earlier work in [1]. The reason is that some of the elements in the proof will be used again later in this paper.

**Proposition 1.2** *For any inhomogeneous Galton-Watson process with offspring distributions* $\{L_n\}_{n=1}^{\infty}$ *we have that*

$$\liminf_{n\to\infty}(\mathbb{E}[T_n])^{1/n} < 1 \Rightarrow \lim_{n\to\infty} \mathbb{P}(T_n > 0) = 0.$$

*Furthermore, if*

$$\sup_n \mathbb{E}[L_n^2] = C_1 < \infty \tag{1}$$

*and*

$$\inf_n \mathbb{E}[L_n] = C_2 > 0, \tag{2}$$

*then we also have that*

$$\liminf_{n\to\infty}(\mathbb{E}[T_n])^{1/n} > 1 \Rightarrow \lim_{n\to\infty} \mathbb{P}(T_n > 0) > 0.$$

Next we have a result concerning the branching number of $\overline{T}$. A priori this is a random variable, but it turns out that $\mathrm{br}\overline{T}$ is an almost sure constant (under mild conditions).

**Theorem 1.3** *Consider an inhomogeneous Galton-Watson process with offspring distributions* $\{L_n\}_{n=1}^{\infty}$ *satisfying (1) and (2). Assume also that*

$$\liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n} > 1. \tag{3}$$

*Then we have that* $\mathrm{br}\overline{T} = \liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n}$, $[\overline{T}]$*-a.s.*

We make some remarks about this result.

1. In [7], it is proved that a.s.,

$$\mathrm{br}\overline{T} = \liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n}$$

under the assumption that $\sup_n ||L_n||_\infty < \infty$. It is claimed in [7] that this assumption cannot be weakened much; our results show that if one adds the very natural condition (3), then in fact one can significantly weaken the assumptions.

2. Naively one might believe that this result would follow from easy arguments. For instance one might try the following approach: Define a new inhomogeneous Galton-Watson tree $T'$ by performing percolation on $T$ with probability for being open equal to $p$. Depending on whether $p$ was smaller or greater than $1/(\liminf E[T_n]^{1/n})$, we get from Proposition 1.2 that $T'$ dies out a.s./survives with positive probability (respectively), concluding the argument. However one then misses the point that the fact that $T'$ survives with positive probability if $p > 1/(\liminf E[T_n]^{1/n})$ does not lead to the conclusion that $\mathrm{brT} \geq \liminf E[T_n]^{1/n}$. Indeed, it is imaginable that with positive probability $\mathrm{brT} = \liminf E[T_n]^{1/n} - \delta$ for some positive $\delta$ and with positive probability $\mathrm{brT} = \liminf E[T_n]^{1/n}$. If this were true, $T$' would still survive with positive probability for the indicated $p$.

The following result about the behaviour of $\theta_n$ will be needed in the proof of Theorem 1.3 but is also quite interesting in its own right. It is not to be expected that $\theta_n$ is in general bounded away from 0 since one can always insert any finite number of generations of degree one in the tree. However, it is the case that there is a subsequence along which $\theta_n$ is bounded away from 0.

**Proposition 1.4** *Consider an inhomogeneous Galton-Watson process with offspring distributions $\{L_n\}_{n=1}^\infty$ satisfying (1), (2) and (3). Then there exists a sequence $\{n_k\}_{k=1}^\infty$ of increasing integers and a constant $C > 0$ such that for all $k \geq 1$,*

$$\theta_{n_k} \geq C.$$

Next, we study bond percolation on $I$. Note that $p_c(I) = p_c(\overline{T})$, since pruning a tree does not change its critical probability. We already noted that

$I$ itself is the family tree of an inhomogeneous Galton-Watson process, and when we perform independent bond percolation on $I$, the resulting component of the origin, to be denoted by $I^p$, also constitutes a family tree of an inhomogeneous Galton-Watson process. Therefore, general results about inhomogeneous Galton-Watson processes automatically apply to percolation on $I$. However, being equipped with a parameter $p$ now, we will derive survival estimates *uniformly* in $p$. We remark that a special case of inhomogeneous Galton-Watson processes results from starting with a deterministic spherically symmetric tree and performing percolation on that tree. One more piece of notation: the number of vertices in $I^p$ at distance $n$ from the root is denoted by $I^p_n$. Also, in this paper, we use various coupling constructions. To facilitate this, all processes, for all values of $p$, are jointly constructed in the obvious way. Consequently, as in the previous example, we will express the $p$-dependence in the events rather than in the measure.

In light of Theorem 1.3, one might expect that for any $\epsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}(0 < I^p_n < ((1-\epsilon)\mathrm{br}\overline{I^p})^n) = 0.$$

In fact, we have the next, much stronger statement.

**Theorem 1.5** *Consider an inhomogeneous Galton-Watson process satisfying (1), (2) and (3), with family tree $T$, and let $\epsilon > 0$. For $p_c(\overline{T}) < p_1 \leq 1$ it is the case that*

$$\lim_{n\to\infty} \mathbb{P}(0 < I^p_n < ((1-\epsilon)\mathrm{br}\overline{I^{p_1}})^n) = 0,$$

*uniformly in $p \in [p_1, 1]$.*

Note that the pointwise (in $p$) convergence in Theorem 1.5 is almost a triviality. The whole point of the theorem is proving the uniform convergence.

Theorem 1.5 combined with Proposition 1.4 will in turn lead us to our next result. Here we define

$$\theta(p) := \mathbb{P}(|I^p| = \infty).$$

**Proposition 1.6** *Consider an inhomogeneous Galton-Watson process satisfying (1), (2) and (3), and let $p_1 > p_c(\overline{T})$. Then there exists a sequence of increasing integers $\{n_k\}_{k=1}^{\infty}$ such that*

$$\theta(p) = \lim_{k\to\infty} \mathbb{P}(I^p_{n_k} > 0),$$

*uniformly on $[p_1, 1]$.*

This result also leads to continuity of the percolation function above $p_c$ for random trees.

