Problem - Lösningar

1. Givet 16 fotbollslag och att varje lag möter varje annat lag, visa att det vid varje tidpunkt finns minst två lag som har spelat samma antal matcher.

Lösning: Varje lag kan ha spelat minst 0 och högst 15 matcher. Numrera 16 "lådor" med talen 0 till 15 och placera varje lag i lådan med nummer antalet matcher laget har spelat. Det kan inte finnas lag i lådorna 0 och 15 samtidigt (om något lag inte har spelat alls kan det inte finnas lag som har mött alla övriga). De 16 lagen är alltså placerade i högst 15 lådor. Lådprincipen säger då att det finns minst en låda som innehåller två lag, d.v.s. det finns minst två lag som har spelat samma antal matcher.

2. Finn alla heltalslösningar till ekvationen $xy = 2x - y$.

Lösning: Ekvationen är ekvivalent med ekvationen $(x+1)(y-2) = -2$. Eftersom det är heltalslösningar som efterfrågas och 2 är ett primtal, betyder det att lösningsmängden är unionen av lösningarna till följande fyra ekvationssystem: $x+1 = 1,\ y-2 = -2;\quad x+1 = -1,\ y-2 = 2;\quad x+1 = -2,\ y-2 = 1;\quad x+1 = 2,\ y-2 = -1$. Lösningarna är alltså $x = 0,\ y = 0;\quad x = -2,\ y = 4;\quad x = -3,\ y = 3;\quad x = 1,\ y = 1$.

3. Talet x bildas genom att man på ett godtyckligt sätt blandar siffrorna i 111 exemplar av talet 2000. Visa att x inte är kvadraten till något heltal.

Lösning: Summan av siffrorna i talet x är $111 \cdot 2 = 222$. Det medför att x är delbart med 3, men inte med 9 och x kan därmed inte vara kvadraten till något heltal.

4. En stege lutas mot ett hus och når 5 m upp på väggen. Samma stege svängs 60° utan att man flyttar dess stödpunkter och når då 3 m upp på väggen på andra sidan gränden. Hur bred är gränden?

Lösning: Låt b vara grändens bredd och l vara stegens längd. Betrakta ett tvärsnitt av gränden (och stegen). Inför beteckningar som i figuren. Vi har då $|AB| = b,\ |PS| = |SQ| = l,\ |AP| = 5,\ |BQ| = 3$ (i meter). Triangeln $\triangle PSQ$ är likbent med toppvinkel 60° och alltså liksidig, d.v.s. $|PQ| = l$. Drag en linje genom Q, som är parallell med AB; den skär sträckan AP i C. Beteckna med M sträckan PQ:s mittpunkt och drag MN vinkelrätt mot AB ($N \in AB$).

Punkten N är AB:s mittpunkt enligt transversalsatsen; samma sats medför att $MN = \frac{5+3}{2} = 4\,\text{m}$. Trianglarna $\triangle SNM$ och $\triangle PCQ$ är likformiga (enligt tredje likformighetsfallet; vinklarnas likhet följer av att motsvarande sidor i de två trianglarna är ortogonala mot varandra). Vi har då

$$\frac{|SM|}{|PQ|} = \frac{\frac{l\sqrt{3}}{2}}{l} = \frac{|MN|}{|QC|} = \frac{4}{b},$$

varav följer att $b = \frac{8}{\sqrt{3}}\,\text{m}$.

Uppgiften kan också lösas m.h.a. endast Pythagoras' sats. Här följer en skiss:

Inför beteckningen $x = |AS|$. Ur trianglarna $\triangle PCQ$, $\triangle PSA$, $\triangle QBS$ får vi $b^2 + 4 = l^2$, $x^2 + 25 = l^2$, $(b - x)^2 + 9 = l^2$. Vi kan då lösa ut x i termer av b och får $x = \frac{b^2-16}{2b}$. De två första likheterna ger då $\left(\frac{b^2-16}{2b}\right)^2 + 25 = l^2 = b^2 + 4$ och vi får ekvationen $3b^4 - 52b^2 - 256 = 0$, vars enda positiva lösning är $b = \frac{8}{\sqrt{3}}\,\text{m}$.

5. Visa att

$$\sqrt[3]{413} > 6 + \sqrt[3]{3}.$$

Lösning: Olikheten ovan är ekvivalent med $97 > 9\cdot\sqrt[3]{3}(6+\sqrt[3]{3})$. Sätt $A = 6+\sqrt[3]{3}$. Ovanstående olikhet kan då skrivas som $97 > 9(A-6)A$, vilket är ekvivalent med $9A^2 - 54A - 97 < 0$, vilket i sin tur är ekvivalent med att A ligger mellan de två nollställena till funktionen $9x^2 - 54x - 97$. Dessa två nollställen är $3 - \frac{1}{9}\sqrt{1602}$ (uppenbarligen mindre än A) och $3 + \frac{1}{9}\sqrt{1602}$. Den andra roten är större än $3 + \frac{40}{9}$ och slutligen visar en enkel kalkyl att $\sqrt[3]{3} < \frac{40}{9} - 3$.

6. Givet är sex kongruenta cirkelskivor i planet med icke-tomt snitt. Visa att minst en av dem innehåller medelpunkten till en av de andra.

Lösning: Kalla medelpunkterna för $O_1, \ldots, O_6$, radien för R och en av de gemensamma punkterna för P. Vi har då att $|O_iP| \leq R$, $i = 1, \ldots, 6$, samt att minst en av vinklarna $\angle O_iPO_j \leq 60°$. Då måste en av de andra vinklarna i triangeln $\triangle O_iPO_j$ vara större än eller lika med $\angle O_iPO_j$, varav följer att $|O_iO_j| \leq \max(|O_iP|, |O_jP|) \leq R$.

7. Basen till en pyramid är en regelbunden n-hörning med sidlängd $2a$. Pyramidens övriga sidor är likbenta trianglar med area a^2. Bestäm n.

Lösning: Pyramidens "övriga sidor" är kongruenta likbenta trianglar med bas $2a$ och höjd a. Av symmetriskäl kommer pyramidens topp P att projiceras på n-hörningens centrum O. Låt AB vara en av n-hörningens sidor och låt M vara AB:s mittpunkt. Vi får då att radien till den inskrivna cirkeln $r = |OM|$ kan beräknas ur de två rätvinkliga trianglarna $\triangle AMO$ och $\triangle MOP$: $r = a \tan \frac{180°(n-2)}{2n} = a \sin \angle MPO$. Därav följer att $0 < \tan \frac{180°(n-2)}{2n} < 1$, vilket ger $0 < \frac{180°(n-2)}{2n} < 45°$. Olikheten ger $2 < n < 4$, så den enda möjligheten är alltså $n = 3$.

8. Givet en vinkel (mindre än 180°) och en punkt P i dess inre, drag (med hjälp av passare och omarkerad linjal) en linje genom P sådan att sträckan som vinkeln skär av linjen halveras av P.

Lösning: Kalla vinkelns spets för O. Sträckan OP är då median i den triangel som bildas när linjen skär vinkelns ben och OP är därmed halva diagonalen i en parallellogram som har ett hörn i O, två sidor på vinkelns ben och vars diagonaler möts i P. Konstruktion: Avsätt PQ på strålen OP, $|PQ| = |OP|$, $Q \neq O$. Drag genom Q linjer parallella med vinkelns ben; dessa linjer skär vinkelns ben i punkter A, B. Då är $OAQB$ en parallellogram, P är den ena diagonalens mittpunkt (per konstruktion), alltså halverar P även den andra diagonalen AB. Linjen genom A och B är den sökta.

9. Givet är en vinkel med spets i punkten O och ben p och q. Låt $A, B \in p$ och $C, D \in q$. Finn mängden av alla punkter M i vinkelns inre sådana att summan av areorna av trianglarna ABM och CDM är lika med S (konstant).

Lösning: Antag att M är en punkt som uppfyller villkoret. Avsätt sträckor OP och OQ på p resp. q s.a. $|OP| = |AB|$, $|OQ| = |CD|$. Då har triangeln OPM samma area som triangeln ABM och $\triangle OQM$ har samma area som $\triangle CDM$. Summan av de två areorna är lika med arean av fyrhörningen $OPMQ$. Samtidigt är fyrhörningens area lika med summan (skillnaden) av arean S_1 av $\triangle OPQ$, som inte påverkas av M:s läge, och arean S_2 av $\triangle PMQ$. Det som krävs är alltså att bestämma var M ska ligga så att arean av $\triangle PMQ$ är $S_2 = |S - S_1|$. Sträckan PQ är fix, det betyder att M måste ligga på en (öppen) sträcka i vinkelns inre på avstånd $2S_2/|PQ|$ från PQ, i halvplanet som innehåller O om $S < S_1$ och i det andra halvplanet om $S > S_1$. (Om $S = S_1$ är lösningen själva (öppna) sträckan PQ.)