**Corollary 1.7** *Consider an inhomogeneous Galton-Watson process satisfying (1), (2) and (3). Then the percolation function $\theta(p)$ is continuous above $p_c(\overline{T})$. In particular, on any spherically symmetric tree $S$ with uniformly bounded degrees, the percolation function is continuous above $p_c(S)$.*

In fact, one can also use Theorem 1.3 to construct a more or less classical proof of this result. As an interesting side remark, we mention that the route via Proposition 1.6 also has a counterpart on $\mathbb{Z}^d$, and gives a new proof for the continuity of the percolation function in that context. This proof does in fact give a rate of convergence for the natural approximations of the percolation function; we discuss these continuity matters in the last section.

In contrast to our last corollary, we have the following example of a tree for which the percolation function is not continuous above $p_c$. To construct such a tree, we use a result in [7], a special case of which says that there is percolation with positive probability on a spherically symmetric tree $S$ with parameter $p$, if and only if

$$\sum_{n=1}^{\infty} \frac{p^{-n}}{S_n} < \infty.$$

To construct an example, we first take a spherically symmetric tree $S$ which is such that $S_n$ is of the order $2^n n^2$. It follows from the above that $p_c(S) = 1/2$ and that $\theta_S(1/2) > 0$. Next, we take a regular tree $S'$ with common degree 4. It is well-known that $p_c(S') = 1/3$. We then construct a tree $S''$ by joining the roots of $S$ and $S'$ by a single edge. It is easy to see that $p_c(S'') = 1/3$ and that $\theta_{S''}$ is discontinuous at $1/2$.

Theorem 1.3 along with Propositions 1.2 and 1.4 will be proved in the next section. All the other results are proved in Section 3. The issues about continuity of the percolation function are discussed in Section 4.

## 2 Proof of Proposition 1.2, Theorem 1.3 and Proposition 1.4

We start by defining a useful probability generating function by

$$h(n,s) := \sum_{j=0}^{\infty} \mathbb{P}(L_n = j)s^j, \ \forall n \geq 1.$$

It is known (see [1]) that if $h''(n,1) < \infty$, for every $n$, then for all $n \geq 1$ we have

$$\left[ \mathbb{E}[T_n]^{-1} + \sum_{j=1}^{n} \frac{h''(j,1)}{h'(j,1)} \mathbb{E}[T_j]^{-1} \right]^{-1} \leq \mathbb{P}(T_n > 0). \tag{4}$$

Of course we have

$$h'(n,1) = \sum_{j=0}^{\infty} j\mathbb{P}(L_n = j) = \mathbb{E}[L_n],$$

and

$$h''(n,1) = \sum_{j=0}^{\infty} j(j-1)\mathbb{P}(L_n = j) = \mathbb{E}[L_n^2] - \mathbb{E}[L_n].$$

We can now proceed with the proof of Proposition 1.2.

**Proof of Proposition 1.2.** The proof of the first statement is easy. Assume that

$$\liminf_{n\to\infty} (\mathbb{E}[T_n])^{1/n} = a < 1,$$

then we get that for any $\epsilon > 0$ such that $a(1+\epsilon) < 1$, there exists a sequence $\{n_k\}_{k=1}^{\infty}$ such that

$$\mathbb{P}(T_{n_k} > 0) \leq \mathbb{E}[T_{n_k}] \leq (a(1+\epsilon))^{n_k},$$

so that

$$\lim_{n\to\infty} \mathbb{P}(T_n > 0) = 0.$$

For the second statement we start by observing that condition (1) gives us that $h''(n,1) = \mathbb{E}[L_n^2] - \mathbb{E}[L_n] < \infty$ for every $n$. Of course this does not require the full statement of equation (1) which will be needed later. In turn, this gives us that inequality (4) is valid for every $n$, and therefore we need to show that

$$\limsup_{n\to\infty}\left[\mathbb{E}[T_n]^{-1}+\sum_{j=1}^{n}\frac{h''(j,1)}{h'(j,1)}\mathbb{E}[T_j]^{-1}\right]^{-1} \tag{5}$$

$$=\limsup_{n\to\infty}\left[\mathbb{E}[T_n]^{-1}+\sum_{j=1}^{n}\frac{\mathbb{E}[L_j^2]-\mathbb{E}[L_j]}{\mathbb{E}[L_j]}\mathbb{E}[T_j]^{-1}\right]^{-1}>0.$$

To this end, we observe that by equations (1) and (2)

$$\sup_j \frac{\mathbb{E}[L_j^2]-\mathbb{E}[L_j]}{\mathbb{E}[L_j]}\le\frac{C_1}{C_2}=C<\infty.$$

Since $\liminf_{n\to\infty}(\mathbb{E}[T_n])^{1/n}>1$ there exists a constant $b>1$ and an $N$ such that for all $n\ge N$,

$$\mathbb{E}[T_n]>b^n.$$

Therefore, for some constant $D<\infty$,

$$\mathbb{E}[T_n]^{-1}+\sum_{j=1}^{n}\frac{h''(j,1)}{h'(j,1)}\mathbb{E}[T_j]^{-1}$$

$$\le\mathbb{E}[T_n]^{-1}+C\sum_{j=1}^{n}\mathbb{E}[T_j]^{-1}\le D+C\sum_{j=N}^{\infty}b^{-j}<\infty.$$

Since the right hand side of the above inequality is independent of $n$, inequality (5) is valid and that concludes the proof. □

We continue by proving Proposition 1.4.

**Proof of Proposition 1.4.** Let $\{X_i\}_{i\ge1}$ be i.i.d. with distribution according to $T_n$ conditioned on the event that $T_\ell=1$. Observe that for $n\ge\ell$

$$T_n=\sum_{k=1}^{T_\ell}X_k,$$

so that (using Wald's lemma)

$$\mathbb{E}[X_1]=\mathbb{E}[T_n\big|T_\ell=1]=\frac{\mathbb{E}[T_n]}{\mathbb{E}[T_\ell]}. \tag{6}$$

Observe that by condition (1) we can use inequality (4) to conclude that for $n \geq \ell$,

$$\left[\mathbb{E}[T_n \Big| T_\ell = 1]^{-1} + \sum_{j=\ell+1}^{n} \frac{h''(j,1)}{h'(j,1)} \mathbb{E}[T_j \Big| T_\ell = 1]^{-1}\right]^{-1} \leq \mathbb{P}(T_n > 0 \Big| T_\ell = 1). \quad (7)$$

We will show that there exists a sequence $\{n_k\}_{k=1}^{\infty}$ of increasing integers and a constant $C < \infty$ such that for all $k \geq 1$ and for all $n \geq n_k$,

$$\mathbb{E}[T_n \Big| T_{n_k} = 1]^{-1} + \sum_{j=n_k+1}^{n} \frac{h''(j,1)}{h'(j,1)} \mathbb{E}[T_j \Big| T_{n_k} = 1]^{-1} \leq C. \quad (8)$$

This will give us that for all $k \geq 1$ we have

$$\lim_{n \to \infty} \mathbb{P}(T_n > 0 \Big| T_{n_k} = 1) \geq \frac{1}{C},$$

proving the lemma. To that end, observe that as in the proof of Lemma 1.2 there exists a constant $C_3$ such that for $n \geq \ell$,

$$\begin{aligned} &\mathbb{E}[T_n \Big| T_\ell = 1]^{-1} + \sum_{j=\ell+1}^{n} \frac{h''(j,1)}{h'(j,1)} \mathbb{E}[T_j \Big| T_\ell = 1]^{-1} \quad (9) \\ &\leq \mathbb{E}[T_n \Big| T_\ell = 1]^{-1} + C_3 \sum_{j=\ell+1}^{n} \mathbb{E}[T_j \Big| T_\ell = 1]^{-1} \\ &\leq (C_3 + 1) \sum_{j=\ell+1}^{n} \mathbb{E}[T_j \Big| T_\ell = 1]^{-1} = (C_3 + 1)\mathbb{E}[T_\ell] \sum_{j=\ell+1}^{n} \frac{1}{\mathbb{E}[T_j]}, \end{aligned}$$

where we use equation (6) in the last equality. Therefore, showing that there exists a sequence $\{n_k\}_{k=1}^{\infty}$ of increasing integers and a constant $C < \infty$ such that for all $k$ we have

$$\mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} \leq C$$

will give us equation (8).

We divide the proof into three cases. First however, define

$$m := \liminf_{n \to \infty} \mathbb{E}[T_n]^{1/n} > 1.$$

In the first case, we have that $\mathbb{E}[T_n]^{1/n} < m$ for infinitely many $n$. We can then conclude that there exists $n_1$, defined to be the largest integer such that $\mathbb{E}[T_{n_1}]^{1/n_1} = \min_{n\geq 1} \mathbb{E}[T_n]^{1/n}$. Having defined $n_k$, we can then define $n_{k+1}$ to be the largest integer greater than $n_k$ such that $\mathbb{E}[T_{n_{k+1}}]^{1/n_{k+1}} = \min_{n>n_k} \mathbb{E}[T_n]^{1/n}$. Let $\epsilon_k$ be defined through $\mathbb{E}[T_{n_k}]^{1/n_k} = m(1-\epsilon_k)$. Observe that by definition of $n_k$, $\mathbb{E}[T_n]^{1/n} \geq m(1-\epsilon_k)$ for every $n \geq n_k$ and also that $\epsilon_k > 0$ for every $k$, and finally that $\epsilon_k \to 0$, as $k \to \infty$. Therefore,

$$\begin{aligned}\mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} &\leq \mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{(m(1-\epsilon_k))^j} \\ &= (m(1-\epsilon_k))^{n_k} \sum_{j=1}^{\infty} \frac{1}{(m(1-\epsilon_k))^{n_k+j}} = \sum_{j=1}^{\infty} \frac{1}{(m(1-\epsilon_k))^j}.\end{aligned}$$

There exists a $K$ such that $m(1-\epsilon_k) > 1$ for every $k \geq K$. For $k \geq K$, the right hand side of the above equation is then bounded by some constant $D_k < \infty$. Furthermore, we can take $D_k \geq D_{k+1}$ and conclude that for all $k \geq K$,

$$\mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} \leq D_K < \infty.$$

For the second and third case, we have that $\mathbb{E}[T_n]^{1/n} < m$ for only finitely many $n$. We can therefore find $N$ large enough so that $\mathbb{E}[T_n]^{1/n} \geq m$ for every $n \geq N$. We have that for every $n$, $\mathbb{E}[T_n]^{1/n} = m(1+a(n))$, where the sequence of numbers $\{a(n)\}_{n=1}^{\infty}$ is such that $a(n) \geq 0$ for every $n \geq N$.

The second case is if $\liminf_{n\to\infty}(1+a(n))^n = C_4$ for some constant $C_4 < \infty$. Then there exists a sequence of strictly increasing integers $\{n_k\}_{k=1}^{\infty}$ such that $(1+a(n_k))^{n_k} \leq 2C_4$ for every $k \geq 1$. By also requiring that $n_1 \geq N$, we get that

$$\begin{aligned}\mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} &\leq \mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{m^j} \\ &= m^{n_k}(1+a(n_k))^{n_k} \sum_{j=1}^{\infty} \frac{1}{m^{n_k+j}} \leq 2C_4 \sum_{j=1}^{\infty} \frac{1}{m^j} < \infty.\end{aligned}$$

The third case is if $\lim_{n\to\infty}(1+a(n))^n = \infty$. We can then find a sequence $\{n_k\}_{k=1}^{\infty}$ (much as in the first case) such that for every $k \geq 1$, $(1+a(n))^n \geq$

$(1+a(n_k))^{n_k}$ for every $n \geq n_k$. By again requiring that $n_1 \geq N$, we get that

$$\mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} \leq \mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{(m(1+a(n_k)))^j}$$
$$= \frac{(m(1+a(n_k)))^{n_k}}{(m(1+a(n_k)))^{n_k}} \sum_{j=1}^{\infty} \frac{1}{(m(1+a(n_k)))^j} \leq \sum_{j=1}^{\infty} \frac{1}{m^j} < \infty.$$