10. Finn alla heltalslösningar till ekvationen

$$x^y = y^x.$$

"Analytisk" lösning: För $x, y > 0$ är ekvationen ekvivalent med $\frac{\ln x}{x} = \frac{\ln y}{y}$. Låt $f(t) = \frac{\ln t}{t}$ och anta att $x > y$ är sådana att $f(x) = f(y)$. Eftersom f inte är konstant, måste den ha antingen lokalt maximum eller lokalt minimum i intervallet (y, x). Det måste alltså finnas en punkt t_0 mellan x och y sådan att $f'(t_0) = 0$. Kalkyl visar att derivatan har ett enda nollställe, $t_0 = e$. Det betyder att givet ett par positiva tal $x \neq y$ s.a. $f(x) = f(y)$, måste det ena vara större än e och det andra mindre. Man ser direkt att $x = 4,\ y = 2$ är ett sådant heltalspar, medan $x = n,\ y = 1, n \geq 3$ inte är det (och enligt analysen ovan finns inga heltalslösningar utom $x = 4,\ y = 2$). Negativa $x,\ y$; $x = y$ - se lösningen nedan.

Lösning: Uppenbarligen är alla par heltal $x = y\,(\neq 0)$ lösningar. Det intressanta är att se om det finns fler par heltal som satisfierar ekvationen. Om $x \neq y$ kan vi av symmetriskäl anta att $x > y$. Låt oss först titta på fallet $x > y > 0$. Det är uppenbart att y måste vara större än 1. Sätt $x = n,\ y = m,\ n > m > 1$ och låt $n = m + p,\ p > 0$. Ekvationen kan då skrivas som $(1 + p/m)^{m/p} = m$. Vi har

$$m^p = \left(1 + \frac{p}{m}\right)^m = \left(\frac{m+p}{m+p-1}\right)^m \left(\frac{m+p-1}{m+p-2}\right)^m \cdots \left(\frac{m+1}{m}\right)^m =$$

$$= \left(1 + \frac{1}{m+p-1}\right)^m \left(1 + \frac{1}{m+p-2}\right)^m \cdots \left(1 + \frac{1}{m}\right)^m < \left(\left(1 + \frac{1}{m}\right)^m\right)^p <$$

$$< \{\text{binomialsatsen \& uppskattning med geometrisk summa}\} < 3^p.$$

Det betyder att $1 < m < 3$, alltså måste m vara lika med 2. Återstår att bestämma n ur ekvationen $n^2 = 2^n$ med bivillkoret $n > 2$. Högerledet är delbart med 8, alltså måste n vara delbart med 4. Uppenbarligen är $n = 4$ en lösning. Antag att $n > 4$. Då är högerledet delbart med 2^8 och då måste n vara delbart med $2^4 = 16$. Induktivt fås att n måste vara delbart med godtyckligt höga potenser av två, vilket är omöjligt, alltså är det enda paret olika positiva heltal som satisfierar ekvationen $n = 4,\ m = 2$.

Det är omöjligt att ha $x > 0 > y$ (man får att ett tal mellan 0 och 1 är lika med ett tal som har belopp större än 1). Båda leden ska ha samma tecken, alltså kan inte det ena talet vara jämnt och det andra udda. Då fås att om $x,\ y$ är ett par lösningar, så är $|x|,\ |y|$ också det, alltså är det enda paret negativa lösningar $-2,\ -4$.

11. Det naturliga talet n är sådant att det finns en rätvinklig triangel med hypotenusa $2n$ och kateter naturliga tal. Visa att det även finns en rätvinklig triangel med hypotenusa n och kateter naturliga tal.

Lösning: Givet är alltså att det finns naturliga tal k och m sådana att $k^2+m^2 = 4n^2$. Det betyder att k och m antingen båda är jämna, eller båda är udda. Om båda är jämna är vi klara. Antag att båda är udda, d.v.s. $k = 2p+1,\ m = 2s+1$. Vi får

$$k^2 + m^2 = (2p + 1)^2 + (2s + 1)^2 = 4(p^2 + p + s^2 + s) + 2 = 4n^2,$$

vilket är omöjligt.

12. En biljardboll ligger vid kanten till ett runt biljardbord med radie R. Efter ett slag reflekteras bollen i kanten sex gånger, utan att på vägen ha kommit närmare bordets centrum än $\frac{9R}{10}$. Kan bollen vid den sjätte reflexionen ha fullbordat ett helt varv kring bordets centrum? Motivera!

Lösning: Nej. Kalla vinkeln mellan tangenten i begynnelsepunkten och bollens riktningsvektor efter första slaget för $\alpha,\ 0 < \alpha \leq \frac{\pi}{2}$. Efter varje reflexion kommer vinkeln mellan tangenten i reflexionspunkten och den nya riktningsvektorn att vara lika med α, eftersom kordan mellan två på varandra följande reflexionspunkter och tangenterna i dessa bildar en likbent triangel och p.g.a. likheten mellan den infallande och den reflekterade vinkeln. Bollen kommer att vara närmast bordets centrum när den befinner sig i mittpunkten på varje korda och, enligt ovan, kommer alla sådana avstånd till origo att vara lika med varandra. Betrakta en triangel som bildas av två reflexionspunkter och bordets centrum. Vinklarna vid bordets kant är båda lika med $\frac{\pi}{2} - \alpha$, medan vinkeln vid bordets centrum blir 2α. För det kortaste avståndet till bordets centrum får vi då $d = R\cos\alpha \geq \frac{9}{10}$. För att bollen vid den sjätte reflexionen ska ha fullbordat ett helt varv kring bordets centrum krävs dock att $6 \cdot 2\alpha \geq 2\pi$, d.v.s. att $\alpha \geq \frac{\pi}{6}$, vilket (eftersom $\alpha \in (0, \frac{\pi}{2}]$) är ekvivalent med $\cos\alpha \leq \frac{\sqrt{3}}{2} < \frac{9}{10}$.

13. Bestäm alla p sådana att ekvationen

$$x^2 + 2x + \frac{1}{\cos^2(p \cdot 180^\circ)} = 0$$

har minst en reell rot.

Lösning: Vi har

$$x^2 + 2x + \frac{1}{\cos^2(p \cdot 180^\circ)} \geq x^2 + 2x + 1 = (x + 1)^2 \geq 0.$$

Enda chansen för ekvationen att ha reella rötter (i själva verket en reell dubbelrot $x_{1,2} = -1$) är att $\cos^2(p \cdot 180°) = 1$, d.v.s. att $\cos(p \cdot 180°) = \pm 1$, vilket inträffar om och endast om p är ett heltal.

14. I varje punkt med heltalskoordinater i planet har man placerat en säck med bollar så att (a) det finns inte fler än 2005 bollar i någon säck; (b) antalet bollar i varje säck är lika med (det aritmetiska) medelvärdet av antalet bollar i säckarna i punktens fyra närmsta grannpunkter. Visa att det finns lika många bollar i alla säckar.

Lösning: Låt $u(i, j)$ vara antalet bollar i punkten med koordinater (i, j), $i, j \in \mathbb{Z}$. Vi har fått givet att

$$0 \leq u(i,j) \leq 2005, \qquad u(i,j) = \frac{u(i+1,j) + u(i-1,j) + u(i,j+1) + u(i,j-1)}{4}.$$

Låt $M = \max u(i, j) = u(i_0, j_0)$ (både minimum och maximum antas, eftersom $u(i, j)$ antar heltalsvärden mellan 0 och 2005). Eftersom $u(i, j) \leq M$ för alla $i, j \in \mathbb{Z}$ och

$$M = u(i_0, j_0) = \frac{u(i_0+1,j_0) + u(i_0-1,j_0) + u(i_0,j_0+1) + u(i_0,j_0-1)}{4},$$

får vi att $u(i_0+1, j_0) = u(i_0-1, j_0) = u(i_0, j_0+1) = u(i_0, j_0-1) = M$. Därav följer att det är exakt M bollar i dessa fyra punkters alla grannar o.s.v. Resultatet följer nu av att en godtycklig punkt med heltalskoordinater (i, j) kan sammanbindas med (i_0, j_0) m.h.a. ändligt många lodräta och vågräta sträckor med längd 1.