We can therefore conclude that there exists a constant $C = C(\{L_n\}_{n=1}^{\infty}) < \infty$ and a sequence of strictly increasing integers $\{n_k\}_{k=1}^{\infty}$ such that for all $k \geq 1$,

$$\mathbb{E}[T_{n_k}] \sum_{j=n_k+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} \leq C.$$

This concludes the proof. $\square$

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** Using that $p_c(I)^{-1} = p_c(\overline{T})^{-1} = \mathrm{br}\overline{\mathrm{T}}$, we need to show that

$$p_c(I)^{-1} = \liminf_{n\to\infty} (\mathbb{E}[T_n])^{1/n}.$$

We will do this by first proving that $p_c(I)^{-1} = \liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$ and then proving that

$$\liminf_{n\to\infty} (\theta_n \mathbb{E}[T_n])^{1/n} = \liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n}.$$

Consider the offspring distribution $L_1'$ of the root of $I$. Let $T^i$ be the tree consisting of child number $i \in \{1, \ldots, L_1\}$ of the root of $T$ and all the descendents of this child. Define also $N_{1,\infty} = |\{T^i : |T^i| = \infty, i = 1, \ldots, L_1\}|$. It is not hard to see that for $k \geq 1$,

$$\mathbb{P}(L_1' = k) = \mathbb{P}(N_{1,\infty} = k \Big| N_{1,\infty} \geq 1) = \frac{\mathbb{P}(N_{1,\infty} = k)}{\theta}.$$

Furthermore, letting $Y_i$ be i.i.d. $\mathrm{Bin}(1, \theta_1)$ random variables and using Wald's lemma we get that

$$\mathbb{E}[L_1'] = \frac{1}{\theta}\mathbb{E}[N_{1,\infty}] = \frac{1}{\theta}\mathbb{E}\left[\sum_{i=1}^{L_1} Y_i\right] = \frac{1}{\theta}\mathbb{E}[Y_1]\,\mathbb{E}[L_1] = \frac{\theta_1}{\theta}\mathbb{E}[L_1].$$

Furthermore, this argument holds for any generation $n$ and therefore we have for all $n \geq 1$,

$$\mathbb{E}[L'_n] = \frac{\theta_n}{\theta_{n-1}} \mathbb{E}[L_n]. \tag{10}$$

Now, perform independent percolation on $I$ with parameter $p$, thus creating a random graph that we denote by $\mathcal{I}^p$. Recall that $I^p$ is the component of the root of this graph. Obviously, $I^p$ is the family tree of an inhomogeneous Galton-Watson process with some offspring distributions $\{L''_n\}_{n=1}^{\infty}$. Furthermore, trivially

$$\mathbb{E}[L''_n] = p\mathbb{E}[L'_n] = p\frac{\theta_n}{\theta_{n-1}} \mathbb{E}[L_n] \quad \forall n \geq 1.$$

Recall that $I^p_n$ is the number of vertices in $I^p$ at distance $n$ from the root and recall that we defined $I_n$ similarly. We have, using a standard result from the theory of branching processes and (10), that

$$\mathbb{E}[I^p_n] = p^n \mathbb{E}[I_n] = p^n \prod_{i=1}^{n} \mathbb{E}[L'_i] = p^n \prod_{i=1}^{n} \frac{\theta_i}{\theta_{i-1}} \mathbb{E}[L_i] = p^n \frac{\theta_n}{\theta} \mathbb{E}[T_n]. \tag{11}$$

Therefore,

$$\liminf_{n\to\infty} (\mathbb{E}[I^p_n])^{1/n} = p \liminf_{n\to\infty} (\frac{\theta_n}{\theta} \mathbb{E}[T_n])^{1/n} = p \liminf_{n\to\infty} (\theta_n \mathbb{E}[T_n])^{1/n}. \tag{12}$$

We would like to use Proposition 1.2 and Proposition 1.4 on $I^p$. However, before we can do that we need to show that the offspring distributions $\{L''_n\}_{n=1}^{\infty}$ satisfies conditions (1) and (2). When we use Proposition 1.4 we will assume that condition (3) is satisfied; the details will become clear.

For some vertex $x$ in generation $n-1$, let $T^i_x$ be the tree consisting of child number $i \in \{1, \ldots, L_n\}$ of $x$ and all the descendents of this child. Define $N_{n,\infty} = |\{T^i_x : |T^i_x| = \infty, i = 1, \ldots, L_n\}|$, and observe that the distribution of this random variable is trivially independent of the specific choice of $x$ in generation $n-1$. Let $Y^n_i$ be i.i.d. $\mathrm{Bin}(1, \theta_n)$ and observe that

$$\begin{aligned} \mathbb{E}[(L''_n)^2] &\leq \mathbb{E}[(L'_n)^2] = \sum_{j=1}^{\infty} j^2 \mathbb{P}(L'_n = j) \\ &= \sum_{j=1}^{\infty} j^2 \mathbb{P}(N_{n,\infty} = j \Big| N_{n,\infty} \geq 1) \end{aligned}$$

$$
\begin{aligned}
&= \frac{\mathbb{E}[N_{n,\infty}^2]}{\theta_{n-1}} = \frac{\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{L_n} Y_i^n\right)^2 \Big| L_n\right]\right]}{\theta_{n-1}} \\
&\leq \frac{\mathbb{E}\left[\mathbb{E}\left[L_n \sum_{i=1}^{L_n} (Y_i^n)^2 \Big| L_n\right]\right]}{\theta_{n-1}} \\
&= \frac{\mathbb{E}\left[L_n \sum_{i=1}^{L_n} \mathbb{E}\left[Y_i^n \Big| L_n\right]\right]}{\theta_{n-1}} = \frac{\mathbb{E}\left[L_n^2 \theta_n\right]}{\theta_{n-1}}.
\end{aligned}
$$