15. Låt f vara en kongruensavbildning i planet som är sammansättning (i valfri ordning) av en translation och en rotation (ej ett helt antal varv). Visa att det finns en entydigt bestämd punkt P i planet sådan att $|PA| = |Pf(A)|$ för varje punkt A i samma plan.

Lösning: Notera att det räcker att visa att f har en entydigt bestämd fix punkt P. Eftersom f är en kongruensavbildning gäller för en sådan att $|PA| = |f(P)f(A)| = |pf(A)|$. Å andra sidan, om en punkt uppfyller kravet i uppgiften får vi $|Pf(P)| = |PP| = 0$ och $P = f(P)$.

Låt O och θ vara rotationscentrum och rotationsvinkel för den rotation som ingår i f, och låt a vara translationsvektorn. Geometriskt kan vi lösa problemet genom att inse att det finns en cirkel med medelpunkt i O sådan att en korda med längden $|a|$ motsvaras av centralvinkel just θ. Två sådana kordor kan väljas så att de ger vektorn $-a$. Punkten P är då den ena eller den andra begynnelsepunkten, beroende på om rotationen eller translationen kommer först i f.

Den mest naturliga lösningen (?) är den som använder komplexa tal. Om $O = 0$, så ges rotationen av multiplikation med $e^{i\theta}$. Om vi identifierar a med ett komplext tal (som också får heta a) och z är en godtycklig punkt i det komplexa talplanet, får vi att

$$f(z) = e^{i\theta}z + a \qquad \text{eller} \qquad f(z) = e^{i\theta}(z + a).$$

Eftersom $e^{i\theta} \neq 1$ är det nu elementärt att visa att det i båda fallen existerar ett entydigt bestämt tal $z_0 \in \mathbb{C}$ så att $z_0 = f(z_0)$.

Ur "Korrespondenskurs 2002/2003" - Lösningar

PROBLEM 1. Let $a_1, a_2, \ldots, a_n$ be positive real numbers in arithmetic progression. Prove that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \frac{4n}{(a_1 + a_n)^2}.$$

I. Solution

The strict inequality should be replaced by "$\geq$" since equality is possible. (Equality holds for $a_1 = a_2 = \cdots = a_n$.) Let d denote the common difference of the arithmetic progression. Then for all $k = 1, 2, \ldots, n$, we have $a_k = a_1 + (k-1)d$ and $a_{n-k+1} = a_1 + (n-k)d$. Hence, $a_k + a_{n-k+1} = 2a_1 + (n-1)d = a_1 + a_n$ which implies

$$\sum_{k=1}^{n} \frac{1}{(a_k + a_{n-k+1})^2} = \frac{n}{(a_1 + a_n)^2} \tag{1}$$

Since $4a_k a_{n-k+1} \leq (a_k + a_{n-k+1})^2$, we have

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} \geq \sum_{k=1}^{n} \frac{4}{(a_k + a_{n-k+1})^2} \tag{2}$$

From (1) and (2), the result follows.

II. Solution

Suppose that $n > 1$ and $d \neq 0$.. By the AM-GM (Arithmetic Mean - Geometric Mean) Inequality, since the a_k's are different numbers, we have,

$$\sqrt{a_k a_{n-k+1}} < \frac{a_k + a_{n-k+1}}{2} = \frac{2a_1 + (n-1)d}{2} = \frac{a_1 + a_n}{2},$$

whence it follows that

$$\sum_{k=1}^{n} \frac{1}{a_k a_{n-k+1}} > \sum_{k=1}^{n} \frac{4}{(a_1 + a_n)^2} = \frac{4n}{(a_1 + a_n)^2}.$$

[Note that, as pointed out, the a_k's do not have to be distinct; thus, the "$>$" must be replaced by "$\geq$".]

PROBLEM 2. A mouse eats his way through a $3 \times 3 \times 3$ cube of cheese by tunnelling through all of the 27 $1 \times 1 \times 1$ sub-cubes. If he starts at one of the corner sub-cubes and always moves onto an uneaten adjacent sub-cube can he finish at the center of the cube? (Assume that he can tunnel through walls but not edges or corners.)

Solution. Colour the 3×3 cube in this manner: colour each corner sub-cube black. Every other sub-cube is coloured black or white so that each sub-cube is a different colour than all the other sub-cubes that it shares a face with.

Now, notice that the corner sub-cube is black, and the centre sub-cube is white. But as the mouse goes through from sub-cube to sub-cube, the destination sub-cube is a different colour from his original cube. Since there are 13 white cubes and 14 black cubes, the mouse's path must go $BWBW \dots BWB$. The last cube must be black. Thus, he cannot end up in the centre sub-cube last.

PROBLEM 3. Find all positive real solutions of the system

$$\begin{aligned} x_1 + x_2 + \cdots + x_n &= 9, \\ \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n} &= 1. \end{aligned}$$

SOLUTION. Multiplication gives

$$\begin{aligned} 9 = (x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) &= \\ = 1 + 1 \cdots + 1 + \left(\frac{x_1}{x_2} + \frac{x_2}{x_1} + \cdots + \frac{x_n}{x_{n-1}} + \frac{x_{n-1}}{x_n}\right) &\geq \\ \geq n + 2\frac{n(n+1)}{2} &= n^2, \end{aligned}$$

since $t + \frac{1}{t} \geq 2$ for all $t \neq 0$ (with equality if and only if $t = 1$). Hence $n = 1$, or $n = 2$, or $n = 3$.

Let $n = 1$. It is obvious that the system of equations $x_1 = 9$, $\frac{1}{x_1} = 1$ has no solution.

Let $n = 2$. We get $x_1 + x_2 = 9$, $\frac{1}{x_1} + \frac{1}{x_2} = \frac{x_1+x_2}{x_1x_2} = \frac{9}{x_1x_2} = 1$, which leads to the quadratic equation $x_1^2 - 9x_1 + 9 = 0$. The solutions are $x_{1,2} = \frac{1}{2}(9 \pm 3\sqrt{5})$.

Let $n = 3$. It means that

$$\begin{aligned} 9 = (x_1 + x_2 + x_3)\left(\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3}\right) &= \\ = 3 + \left(\frac{x_1}{x_2} + \frac{x_2}{x_1} + \frac{x_3}{x_1} + \frac{x_1}{x_3} + \frac{x_3}{x_2} + \frac{x_2}{x_3}\right) &= 3^2, \end{aligned}$$

and hence $\frac{x_1}{x_2} + \frac{x_2}{x_1} = \frac{x_3}{x_1} + \frac{x_1}{x_3} = \frac{x_3}{x_2} + \frac{x_2}{x_3} = 2$. It then follows that $\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_3}{x_1} = 1$, i.e. $x_1 = x_2 = x_3 = 3$.

Problem 4. Prove that for any real numbers a, b, c such that $0 < a, b, c < 1$, the following inequality holds

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < 1.$$

Solution. Observe first that $\sqrt{x} < \sqrt[3]{x}$ for $x \in [0, 1]$. Thus. we have $\sqrt{abc} < \sqrt[3]{abc}$ and

$$\sqrt{(1-a)(1-b)(1-c)} < \sqrt[3]{(1-a)(1-b)(1-c)}.$$

By the AM-GM inequality, we get

$$\sqrt{abc} < \sqrt[3]{abc} \leqslant \frac{a+b+c}{3},$$

and

$$\sqrt{(1-a)(1-b)(1-c)} \leqslant \sqrt[3]{(1-a)(1-b)(1-c)} \leqslant \frac{(1-a)+(1-b)+(1-c)}{3}.$$

Summing up, we obtain

$$\sqrt{abc} + \sqrt{(1-a)(1-b)(1-c)} < \frac{1+1-a+b+1-b+c+1-c}{3} = 1,$$

as desired.

Problem 5. In the triangle ABC, the midpoint of BC is D. Given that $\angle ADB = 45°$ and $\angle ACB = 30°$, determine $\angle BAD$.

Solution. (Draw a figure!) Let BH be the altitude from the point B, H belongs to AC. The triangle BCH has a right angle at H and an angle of 30° at the vertex C. It follows that $BH = \frac{1}{2}BC = BD = DC$. We also have that $HD = \frac{1}{2}BC$ as the median to the hypotenuse. Hence the triangle BDH is equilateral. Then $\angle HDA = 60° - 45° = 15° = \angle HAD = 180° - 135° - 30°$ and it follows that $AH = HD = HB$. We then have $\angle BAD = 180° - 45° - \angle DBH - \angle ABH = 180° - 45° - 60° - 45° = 30°$.

Problem 6. Five points are given in the plane such that each of the 10 triangles they define has area greater than 2. Prove that there exists a triangle of area greater than 3.

Solution. Denote by A, B, C, D, E the five given points. If the pentagon $ABCDE$ is concave, we can suppose that D is situated inside the triangle ABC or inside the quadrilateral $ABCE$.

In the first case $\text{area}(ABC) = \text{area}(ABD) + \text{area}(DBC) + \text{area}(DAC) > 6 > 3$.

In the second case, D is inside one of the triangles BCE, ACE, ABC or ABE. Suppose, without loss of generality, that D is inside the triangle BCE. Then

$$\text{area}(BCE) \geqslant \text{area}(BDC) + \text{area}(CDE) > 4 > 3.$$

Consider now the case when $ABCD$ is a convex pentagon. Let M and N be the intersection points of BE with AC and AD respectively.

The following result will be useful.

Lemma. Let $PQRS$ be a quadrilateral and T a point on the side PQ. Then

$$\text{area}(TRS) \geqslant \min(\text{area}(PSR), \text{area}(QSR)).$$

The proof consists of simply observing that the distance from T to SR is bounded up and below by the distances from P and Q to SR.

In our case, suppose that $BM \geqslant \frac{1}{3}BE$, which yields $BM \geqslant \frac{1}{2}ME$. Then

$$\begin{aligned}\text{area}(BDE) &= \text{area}(BDM) + \text{area}(MDE) \geqslant \frac{1}{2}\text{area}(MDE) + \text{area}(MDE)\\ &= \frac{3}{2}\text{area}(MDE) \geqslant \frac{3}{2}\min(\text{area}(CDE), \text{area}(ADE)) > \frac{3}{2}\cdot 2 = 3.\end{aligned}$$

The case when $NE \geqslant \frac{1}{3}BE$ is similar. It is left to consider the case when $MN \geqslant \frac{1}{3}BE$. We then have:

$$\text{area}(AMN) \geqslant \frac{1}{3}\text{area}(ABE) > \frac{2}{3},$$

$$\text{area}(MND) \geqslant \frac{1}{3}\text{area}(BED) > \frac{2}{3}$$

and

$$\text{area}(MCD) \geqslant \min(\text{area}(BCD), \text{area}(ECD)) > 2.$$

Summing up, we conclude $\text{area}(ACD) > 2 + \frac{2}{3} + \frac{2}{3} > 3$ and the proof is complete.

Ur "Korrespondenskurs 2003/2004" - Lösningar

Problem 1. Let $p(3) = 2$ where $p(x)$ is a polynomial with integer coefficients. Is it possible that $p(2003)$ is a perfect square?

Solution. As the polynomial p has integer coefficients, $x-y$ divides $p(x)-p(y)$. We have $p(2003) - p(3) = (2003 - 3)k$, where k is an integer, so that $p(2003) = 2000k + 2$. It follows that $p(2003)$ is an even number not divisible by 4, as $p(2003) \equiv 2 \mod 4$. Hence $p(2003)$ cannot be a perfect square.

Problem 2. The cat from the neighbouring village keeps coming to Dunce-village to irritate their dogs. Every night when all of them are sleeping, the cat sneaks up to Dunce-village, mews out loud and runs back home. When the cat mews, all dogs that are up to 90 m away from the cat, start barking. As Dunce-village is small, any two dogs in the village are up to 100 m from each other. Is it possible for the cat to take a position such that all dogs start barking at the same time?

Solution. Yes, the cat can take a position such that all dogs start barking at the same time.

Let A and B be two dogs such that the distance between them is maximal. Consider the intersection of the disks of radii $|AB|$ with centres A and B. There are no dogs out of this intersection. If the cat positions itself at the midpoint M of AB, then all dogs will be within $\frac{\sqrt{3}}{2} \cdot |AB| \leq 50 \cdot \sqrt{3} < 90\,\text{m}$ from it.

Problem 3. Let D be the midpoint of the hypotenuse AB of the right triangle ABC. Denote by O_1 and O_2 the circumcenters of the triangles ADC and DBC, respectively. Prove that AB is tangent to the circle with diameter O_1O_2.

Solution. Both O_1 and O_2 lie on the perpendicular bisector of the segment CD. Apart from that O_1 lies on the perpendicular bisector of AC and O_2 on the perpendicular bisector of BC. Consider the perpendicular bisector of AC. It is parallel to BC and passes through the midpoint of AC, hence it must pass through the midpoint of AB which is D. By analogy the same goes for the perpendicular bisector of BC. This means that the two intersect at D. Let k be the circle with diameter O_1O_2. Denote its center by O.We have that $\angle O_1DO_2$ is a right angle, since the perpendicular bisectors are parallel to AC and BC respectively. This means that D lies on k. We have $\angle BAC = \angle ACD = \angle DO_1O = \angle O_1DO$ (since angles at the base of an isosceles triangle are equal and since $O_1D \perp AC$, $O_1O \perp CD$) and $\angle ADO_1 = \angle ABC$ (since $O_1D||BC$). Hence we get $\angle ADO = \angle BAC + \angle ABC = \frac{\pi}{2}$, which means that K is tangent to AB at D.

Alternative solution. Denote by k the circle that is tangent to the side AB in the point D and that passes through C. Let O be the center of k. Then the points O, O_1 and O_2 are collinear as they all lie on the perpendicular bisector of the segment CD. Denote $\angle BAC = \alpha$. The center angle $\angle DO_1C$ is then equal to 2α. The triangle ADC is isosceles, so that $\angle BDC = 2\alpha$ and therefore O_1 lies on the circle k (DC is a chord and AB is tangent to k, hence $\angle BDC = \frac{1}{2}\angle DOC$; it is possible that $\angle DOC > \pi$). Similarly, the point O_2 lies on the circle k. As we have already proved that the points O, O_1 and O_2 are collinear, the segment O_1O_2 is in fact the diameter of the circle k.

Problem 4. Prove the inequality

$$\frac{a^2+b^2}{c^2+ab}+\frac{b^2+c^2}{a^2+bc}+\frac{c^2+a^2}{b^2+ca}\geq 3$$

for all positive numbers a, b, c.

Solution. From the Cauchy-Schwarz inequality we get

$$\begin{aligned} c^2+ab &\leq \sqrt{c^2+a^2}\sqrt{c^2+b^2}, \\ a^2+bc &\leq \sqrt{a^2+b^2}\sqrt{a^2+c^2}, \\ b^2+ca &\leq \sqrt{b^2+c^2}\sqrt{b^2+a^2}. \end{aligned}$$

According to the above inequalities and the inequality between the arithmetic and the geometric means we have:

$$\begin{aligned} &\frac{a^2+b^2}{c^2+ab}+\frac{b^2+c^2}{a^2+bc}+\frac{c^2+a^2}{b^2+ca}\geq \\ &\geq \frac{a^2+b^2}{\sqrt{c^2+a^2}\sqrt{c^2+b^2}}+\frac{b^2+c^2}{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}+\frac{c^2+a^2}{\sqrt{b^2+c^2}\sqrt{b^2+a^2}}\geq \\ &\geq 3\left(\frac{a^2+b^2}{\sqrt{c^2+a^2}\sqrt{c^2+b^2}}\cdot\frac{b^2+c^2}{\sqrt{a^2+b^2}\sqrt{a^2+c^2}}\cdot\frac{c^2+a^2}{\sqrt{b^2+c^2}\sqrt{b^2+a^2}}\right)^{\frac{1}{3}}=3. \end{aligned}$$

Problem 5. Solve the following system in real numbers

$$\begin{array}{ccccccc} x^2 & + & y^2 & - & z(x+y) & = & 2 \\ y^2 & + & z^2 & - & x(y+z) & = & 4 \\ z^2 & + & x^2 & - & y(z+x) & = & 8 \end{array}.$$

Solution. Using subtraction between pairs of equations of the system we get the system

$$\begin{aligned} (z-x)(x+y+z) &= 2 \\ (y-z)(x+y+z) &= -6 \\ (x-y)(x+y+z) &= 4 \end{aligned}$$

or, equivalently (since $x+y+x \neq 0$ for all solutions),

$$\frac{1}{x+y+z} = \frac{x-y}{4} = \frac{y-z}{-6} = \frac{z-x}{2} = t.$$

By adding up the equations of the given system we find

$$(x-y)^2 + (y-z)^2 + (z-x)^2 = 14$$

The two last equalities give $t = \frac{1}{2}$ or $t = -\frac{1}{2}$. Finally, for $t = \frac{1}{2}$ we find $(x, y, z) = (1, -1, 2)$ and for $t = -\frac{1}{2}$ we find $(x, y, z) = (-1, 1, -2)$.