In the second inequality we use that for any real numbers $a_1, \ldots, a_n$ we have that $(a_1 + \cdots + a_n)^2 \leq n(a_1^2 + \cdots + a_n^2)$. Obviously we must also have that

$$
\theta_{n-1} \geq \mathbb{P}(L_n > 0)\theta_n,
$$

and we can use Cauchy-Schwarz to see that

$$
\mathbb{E}[L_n]^2 = \mathbb{E}[L_n I_{\{L_n > 0\}}]^2 \leq \mathbb{P}(L_n > 0)\mathbb{E}[L_n^2].
$$

Therefore,

$$
\frac{\mathbb{E}\left[L_n^2 \theta_n\right]}{\theta_{n-1}} \leq \frac{\mathbb{E}\left[L_n^2\right]}{\mathbb{P}(L_n > 0)} \leq \mathbb{E}\left[L_n^2\right] \frac{\mathbb{E}\left[L_n^2\right]}{\mathbb{E}\left[L_n\right]^2} \leq \frac{C_1^2}{C_2^2} < \infty.
$$

Furthermore

$$
\inf_n \mathbb{E}[L_n''] = p \inf_n \mathbb{E}[L_n'] \geq p,
$$

since $\mathbb{E}[L_n'] \geq 1$ for every $n$.

We can now proceed to use Proposition 1.2 with equation (12) to see that $I^p$ survives with positive probability if $p > (\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n})^{-1}$ while it dies out a.s. if $p < (\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n})^{-1}$.

This is not quite enough for our purposes: it could be the case that with positive probability, $I$ is such that $I^p$ a.s. dies out. Since we want to make a statememt about almost all trees $I$, we argue that in fact, if $p > 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$, then $\mathcal{I}^p$ contains an infinite component with probability 1 as our next argument shows.

Assume therefore that $\liminf_{n\to\infty}(\mathbb{E}[I_n^p])^{1/n} = p \liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n} > 1$. This is condition (3) for $I^p$. Construct the tree $I^p$ by letting $I_1^p$ have distribution equal to $L_1''$. Proceed by letting $I_2^p$ be the sum $\sum_{i=1}^{I_1^p} L_{2,i}''$, where $\{L_{2,i}''\}_{i=1}^\infty$ are i.i.d. with distribution equal to $L_2''$ and let them also be independent of everything else. Continuing in this fashion, we have two possibilities.

First we may find that $I_n^p > 0$ for every $n$. Second, we might instead find that for some $n$, we have $I_n^p = 0$. If this is the case, there exists some integer $n_{k_1} > n$ in the subsequence dictated by Proposition 1.4. However, since $I$ is infinite we must have that $\mathcal{I}^p$ contains a subtree (possibly consisting of only one vertex ) with the root being some vertex at level $n_{k_1}$. Construct this subtree in the same way as we constructed $I^p$ above. This subtree has some probability to survive which is by Proposition 1.4 uniformly bounded away from 0. It is also easy to see that the event of survival of this subtree is conditionally independent of the part of $\mathcal{I}^p$ examined so far (up to generation $n$).

If again we find that this subtree is finite, we continue in the same way. Since all the subtrees that we pick have uniformly positive probability to survive by Proposition 1.4 and the survival of them are conditionally independent we see that $\mathcal{I}^p$ must contain an infinite component with probability 1. We therefore conclude that

$$\mathbb{P}(\mathcal{I}^p \text{ has an infinite component}) = \begin{cases} 1, & p > 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}, \\ 0, & p < 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}. \end{cases}$$

This is the same as saying that for almost every $I$, we will after performing percolation with parameter $p$ on $I$ a.s. get an infinite component if $p > 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$ while we will a.s. not get an infinite component if $p < 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$. It follows that for almost every $I$ the probability that the component of the root is infinite is positive if $p > 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$ while it is 0 if $p < 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$. This gives us that $p_c(I) = 1/\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$ from which it follows that $\text{br}I = \liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n}$ (recall that $p_c(I) = 1/\text{br}I$).

We now proceed with the final step in proving that

$$\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n} = \liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n}.$$

Obviously, $\theta_n \mathbb{E}[T_n] \leq \mathbb{E}[T_n]$ for every $n$, so we only need to show that

$$\liminf_{n\to\infty}(\theta_n \mathbb{E}[T_n])^{1/n} \geq \liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n}.$$

As before, let $m = \liminf_{n\to\infty} \mathbb{E}[T_n]^{1/n} > 1$ and choose $\epsilon > 0$, so that $m(1-\epsilon) > 1$. Furthermore, we can choose an $N$ such that $\mathbb{E}[T_n]^{1/n} \geq m(1-\epsilon)$ for every $n \geq N$. Using inequalities (7) and (9) we get that for some constant $C$

and $m \geq n$,

$$\mathbb{P}(T_m > 0 \Big| T_n = 1) \geq \left[ C\mathbb{E}[T_n] \sum_{j=n+1}^{m} \frac{1}{\mathbb{E}[T_j]} \right]^{-1} \geq \left[ C\mathbb{E}[T_n] \sum_{j=n+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} \right]^{-1}.$$

Therefore, for $n \geq N$,

$$\begin{aligned}
\theta_n \mathbb{E}[T_n] &= \lim_{m\to\infty} \mathbb{P}(T_m > 0 \Big| T_n = 1)\mathbb{E}[T_n] \\
&\geq \left[ C \sum_{j=n+1}^{\infty} \frac{1}{\mathbb{E}[T_j]} \right]^{-1} \geq \left[ C \sum_{j=n+1}^{\infty} \frac{1}{(m(1-\epsilon))^j} \right]^{-1} \\
&= \left[ \frac{C}{(m(1-\epsilon))^n} \sum_{j=1}^{\infty} \frac{1}{(m(1-\epsilon))^j} \right]^{-1} = (m(1-\epsilon))^n C',
\end{aligned}$$

where $C' > 0$. Therefore, for all $n \geq N$ we have

$$(\theta_n \mathbb{E}[T_n])^{1/n} \geq m(1-\epsilon) C'^{1/n},$$

so that

$$\liminf_{n\to\infty} (\theta_n \mathbb{E}[T_n])^{1/n} \geq m(1-\epsilon).$$

Since $\epsilon > 0$ can be choosen arbitrarily small, we are done.