Problem 6. Given a triangle such that the sines of all three angles are rational numbers, prove that the cosines of all three angles are rational too.

Solution. Denote the angles of the triangle by α, β, γ and the side lengths by a, b, c (a lies opposite α, etc). Similarity does not affect the angles, nor does it affect their sines and cosines, which means we are free to choose the length of one of the sides. Choose $a = 1$. By the law of sines we then have

$$\frac{1}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma},$$

which means all side lengths are rational numbers. Then, by the law of cosines

$$1 = b^2 + c^2 - 2bc\cos\alpha, \quad b^2 = c^2 + 1 - 2c\cos\beta, \quad c^2 = 1 + b^2 - 2b\cos\gamma,$$

and hence all three cosines are rational numbers too.

Ur "Korrespondenskurs 2004/2005" - Lösningar

PROBLEM 1. Each of the players in a tennis tournament played one match against each of the others. If every player won at least one match, show that there is a group A, B, C of three players for which A beat B, B beat C and C beat A.

SOLUTION. Let the number of players be n. The number of matches won by each player is an integer between 1 and $n-1$, so these cannot be distinct as there are fewer than n possibilities. Hence two players A and B won the same number k of matches. Suppose without loss of generality that A beat B. Then even if B beat the remaining $k-1$ players who lost to A, there must be a player C who beat A but was beaten by B. So A beat B, B beat C and C beat A.

PROBLEM 2. Suppose p, q are distinct primes and S is a subset of $\{1, 2, \ldots, p-1\}$. Let $N(S)$ denote the number of solutions of the equation

$$\sum_{i=1}^{q} x_i \equiv 0 \pmod{p},$$

where $x_i \in S,\ i = 1, 2, \ldots, q$. Prove that $N(S)$ is a multiple of q.

SOLUTION. Denote by A the set of all solutions of the equation. Partition A into subsets as follows:

For $(x_1, x_2, \ldots, x_q) \in A$, define $A(x_1, x_2, \ldots, x_q)$ as the set of all $(y_1, y_2, \ldots, y_q)$, where $(y_1, y_2, \ldots, y_q)$ is a permutation of $(x_1, x_2, \ldots, x_q)$. Suppose $a_i \in S,\ 1 \leq i \leq n$, are distinct and $(x_1, x_2, \ldots, x_q)$ is an arrangement of k_1 a_1's, k_2 a_2's, $\ldots$, k_n a_n's with $k_i \geq 1$. The set $A(x_1, x_2, \ldots, x_q)$ has

$$\frac{q!}{k_1!k_2!\ldots k_n!}$$

elements. Hence $\#A(x_1, x_2, \ldots, x_q)$ is a multiple of q unless q is cancelled ($\#A$ denotes the number of elements of A). Since q is a prime this only happens for $n = 1$ and $k_1 = q$. However, in this case we have

$$qa_1 \equiv 0 \pmod{p} \quad \Leftrightarrow \quad a_1 \equiv 0 \pmod{p}.$$

But S does not contain a multiple of p, and so this case does not arise. The result follows.

PROBLEM 3. The points A, B, C, D, E, F on a circle of radius R are such that $AB = CD = EF = R$. Show that the middle points of BC, DE, FA are vertices of an equilateral triangle.

SOLUTION. The length of the radius is clearly irrelevant. We shall therefore assume that $R = 1$. Further, we shall assume that the problem is imbedded in $\mathbb{C}$. Without loss of generality we can choose

$$A = 1, \quad B = e^{i\frac{\pi}{3}}, \quad C = e^{i\alpha}, \quad D = e^{i(\alpha+\frac{\pi}{3})}, \quad E = e^{i\beta}, \quad F = e^{i(\beta+\frac{\pi}{3})},$$

for some real α and β. We then have

$$M_1 = \frac{1}{2}(e^{i\frac{\pi}{3}} + e^{i\alpha}), \qquad M_2 = \frac{1}{2}(e^{i(\alpha+\frac{\pi}{3})} + e^{i\beta}), \qquad M_3 = \frac{1}{2}(e^{i(\beta+\frac{\pi}{3})} + 1),$$

$$M_2 - M_3 = \frac{1}{2}(e^{i\alpha}e^{i\frac{\pi}{3}} + e^{i\beta}e^{-i\frac{\pi}{3}} - 1),$$

$$\begin{aligned} M_1 - M_3 &= \frac{1}{2}(e^{i\frac{\pi}{3}} + e^{i\alpha} - e^{i\beta}e^{i\frac{\pi}{3}} - 1) = \frac{1}{2}(e^{i\alpha} - e^{i\beta}e^{i\frac{\pi}{3}} - e^{-i\frac{\pi}{3}}) = \\ &= \frac{1}{2}e^{-i\frac{\pi}{3}}(e^{i\alpha}e^{i\frac{\pi}{3}} - e^{i\beta}e^{2i\frac{\pi}{3}} - 1) = \frac{1}{2}e^{-i\frac{\pi}{3}}(e^{i\alpha}e^{i\frac{\pi}{3}} + e^{i\beta}e^{-i\frac{\pi}{3}} - 1). \end{aligned}$$

This means that $M_2 - M_3 = e^{i\frac{\pi}{3}}(M_1 - M_3)$. Hence the segments M_1M_3 and M_2M_3 are of the same length and the angle at M_3 is $\frac{\pi}{3}$. Thus the triangle $M_1M_2M_3$ is equilateral.

PROBLEM 4. Find all functions $f : \mathbb{N} \cup \{0\} \to \mathbb{R}$ such that

$$f(n+m) + f(n-m) = f(kn), \qquad \forall n, m \in \mathbb{N} \cup \{0\}, \quad m \leq n,$$

where k is a fixed nonnegative integer.

SOLUTION I. For $m = 0$ we get that $2f(n) = f(kn)$. Further, $m = 1$ gives $f(n+1) + f(n-1) = f(kn) = 2f(n)$, which is a second order linear recurrence equation. Its characteristic equation is $r^2 - 2r + 1 = 0$, with the solutions $r_{1,2} = 1$. All possible solutions are given by $f(n) = an + b$, where $a, b \in \mathbb{R}$. We need to pick out those among the solutions that solve the given equation. We have that $a(kn) + b = 2(an + b)$ and hence $a(k-2)n - b = 0$ for all n. This means that $b = 0$ and $a = 0$ for $k \neq 2$, so that for $k \neq 2$ we only have the s.c. trivial solution, $f \equiv 0$. For $k = 2$ the coefficient a can be an arbitrary real number, so that for $k = 2$ we have the solutions $f(n) = an, \ a \in \mathbb{R}$.

SOLUTION II. We have $2f(n) = f(kn)$, which implies $f(0) = 0$. Also, choosing $m = n$ we get $f(2n) = f(kn) = 2f(n)$. Set $f(1) = a$. We then have $f(2) = 2f(1) = 2a, \ f(3) + f(1) = 2f(2)$, which gives $f(3) = 4a - a = 3a$. By induction $f(n) = na$. Hence $f(k) = ka$, on the other hand $f(k) = 2f(1) = 2a$. This means that $(k-2)a = 0$, so for $k = 2$ we have the solutions $f(n) = an, \ a \in \mathbb{R}$, whereas for $k \neq 2$ the trivial solution is unique.

PROBLEM 5. Let $\triangle ABC$ be an isosceles triangle with $AB = AC$ and $\angle A = 20^\circ$.The point D on AC is such that $AD = BC$. Determine the angle $\angle BDC$.

SOLUTION. Since the triangle is isosceles, we have that $\angle B = \angle C = 80°$. Denote $AD = BC = a$. Choose the point E on AC so that $\angle CBE = 20°$. It follows that $\angle BEC = 80°$ and hence $BE = BC = a$. Choose now F on AB so that $\angle FEB = 60°$. This means that $\triangle FEB$ is equilateral and $EF = BF = a$. Now, choose G on AC so that $\angle FGE = 40°$. Since $\angle GEF = 180° - 80° - 60° = 40°$, it follows that $GF = EF = a$. The angle FGE is an exteriour angle of $\triangle AFG$. This implies $\angle AFG = 40° - 20° = 20°$ and hence $AG = GF = a$. This is possible only if the points D and G coincide. We also have that $\angle DFA$ is an exteriour angle of $\triangle FBD$, which is isosceles. It follows that $\angle FDB = 10°$ and hence $\angle BDC = 30°$.