□

**Remark** In fact, the proof of Theorem 1.3 shows that if the family tree $T$ of a Galton-Watson process satisfies (1), (2) and (3), and $p > p_c(\overline{T})$ , then so does the family tree associated with the $I^p$ process.

# 3 Proof of Theorem 1.5 and Proposition 1.6

Before we can prove Theorem 1.5, we need the following domination lemmas. The first one appears (without proof) in [3]. The proof we give is due to Olle Häggström (unpublished).

**Lemma 3.1** *For $k \geq 1$, $p \in (0,1)$ and $0 \leq m \leq k$, write $\rho_{k,p,m}$ for the distribution of a Binomial$(k,p)$ random variable conditioned on taking value at least $m$. For $p_1 \leq p_2$, we have*

$$\rho_{k,p_1,m} \preceq \rho_{k,p_2,m} ,$$

*where $\preceq$ denotes stochastic domination.*

**Proof.** For $i = 1, 2$, let $Y_i$ be a $\mathrm{Bin}(k, p_i)$ random variable, and let $X_i$ be a random variable with distribution $\rho_{k,p_i,m}$. Since $x/(1-x) < y/(1-y)$ for $0 < x < y < 1$, it is enough to show that for any $n \in \{m+1, \ldots, k\}$ we have

$$\frac{\mathbb{P}(X_1 \geq n)}{\mathbb{P}(X_1 < n)} \leq \frac{\mathbb{P}(X_2 \geq n)}{\mathbb{P}(X_2 < n)},$$

which is the same as showing that

$$\frac{\mathbb{P}(X_2 \geq n)}{\mathbb{P}(X_1 \geq n)} \cdot \frac{\mathbb{P}(X_1 < n)}{\mathbb{P}(X_2 < n)} \geq 1\,. \tag{13}$$

Writing $Z_1$ and $Z_2$ for the probabilities that $Y_1 \geq m$ and $Y_2 \geq m$, respectively, the left-hand side of (13) becomes

$$\frac{\frac{1}{Z_2}\sum_{j=n}^{k}\binom{k}{j}p_2^j(1-p_2)^{k-j}}{\frac{1}{Z_1}\sum_{j=n}^{k}\binom{k}{j}p_1^j(1-p_1)^{k-j}} \cdot \frac{\frac{1}{Z_1}\sum_{j=m}^{n-1}\binom{k}{j}p_1^j(1-p_1)^{k-j}}{\frac{1}{Z_2}\sum_{j=m}^{n-1}\binom{k}{j}p_2^j(1-p_2)^{k-j}}\,. \tag{14}$$

Cancelling the $Z_i$'s and introducing the notation $\phi_i = \frac{p_i}{1-p_i}$ for $i = 1, 2$, the expression in (14) may further be rewritten as

$$\begin{aligned}\frac{p_2^n(1-p_2)^{k-n}\sum_{j=n}^{k}\binom{k}{j}\phi_2^{j-n}}{p_1^n(1-p_1)^{k-n}\sum_{j=n}^{k}\binom{k}{j}\phi_1^{j-n}} \cdot \frac{p_1^n(1-p_1)^{k-n}\sum_{j=m}^{n-1}\binom{k}{j}\phi_1^{j-n}}{p_2^n(1-p_2)^{k-n}\sum_{j=m}^{n-1}\binom{k}{j}\phi_2^{j-n}} = \\ = \frac{\sum_{j=n}^{k}\binom{k}{j}\phi_2^{j-n}}{\sum_{j=n}^{k}\binom{k}{j}\phi_1^{j-n}} \cdot \frac{\sum_{j=m}^{n-1}\binom{k}{j}\phi_1^{j-n}}{\sum_{j=m}^{n-1}\binom{k}{j}\phi_2^{j-n}}\,.\end{aligned} \tag{15}$$

Now note that $\phi_1 \leq \phi_2$, so that

$$\sum_{j=n}^{k}\binom{k}{j}\phi_2^{j-n} \geq \sum_{j=n}^{k}\binom{k}{j}\phi_1^{j-n}$$

and

$$\sum_{j=m}^{n-1}\binom{k}{j}\phi_1^{j-n} \geq \sum_{j=m}^{n-1}\binom{k}{j}\phi_2^{j-n}\,.$$

Hence, the expression in (15) is greater than or equal to 1, so (13) is verified and the lemma is established. □

We proceed with the following lemma. We will in fact only use it in the case $m = 1$, but we nevertheless provide a proof of the general statement.

**Lemma 3.2** *In the notation of Lemma 3.1, it is the case that for any* $1 \leq k \leq l$ *and* $0 \leq m \leq k$

$$\rho_{k,p,m} \preceq \rho_{l,p,m},$$

*for all* $0 < p < 1$.

**Proof.** It is obvious that we only need to prove the lemma in the case $l = k{+}1$. Therefore, let $Y_1, \ldots, Y_{k+1}$ and $X_1, \ldots, X_k$ be i.i.d. Bernoulli random variables with expectation $p$ and let $Y = \sum_{i=1}^{k+1} Y_i$ and $X = \sum_{j=1}^{k} X_j$. We need to show that $\mathbb{P}(X \geq n | X \geq m) \leq \mathbb{P}(Y \geq n | Y \geq m)$, for all $n = m, m+1, \ldots, k$. To this end we write

$$\begin{aligned}
\mathbb{P}(Y \geq n | Y \geq m) &= \mathbb{P}(Y \geq n | Y \geq m, Y_{k+1} = 0)\mathbb{P}(Y_{k+1} = 0 | Y \geq m) \\
&\quad + \mathbb{P}(Y \geq n | Y \geq m, Y_{k+1} = 1)\mathbb{P}(Y_{k+1} = 1 | Y \geq m) \\
&= \mathbb{P}(X \geq n | X \geq m)\mathbb{P}(Y_{k+1} = 0 | Y \geq m) \\
&\quad + \mathbb{P}(X \geq n-1 | X \geq m-1)\mathbb{P}(Y_{k+1} = 1 | Y \geq m).
\end{aligned}$$