PROBLEM 6. The numbers $a_1, a_2, \ldots, a_n$ are positive and such that $a_1 + a_2 + \ldots + a_n = 1$. Show that

$$\frac{a_1}{1 + a_1\sqrt{2}} + \frac{a_2}{1 + a_2\sqrt{2}} + \ldots + \frac{a_n}{1 + a_n\sqrt{2}} \le \frac{n}{n + \sqrt{2}}.$$

When does equality occur?

SOLUTION. We shall prove the inequality

$$\frac{a_1}{1 + xa_1} + \frac{a_2}{1 + xa_2} + \ldots + \frac{a_n}{1 + xa_n} \le \frac{n}{n + x},$$

for all $x \ge 0$. It is obvious that equality holds for $x = 0$. Let $x > 0$. We have

$$\frac{a_1}{1 + xa_1} + \frac{a_2}{1 + xa_2} + \ldots + \frac{a_n}{1 + xa_n} =$$
$$= \frac{1}{x}\left(\frac{1 + xa_1 - 1}{1 + xa_1} + \frac{1 + xa_2 - 1}{1 + xa_2} + \ldots + \frac{1 + xa_n - 1}{1 + xa_n}\right) =$$
$$= \frac{1}{x}\left(n - \frac{1}{1 + xa_1} - \frac{1}{1 + xa_2} - \ldots - \frac{1}{1 + xa_n}\right).$$

What we need to prove is that

$$\frac{1}{1 + xa_1} + \frac{1}{1 + xa_2} + \ldots + \frac{1}{1 + xa_n} \ge \frac{n^2}{n + x}.$$

The inequality between harmonic and arithmetic means gives

$$\frac{1}{1 + xa_1} + \frac{1}{1 + xa_2} + \ldots + \frac{1}{1 + xa_n} \ge \frac{n^2}{1 + xa_1 + xa_2 + \ldots + xa_n} = \frac{n^2}{n + x}.$$

For $x \ne 0$ equality occurs iff all terms in the sum above are equal, i.e. iff $a_1 = a_2 = \ldots = a_n = \frac{1}{n}$.

Ur "Korrespondenskurs 2005/2006" - Lösningar

PROBLEM 1. The circles $\mathcal{C}_1$ and $\mathcal{C}_2$ intersect at A and B. The tangent line to $\mathcal{C}_2$ at A meets $\mathcal{C}_1$ at the point C and the tangent line to $\mathcal{C}_1$ at A meets $\mathcal{C}_2$ at the point D. A ray from A, interior to the angle $\angle CAD$, intersects $\mathcal{C}_1$ at M, $\mathcal{C}_2$ at N and the circumcircle of the triangle $\triangle ACD$ at P. Prove that $AM = NP$.

SOLUTION. Without loss of generality we can consider the case when the ray is interior to the angle $\angle BAC$. Then

$$\angle CMP = \angle MCA + \angle CAM = \angle MAD + \angle CAM = \angle CAD.$$

Also, $\angle CPM = \angle CDA$, since both subtend the cord AC in the circumcircle ACD. Therefore, the triangles $\triangle ACD$ and $\triangle MCP$ are similar, and thus

$$\frac{MC}{AC} = \frac{MP}{AD}.$$

Furthermore, $\angle ACM = \angle NAD$ and $\angle CAM = \angle ADN$, so $\triangle ACM \sim \triangle DAN$. Hence

$$\frac{AN}{AD} = \frac{MC}{AC}.$$

Thus $MP = AN$ and $AM = NP$, as claimed.

PROBLEM 2. Let h be a positive integer and let $\{a_n\}_{n\in\mathbb{N}\cup\{0\}}$ be the sequence defined by recursion as follows:

$$a_0 = 1; \qquad a_{n+1} = \begin{cases} \frac{a_n}{2} & \text{if } a_n \text{ is even,} \\ a_n + h & \text{if } a_n \text{ is odd.} \end{cases}$$

(For instance, if $h = 27$ one has $a_1 = 28, a_2 = 14, a_3 = 7, a_4 = 34, a_5 = 17, a_6 = 44\ldots$).

For which values of h does there exist $n > 0$ ($n \in \mathbb{N}$) such that $a_n = 1$?

SOLUTION. It is obvious that no such index exists if h is an even number. We shall prove that such an index does exist for all odd natural numbers h.

Let h be an odd number. It is easily seen that all odd numbers in the sequence are less than h and all even numbers are less than $2h$. Since the number of odd numbers, less than h, is finite, there exists an odd number which appears in the sequence more than once. Among all numbers with this property, let N be the one that appears with the smallest index and assume that $N \neq 1$. Consider the first two appearances of N in the sequence: $a_k = N,\ a_m = N,\ k < m$ (it is obvious that the sequence is periodic thereafter). We then have $m > k > 0$ and $a_{k-1} = a_{m-1} = 2N$. If $2N > h$, then $a_{k-2} = a_{m-2} = 2N-h$, which is an odd number, and $k-2 < k$, which is a contradiction to the minimality condition. In general, there is a unique way of reconstructing the sequence backwards: $a_{k-i} = a_{m-i} = 2^i N$, as long as $2^{i-1}N < h$, $a_{k-p} = a_{m-p} = 2^p N - h$, for $2^{p-1}N < h < 2^p N$. As above, $2^p N - h$ is an odd number which appears more than once in the sequence and its first appearance has an index, smaller than the index of N:s first appearance - a contradiction. The only remaining possibility is that $N = 1$.

Problem 3. Find all natural numbers n such that there exists a polynomial p with real coefficients for which

$$p\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n} \qquad \forall x \in \mathbb{R},\ x \neq 0.$$

Solution. We shall prove that such a polynomial exists if and only if n is odd.

Suppose that n is even. Assume that p is a polynomial with the required property. For $a \neq 0$, take $x = a$ and $x = -\frac{1}{a}$, to get

$$p\left(a - \frac{1}{a}\right) = a^n - \frac{1}{a^n} \quad \text{and} \quad p\left(-\frac{1}{a} + a\right) = \frac{1}{a^n} - a^n,$$

an obvious contradiction for $a \neq \pm 1$. Hence n cannot be even.

Now, suppose that n is odd. We shall prove by induction on n that there exists a polynomial $p_n(x)$ such that

$$p_n\left(x - \frac{1}{x}\right) = x^n - \frac{1}{x^n} \qquad \forall x \in \mathbb{R},\ x \neq 0.$$

It is obvious that such a polynomial exists for $n = 1$ and for $n = 3$ (take $p_1(x) = x,\ p_3(x) = x^3 + 3x$). Suppose that p_n and p_{n+2} are polynomials as required, and define

$$p_{n+4}(x) = (x^2 + 2)p_{n+2}(x) - p_n(x).$$

A simple calculation shows that p_{n+4} also satisfies the condition, and we are done.

Alternative solution. Suppose n is odd. The thought that leaps to mind is to compute the powers of $x - \frac{1}{x}$:

$$\left(x - \frac{1}{x}\right)^n = x^n - \frac{1}{x^n} - \binom{n}{1}\left(x^{n-2} - \frac{1}{x^{n-2}}\right) + \ldots + (-1)^{\frac{n-1}{2}}\binom{n}{\frac{n-1}{2}}\left(x - \frac{1}{x}\right).$$

This means that given $p_1, p_3, \ldots, p_{n-2}$ as required, we are able to construct p_n:

$$p_n(x) = x^n + \binom{n}{1}p_{n-2}(x) - \binom{n}{2}p_{n-4}(x) + \ldots + (-1)^{\frac{n+1}{2}}\binom{n}{\frac{n-1}{2}}p_1(x).$$

As it is obvious how to construct p_1, the solution for n odd is complete.

Problem 4. Let S be a set of n points in the plane such that any two points of S are at least 1 unit apart. Prove that there is a subset T of S with at least $n/7$ points such that any two points of T are at least $\sqrt{3}$ units apart.

Solution. Let $X \in S$ be a point on the boundary of the convex hull of S. We shall show that there is a set S_X, consisting of at most 6 points, such that their distances from X are in the range $[1, \sqrt{3})$. Then we can put X in T and consider the set S', obtained by deleting S and S_X. By continuing this process, we can construct a set T with at least $n/7$ points which satisfies the condition.

Let l be the line containing X such that all the points in S lie on one side of l. With X as the centre, draw two semicircles with radii 1 and $\sqrt{3}$. Divide the corresponding semidisks into 6 equal sectors. Since in each of the 6 regions between the two circles and inside one of the 6 sectors, any two points are at most 1 unit apart, the whole region between the circles contains at most 6 points of S.

Problem 5. A number of points on a circle of radius 1 are joined by chords. It is known that any diameter of the circle intersects at most 6 of the chords. Prove that the sum of the lengths of the chords is less than 19.

Solution. The sum of the lengths of the chords will be less than the sum of the lengths of the corresponding arcs. It is therefore sufficient to prove that than the sum of the lengths of the corresponding arcs is less than 19. A diameter will intersect a certain chord if and only if one of its ends lies on the corresponding arc. This means that for each diameter its both ends lie on at most 6 of the arcs. Regard an arbitrary diameter. It may be that its ends split some of the arcs into smaller ones; but, the new set of arcs will still have the required property. Also, if we replace one arc by its centrally symmetric one with respect to the centre of the circle, the new set of arcs will again have the required property. Hence we can replace the original set of arcs by a set on only one of the halves of the circle (with respect to the chosen diameter) and the special property now means that each point of this half of the circle belongs to at most 6 of the arcs. Thus the sum of the lengths of the arcs is at most $6\pi < 19$.

Ur "Korrespondenskurs 2006/2007" - Lösningar

PROBLEM 1. How many five-digit palindromic numbers divisible by 37 are there?

SOLUTION. A five-digit palindromic number can be written as $t = \overline{abcba} = 10001a + 1010b + 100c = 37(270a + 27b + 3c) + 11(a + b - c)$. Hence the number t will be divisible by 37 iff (if and only if) $a + b - c$ is divisible by 37. Since $0 \leq a, b, c \leq 9$ and $a \neq 0$, we have $-8 \leq a+b-c \leq 18$, so that the number $a+b-c$ will be divisible by 37 iff $a + b - c = 0$. This means that $c = a + b$ and it follows that $a + b \leq 9$. For $a = 1$ there are 9 possible choices for b, for $a = 2$ there are 8 possible choices for b and so on. Hence there are $9+8+7+6+5+4+3+2+1 = 45$ possibilities for t.

PROBLEM 2. How many points (x, y) are there on the circle $x^2 + y^2 = 1$ such that x and y have at most (a) two (b) m decimal places? (m is a non-negative integer)

SOLUTION. (a) Let $x^2 + y^2 = 1$, where $x = \frac{a}{100},\ y = x = \frac{b}{100},\ a, b \in \mathbb{Z}$. We then have $a^2 + b^2 = 100^2$ and it follows that a and b have the same parity.

Assume that a and b are odd, i.e. $a = 2s + 1,\ b = 2t + 1$. We get

$$(2s + 1)^2 + (2t + 1)^2 = 2^4 \cdot 5^4,$$

and

$$2(s^2 + t^2 + s + t) + 1 = 2^3 \cdot 5^4,$$

which is impossible since the two sides of the equality have opposite parity.

Assume now that a and b are even, i.e. $a = 2s,\ b = 2t$. We get

$$s^2 + t^2 = 2^2 \cdot 5^4,$$

meaning that s and t have the same parity. Imitating what we did above we get that s and t have to be even numbers, $s = 2u$, $t = 2v$ and that

$$u^2 + v^2 = 625 = 25^2 \qquad (x = \frac{4u}{100},\ y = \frac{4v}{100}).$$

By symmetry, if (x, y) is a solution, then so are $(\pm x, \pm y),\ (\pm x, \mp y)$ and $(\pm y, \pm x),\ (\pm y, \mp x)$. Hence it suffices to look for solutions of the form $(x, y),\ 0 \leq x \leq y \ \Leftrightarrow\ 0 \leq u \leq v$.
We can now proceed using Pythagorean triples, but, since there are at most 25 (actually, quite a bit fewer) possibilities for u, v, we can list them to get $0^2 + 25^2 = 7^2 + 24^2 = 15^2 + 20^2 = 625$, and hence the solutions

$$(0, \pm 1), \quad (\pm 1, 0),$$
$$(\pm 0.28, \pm 0.96), \quad (\pm 0.28, \mp 0.96), \quad (\pm 0.96, \pm 0.28), \quad (\pm 0.96, \mp 0.28),$$
$$(\pm 0.6, \pm 0.8), \quad (\pm 0.6, \mp 0.8), \quad (\pm 0.8, \pm 0.6), \quad (\pm 0.8, \mp 0.6).$$

This means that there are $4+8+8=20$ points (x,y) on the circle $x^2+y^2=1$ such that x and y have at most two decimal places.

(b) (outline) We have $x=\frac{a}{10^m},\ y=\frac{b}{10^m}$. It follows that

$$a^2+b^2=2^{2m}\cdot 5^{2m}=((1+i)(1-i))^{2m}((2+i)(2-i))^{2m},$$

since $2=(1+i)(1-i)$, $5=(2+i)(2-i)$ (prime factorization in Gaussian integers, $\mathbb{Z}[i]$). Identify (x,y) with the complex number $x+iy$. Let z be the product of $2m$ among the factors $(1\pm i)$ and $2m$ among $(2\pm i)$; we then have $a^2+b^2=z\bar{z}$. Let $z=a+bi$. If the point $x+iy$ lies on the unit circle, then so do $\pm x\pm iy$, $\pm x\mp iy$, $\pm y\pm ix$, $\pm y\mp ix$. We are interested in the possible quotients of the real and imaginary parts of z, modulo the above symmetries. Since $(1\pm i)^{2p}$ is either real or purely imaginary, z will be equal to a real or purely imaginary number times $(2\pm i)^{2s}=(3\pm 4i)^s$, $1\le s\le m$. As $x\neq y$ ($\sqrt{2}\notin\mathbb{Q}$), we get $8m+4$ different points

$$(\pm 1,0),\quad (0,\pm 1),\quad \left(\frac{6}{10}+i\frac{8}{10}\right)^s,\qquad 1\le s\le m,$$

and further by the symmetries listed above.

Problem 3. Sixteen points in the plane form a 4×4 grid. Show that any seven of these points contain three which are the vertices of an isosceles (nondegenerate) triangle. Will the statement still hold if we replace seven by six?

Solution. Assume that at least one of the seven points is an "inner point" of the grid (i.e. has four neighbours in the grid). Without loss of generality, we may assume it is the point with "coordinates" $(2,2)$. The rest of the points can be divided into five classes as follows: all remaining inner points $\{(2,3),(3,2),(3,3)\}$; the points $\{(1,1),(1,2),(1,3)\}$; the points $\{(2,1),(3,1)\}$; the points $(2,4),(4,2),(4,4))$; all remaining points $\{(1,4),(3,4),(4,1),(4,3)\}$. According to Dirichlet's Box Principle (a.k.a. the pigeonhole principle), at least two of the six chosen points have to belong to the same class and will thus, together with the chosen inner point, form an isosceles triangle.

Assume that all seven points are boundary points. Divide the 12 boundary points into 3 classes as follows: $\{(1,1),(1,4),(4,1),(4,4)\}$; $\{(1,2),(2,4),(3,1),(4,3)\}$; $\{(1,3),(2,1),(3,4),(4,2)\}$. At least three of the seven chosen points have to belong to the same class and will thus form an isosceles triangle.

If the chosen points are only six, the statement needn't be true (for example, choose $\{(1,1),(1,2),(1,3),(2,4),(3,4),(4,4)\}$).

Problem 4. An (a) $n\times n$ square ((b) $n\times p$ rectangle) is divided into (a) n^2 ((b) np) unit squares. Initially m of these squares are black and all the others are white. The following operation is allowed: If there exists a white square which is adjacent to at least two black squares, we can change the colour of this square from white to black. Find the smallest possible m such that there exists an initial position from which, by applying repeatedly this operation, all unit squares can be made black.

SOLUTION. (a) If the n squares covering one fix diagonal of the $n \times n$ square are initially black, then it is easily seen that all squares can be made black, meaning that $m \leq n$. We shall show that $m = n$ is the smallest possible value. The $n \times n$ square is divided into uniy squares, and the sides of these squares are unit segments (there is a total of $2n(n+1)$ such segments; a segment on the boundary is a side of only one unit square, whereas an interior segment is a side of two unit squares). We consider the number of such segments which are sides of exactly one black unit square (regardless of whether they lie in the interior or on the boundary). We shall call all such segments boundary segments. It is easy to see that if we change from white to black a square which was adjacent to exactly k black squares, then k boundary segments will disappear (the segments shared by the changed square and an adjacent black square), and $4 - k$ new boundary segments will be created (the sides of the changed square which were not boundary segments before the change). Since the operation is allowed only if $k \geq 2$, we have $4 - k \leq k$, so the number of boundary segments cannot increase. Observe that there are initially m black squares, so there are (initially) at most $4m$ boundary segments. When all the squares are turned black there are $4n$ boundary segments, which yields the desired inequality $m \geq n$.