Therefore, we need to show that for $n > m$,

$$\mathbb{P}(X \geq n-1 | X \geq m-1) \geq \mathbb{P}(X \geq n | X \geq m),$$

or equivalently,

$$\mathbb{P}(X \geq n | X \geq n-1) \leq \mathbb{P}(X \geq m | X \geq m-1).$$

It is easy to see that it suffices to prove this for $m = n-1$, or to simplify notation, to show that

$$\mathbb{P}(X \geq n+1 | X \geq n) \leq \mathbb{P}(X \geq n | X \geq n-1).$$

Since

$$\mathbb{P}(X \geq n+1 | X \geq n) = 1 - \mathbb{P}(X = n | X \geq n),$$

we need to show that

$$\mathbb{P}(X = n-1 | X \geq n-1) \leq \mathbb{P}(X = n | X \geq n).$$

Writing $p_n := \mathbb{P}(X = n)$ we rewrite this as

$$\frac{p_n + \cdots + p_k}{p_{n-1} + \cdots + p_k} \leq \frac{p_n}{p_{n-1}},$$

or equivalently that

$$p_{n-1}(p_{n+1}+\cdots+p_k)\le p_n(p_n+\cdots+p_k). \tag{16}$$

It suffices to show that $p_{n-1}p_{n+j}\le p_np_{n+j-1}$, for $1\le j\le k-n$. This however is easily checked by a straightforward calculation. □

We are now ready to prove Theorem 1.5.

**Proof of Theorem 1.5.** For the purpose of this proof, we introduce a new stochastic process $\tilde{I}_n^p$, indexed by $n=1,2,\dots$ as follows. $\tilde{I}_1^p$ is distributed as the number of points in $I_1^p$. If this number of points is 0 however, we resample according to the same distribution, and repeat this until the total number of offspring is at least 1. If we do *not* resample at this first generation, we define $R_0:=1$; if we do resample, we set $R_0=0$.

In an inductive fashion, having defined $\tilde{I}_n^p$, we consider all points in $\tilde{I}_n^p$ and give each of them a random number of offspring distributed as $L''_{n+1}$, independently of each other. However, if the total number of offspring is 0, we resample *all* offsprings using the same distributions, until the total number of offspring is at least 1. If we do not have to resample, we define $R_n:=1$; if we do resample, we set $R_n=0$. Of course, the distribution of the number of points in $\tilde{I}_n^p$ given $\tilde{I}_{n-1}^p=k$ for some $k\ge1$ is the same as the distribution of the number of points in $I_n^p$ given $I_{n-1}^p=k$ conditioned on being at least one.

We can now write, for any $M$,

$$\begin{aligned}\mathbb{P}(0<I_n^p<M)&=\mathbb{P}(\prod_{i=0}^{n-1}R_i=1,0<\tilde{I}_n^p<M)\\&\le\mathbb{P}(0<\tilde{I}_n^p<M)=\mathbb{P}(\tilde{I}_n^p<M).\end{aligned}\tag{17}$$

Now let $p_c(\overline{T})<p<q$. We claim that

$$\tilde{I}_n^p\preceq\tilde{I}_n^q.$$

To see this, we note that the offspring distributions of $I^p$ can be realised by first drawing from the appropriate $L'_n$, and then keep all points in the offspring with probability $p$, independently of each other. Now the combination of Lemma 3.1 and Lemma 3.2 implies that for $k\le\ell$ and $p\le q$ we have

$$\rho_{k,p,1}\preceq\rho_{\ell,q,1}. \tag{18}$$

Clearly, we can couple $\tilde{I}_1^p$ and $\tilde{I}_1^q$ so that $\tilde{I}_1^p \leq \tilde{I}_1^q$, since we can use the same offspring $L_1'$ for them to get $I_1$ and then the domination follows from Lemma 3.1. Let $\{L_{2,i}'\}_{i=1}^{\tilde{I}_1^q}$ be i.i.d. with distribution equal to $L_2'$ and independent of everything else. We can now get $\tilde{I}_2^p$ by letting it be a $\mathrm{Bin}(\sum_{i=1}^{\tilde{I}_1^p} L_{2,i}', p)$ conditioned on being at least one. Similarly we get $\tilde{I}_2^q$ by letting it be a $\mathrm{Bin}(\sum_{i=1}^{\tilde{I}_1^q} L_{2,i}', q)$ conditioned on being at least one. The fact that we can couple $\tilde{I}_2^q$ and $\tilde{I}_2^p$ so that $\tilde{I}_2^q \leq \tilde{I}_2^p$ now follows from (18). Repeating this procedure at every level gives that

$$\mathbb{P}(\tilde{I}_n^p < M) \leq \mathbb{P}(\tilde{I}_n^{p_1} < M), \tag{19}$$

for all $p > p_1$, and this is where the uniformity in $p$ comes from.

Of course letting $M$ above depend on $n$ does not change the validity of the argument. According to (17) and (19) it therefore suffices to show that

$$\mathbb{P}(\tilde{I}_n^{p_1} < ((1-\epsilon)\mathrm{br}\overline{I^{p_1}})^n) \to 0,$$

as $n \to \infty$. For this, we use Theorem 1.3 and Proposition 1.4. Consider the subsequence $\{n_k\}$ and the constant $C > 0$ dictated by applying Proposition 1.4 to $I^{p_1}$. This is allowed according to the remark following the proof of Theorem 1.3. Since each element in the $n_1$th generation of the $I^{p_1}$ process has a probability at least $C$ to survive, there is at least probability $C > 0$ that no resampling is ever going to be necessary in the $\tilde{I}^{p_1}$ process after time $n_1$. There are now two possibilities. Either, at some point resampling is needed, or no resampling is ever needed after time $n_1$.

In the latter case, we have that $\tilde{I}_n^{p_1}$ is at least as large as the number of points in a surviving copy of an $I^{p_1}$ tree with only one vertex at generation $n_1$. It follows from Theorem 1.3 that this surviving tree has branching number $\mathrm{br}\overline{I^{p_1}}$. Using that the lower growth number is at least as large as the branching number we are done in this case.