(b) Without loss of generality, we may assume that $n \leq p$, where n is the number of rows and p is the number of columns of the rectangle. In this case the last observation from above gives $4m \geq 2(n+p)$, and hence, $m \geq \left[\frac{n+p-1}{2}\right] + 1$. To show that the required number is indeed $m = \left[\frac{n+p-1}{2}\right] + 1$, place the initial black squares as follows: an up-left to down-right diagonal in the $n \times n$ square in the left part of the rectangle and then a black square in every second column in the first row starting with the last column. The total number of black squares initially placed will be $n + \frac{p-n}{2}$ in case n and p have the same parity, and $n + \frac{p-n}{2} + 1$ in case n and p have opposite parity, i.e. exactly $\left[\frac{n+p-1}{2}\right] + 1$ in both cases.

PROBLEM 5. Show that the inequality

$$3(a+b+c) \geq 8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}}$$

holds for all positive reals a, b, c. Determine the cases of equality.

SOLUTION. By the inequality between the arithmetic and the cubic means, we have

$$8\sqrt[3]{abc} + \sqrt[3]{\frac{a^3+b^3+c^3}{3}} \le 9\sqrt[3]{\frac{8abc + \frac{a^3+b^3+c^3}{3}}{9}} = 3\sqrt[3]{a^3+b^3+c^3+24abc}\,.$$

Note that $8\sqrt[3]{abc}$ is interpreted as a sum of 8 terms in this context. Therefore, it is sufficient to prove the inequality

$$3(a+b+c) \ge 3\sqrt[3]{a^3+b^3+c^3+24abc}\,,$$

or

$$(a+b+c)^3 \ge a^3+b^3+c^3+24abc.$$

Expanding the left-hand side, we see that this is equivalent to

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \ge 6abc,$$

which is a direct consequence of the AM-GM inequality. Equality holds iff $abc = \frac{a^3+b^3+c^3}{3}$ and $a^2b = ab^2 = b^2c = bc^2 = c^2a = ca^2$, which is obviously the case iff $a = b = c$.

Ur "Korrespondenskurs 2007/2008" - Lösningar

Problem 1. For all real numbers a, let $\lfloor a \rfloor$ denote the largest integer less than or equal to a, and let $\{a\} = a - \lfloor a \rfloor$. Solve the system of equations

$$\begin{cases} x + \lfloor y \rfloor - \{z\} = 2,98 \\ \lfloor x \rfloor + \{y\} + z = 4,05 \\ -\{x\} + y + \lfloor z \rfloor = 5,01 \end{cases}.$$

Solution. Add the third equation to the first, subtract the second one from the result and divide by 2; we get

$$\lfloor y \rfloor - \{z\} = 1,97.$$

Hence $\lfloor y \rfloor = 2$ and $\{z\} = 0,03$. It follows that $x = 1,01$. The third equation now yields

$$2 + \{y\} + \lfloor z \rfloor = 5,02,$$

which means that $\lfloor z \rfloor = 3$ and $\{y\} = 0,02$. Thus $y = 2,02$ and $z = 3,03$.

Problem 2. Let A, B, C, D be points in the plane such that

$$0 < AB, AC, AD < 1 < BC, BD, CD.$$

Show that the four points do not lie on a circle.

Solution. By the symmetry of the problem with respect to B, C, D, it is enough to show that A and C cannot be opposite vertices of an inscribed quadrilateral. Indeed, if such were the case, we would have $\angle A + \angle C = \angle B + \angle D = \pi$.

In any triangle the angle opposite the greater side is greater. (Euclid) Thus we can conclude that

$$\begin{aligned} \angle BAC &> \angle ABC, \\ \angle BAC &> \angle BCA, \\ \angle CAD &> \angle ACD, \\ \angle CAD &> \angle ADC. \end{aligned}$$

Hence $\angle A + \angle C > \angle A = \angle BAC + \angle CAD > \angle ABC + \angle ADC = \angle B + \angle D$ and it follows that A and C cannot be opposite vertices of an inscribed quadrilateral. By the symmetry of the problem with respect to B, C, D this means that the four points cannot lie on a circle.

Problem 3. The sum of n real numbers is positive and the sum of their squares is greater than n^2. Show that at least one of the numbers is greater than 1.

Solution. The statement is obvious if all the numbers are non-negative.

Assume now that there exist n real numbers, $a_1, \dots, a_n$, not all of them non-negative, and such that

$$S = a_1+a_2+\cdots+a_n > 0, \quad a_1^2+a_2^2+\cdots+a_n^2 > n^2, \quad a_k \le 1 \text{ for all } k = 1, 2, \dots, n.$$

Without loss of generality we can assume that $1 \ge a_1 \ge a_2 \ge \ldots \ge a_m \ge 0 > a_{m+1} \ge \ldots \ge a_n$, where $m < n$. This implies

$$0 > a_{m+1} + \cdots + a_n = S - (a_1 + \cdots + a_m) \ge S - m > -m,$$

and hence

$$a_{m+1}^2 + \cdots + a_n^2 + 2a_{m+1}a_{m+2} + \cdots + 2a_{n-1}a_n < m^2$$

We get

$$n^2 < a_1^2+\cdots+a_n^2 \le m+a_{m+1}^2+\cdots+a_n^2 < m+m^2-(2a_{m+1}a_{m+2}+\cdots+2a_{n-1}a_n).$$

Since $a_{m+1},\ a_{m+2}, \dots, a_n < 0$, we have

$$0 < 2a_{m+1}a_{m+2} + \cdots + 2a_{n-1}a_n < -n^2 + m^2 + m,$$

and

$$n^2 < m^2 + m \le (n-1)^2 + n - 1 = n^2 - n < n^2,$$

a contradiction. Hence at least one of the numbers has to be greater than 1.

Problem 4. Between every pair of cities in a country there is a flight in each direction operated by one of several airlines. If an airline has direct flights between A and B and between B and C, then it has no direct flights between A and C. Show that one can travel through the country, passing all the cities exactly once, and in such a way that one changes airlines on each transfer occasion. (The travel begins and ends in different cities.)

Solution. Let us first check the statement for $n = 3$. There has to be a flight from X_1 to X_2 and also one from X_2 to X_3. If the flight from X_2 to X_3 is by the same airline as the flight from X_1 to X_2, we can renumber the cities as follows $X_1' = X_1,\ X_2' = X_3,\ X_3' = X_2$, and by the condition of the problem we get a journey as requested. NB: we kept the airline of the last segment and only changed the first one. Assume now we have a route $X_1 \to \cdots \to X_{n-1}$ and add the last city X_n. If $X_{n-1} \to X_n$ and $X_{n-2} \to X_{n-1}$ are operated by different airlines, we are done. If they are operated by the same airline, we can renumber the last three cities as shown above. This may now lead to a problem one step earlier in the chain, but if it does, we can renumber the last three cities but one. (The problems do not spread forward, as we keep the airline of the last segment unchanged at each step.) This process can then go on until we get a journey which is well-organized, in the terms of the problem.

Problem 5. The infinite sequence $a_1, a_2, \ldots, a_n, \ldots$, consists of natural numbers (i.e. positive integers) and is such that

$$a_1 = 1, \qquad a_n^2 > a_{n-1}a_{n+1} \quad \text{for all} \quad n \in \mathbb{N}.$$

Prove that $a_n \geq n$ for all $n \in \mathbb{N}$.

Solution. We shall prove that the sequence is strictly increasing, i.e. $a_{n+1} > a_n$ for all $n \in \mathbb{N}$. Assume that for some $m \in \mathbb{N}$ we have $a_{m+1} \leq a_m$. The given inequality then implies

$$1 \geq \frac{a_{m+1}}{a_m} > \frac{a_{m+2}}{a_{m+1}} > \cdots ,$$

which means the infinite sequence is strictly decreasing starting at a_{m+1}. But, this is impossible, since all the elements of the sequence are natural numbers. Hence the sequence is strictly increasing, and it follows that $a_n \geq n$ for all $n \in \mathbb{N}$.