On the other hand, if resampling is needed, then we take the first element in the subsequence $\{n_k\}$ after the first resampling, and repeat the reasoning from there. It follows that a.s., $\liminf_{n\to\infty}(\tilde{I}_n^{p_1})^{1/n} \geq \mathrm{br}\overline{I^{p_1}}$, and the proof is complete. □

We can now prove Proposition 1.6

**Proof of Proposition 1.6.** We write

$$\theta(p) = \mathbb{P}(I_n^p > 0) - \mathbb{P}(I_n^p > 0, |I^p| < \infty),$$

recall that $I^p$ denotes the component of the root. We will prove that along a subsequence, the last term tends to zero uniformly in $p \in [p_1, 1]$, where $p_1 > p_c(\overline{T})$, from which the result follows.

Since the $p$-dependence is important now, we write $\theta_n(p)$ for $\theta_n$ in the context of the Galton-Watson process associated with $I^p$. For any $M > 0$ we write, for $p_1 \leq p \leq 1$,

$$\begin{aligned} \mathbb{P}(I_n^p > 0, |I^p| < \infty) &\leq \mathbb{P}(0 < I_n^p < M) + \mathbb{P}(I_n^p \geq M, |I^p| < \infty) \\ &\leq \mathbb{P}(0 < I_n^p < M) + (1 - \theta_n(p))^M \\ &\leq \mathbb{P}(0 < I_n^p < M) + (1 - \theta_n(p_1))^M. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary. We want to apply Proposition 1.4 to $I^p$. According to the remark after the proof of Theorem 1.3, all the assumptions of Proposition 1.4 holds for $I^p$ since $p > p_c(\overline{T})$.

Now let $C$ be the constant in Proposition 1.4 when we apply it to $I^{p_1}$. We choose $M$ so large that $(1 - C)^M < \epsilon/2$. Next choose $n$ in the appropriate subsequence of Proposition 1.4 and at the same time so large that the first term at the right hand side is at most $\epsilon/2$; this is possible according to Theorem 1.5 above. The right hand side is then bounded above by $\epsilon$, uniformly in $p \in [p_1, 1]$. In summary, for any $\epsilon > 0$ we can find $K$ such that

$$\theta(p) \geq \mathbb{P}(I_{n_k}^p > 0) - \epsilon$$

for every $p \in [p_1, 1]$ and every $n_k$ in the subsequence dictated by Proposition 1.4 with $k \geq K$. We see that for all $k \geq K$ and for all $p \in [p_1, 1]$,

$$|\theta(p) - \mathbb{P}(I_{n_k}^p > 0)| \leq \epsilon,$$

which concludes the argument. □

# 4 Continuity of the percolation function

The supercritical continuity of $\theta(p)$ (Corollary 1.7) follows immediately from Proposition 1.6. We point out however that it is possible to obtain the same result by combining Theorem 1.3 with a modified version of the classical argument found in [2]. We provide a sketch.

**Sketch of proof of Corollary 1.7 from Theorem 1.3.** We start by drawing an $I$ from the correct distribution. Associate to every edge $e$ in $I$

an independent $U([0,1])$ random variable, denoted by $U_e$. For $p_c < q < p$, create $\mathcal{I}^q$ and $\mathcal{I}^p$ by keeping every vertex of $I$ and those edges $e \in I$ such that $U_e \leq q, p$ respectively. Consider any infinite subtree $J$ in $\mathcal{I}^p$. Theorem 1.3 gives us that $p_c(J) = 1/\text{br}J = 1/p \liminf_{n\to\infty} \mathbb{E}[I_n]^{1/n} = p_c(I)/p$ a.s. Therefore, performing further percolation on $J$ with density $q/p > p_c(I)/p$ will result in a new graph containing an infinite subgraph a.s. Of course, the distribution of this new graph must be the same as $J \cap \mathcal{I}^q$. Furthermore this holds in particular if $J = I^p$ showing that if $|I^p| = \infty$ then there exists a.s. an infinite subtree of $I^p \cap \mathcal{I}^q$. It is now possible to proceed as in [2]. $\square$

The non-classical way to conclude continuity of the percolation function has an interesting analogy on $\mathbb{Z}^d$. Define $B_n := [-n,n]^d$ and write $\partial B_n$ for the (inner) boundary of $B_n$. Letting $\{0 \leftrightarrow \partial B_n\}$ denote the event that the origin is connected to $\partial B_n$ by a path of open edges, define

$$\varphi_n(p) := \mathbb{P}_p(0 \leftrightarrow \partial B_n).$$

Clearly,

$$\theta(p) = \lim_{n\to\infty} \varphi_n(p), \tag{20}$$

for all $0 \leq p \leq 1$. The inequality of the following equation (valid for every $n \geq 1$) is a part of Theorem 8.18 of [4]:

$$\varphi_n(p) - \theta(p) = \mathbb{P}_p(0 \leftrightarrow B_n, |C| < \infty) \leq A(p,d)n^d e^{-n\sigma(p)}, \tag{21}$$

where we can take

$$A(p,d) = \frac{d^2}{p^2(1-p)^{d-2}}. \tag{22}$$

Furthermore, according to Theorem 8.21 of [4] we can take $\sigma(p)$ to be uniformly bounded away from 0 on any closed sub-interval of $(p_c, 1)$. We point out the following corollary and sketch the proof.

**Corollary 4.1** *The percolation function $\theta(p)$ on $\mathbb{Z}^d$, $d \geq 2$ is continuous for $p > p_c$.*

**Sketch of proof.** Choose $p_c < p_1 < p_2 < 1$. Combining equations (21), (22) and Theorem 8.21 of [4] explained directly above, it is straightforward to prove that there exists constants $C = C(p_1,p_2) < \infty$ and $\delta = \delta(p_1,p_2) > 0$ such that for any $p \in [p_1,p_2]$ and any $n \geq 1$,

$$\varphi_n(p) - \theta(p) \leq Ce^{-n\delta}.$$

Since trivially

$$\theta(p) \leq \varphi_n(p),$$

it follows that $\varphi_n(p) \to \theta(p)$ uniformly on any closed subinterval of $(p_c, 1)$, from which the statement follows. □

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