24th Nordic Mathematical Contest, 13th of April, 2010

Solutions of the preliminary version of the problem set

1. *A function $f\colon \mathbb{Z}_+ \to \mathbb{Z}_+$, where $\mathbb{Z}_+$ is the set of positive integers, is non-decreasing and satisfies $f(mn) = f(m)f(n)$ for all relatively prime positive integers m and n. Prove that $f(8)f(13) \geq (f(10))^2$.*

Solution: Since f is non-decreasing, $f(91) \geq f(90)$, which (by factorization into relatively prime factors) implies $f(13)f(7) \geq f(9)f(10)$. Also $f(72) \geq f(70)$, and therefore $f(8)f(9) \geq f(7)f(10)$. Since all values of f are positive, we get

$$f(8)f(9) \cdot f(13)f(7) \geq f(7)f(10) \cdot f(9)f(10),$$

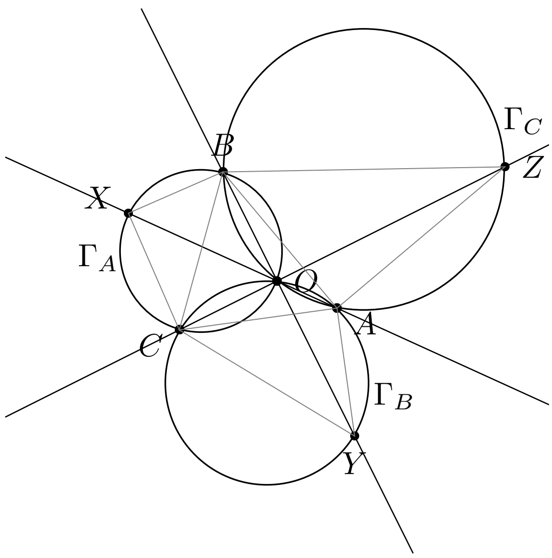
and dividing both sides by $f(7)f(9) > 0$,

$$f(8)f(13) \geq f(10)f(10) = (f(10))^2. \quad \square$$

2. *Three circles Γ_A, Γ_B and Γ_C share a common point of intersection O. The other common point of Γ_A and Γ_B is C, that of Γ_A and Γ_C is B, and that of Γ_C and Γ_B is A. The line AO intersects the circle Γ_A in the point $X \neq O$. Similarly, the line BO intersects the circle Γ_B in the point $Y \neq O$, and the line CO intersects the circle Γ_C in the point $Z \neq O$. Show that*

$$\frac{|AY|\,|BZ|\,|CX|}{|AZ|\,|BX|\,|CY|} = 1.$$

Solution:



Observe $\measuredangle AOY = \measuredangle BOX = \alpha$ (vertical angles). By looking on peripheral angles we further get

$$\alpha = \measuredangle BCX = \measuredangle BOX = \measuredangle AOY = \measuredangle ACY.$$

In the same manner we get

$$\beta = \measuredangle BAZ = \measuredangle BOZ = \measuredangle COY = \measuredangle CAY,$$
$$\gamma = \measuredangle ABZ = \measuredangle AOZ = \measuredangle COX = \measuredangle CBX.$$

We have $\alpha + \beta + \gamma = \measuredangle AOY + \measuredangle YOC + \measuredangle COX = 180°$ and hence all triangels CAY, CXB and ZAB have angles α, β and γ and hence they are similar. This gives $\frac{|AY|}{|CY|} = \frac{|AB|}{|BZ|}$ and $\frac{|CX|}{|BX|} = \frac{|AZ|}{|AB|}$, i.e.

$$\frac{|AY|\,|BZ|\,|CX|}{|AZ|\,|BX|\,|CY|} = \frac{|AB|\,|AZ|\,|BZ|}{|BZ|\,|AB|\,|AZ|} = 1. \quad \square$$

3. *Laura has 2010 lamps connected with 2010 buttons in front of her. For each button, she wants to know the corresponding lamp. In order to do this, she observes which lamps are lit when Richard presses a selection of buttons. Richard always presses the buttons simultaneously, so the lamps are lit simultaneously, too.*

a) *If Richard chooses the buttons to be pressed, what is the maximum number of different combinations of buttons he can press until Laura can assign the buttons to the lamps correctly?*

b) *Supposing that Laura will choose the combinations of buttons to be pressed, what is the minimum number of attempts she has to do until she is able to associate the buttons with the lamps in a correct way?*

Solution: a) Let us say that two lamps are separated if one of the lamps is turned on while the other lamp remains off. Laura can find out which lamps belong to the buttons if every two lamps are separated. Let Richard choose two arbitrary lamps. To begin with, he turns both lamps on and then varies all the other lamps in all possible ways. There are 2^{2008} different combinations for the remaining $2010 - 2 = 2008$ lamps. Then Richard turns the two chosen lamps off. Also, at this time there are 2^{2008} combinations for the remaining lamps. Consequently, for the 2^{2009} combinations in all, it is not possible to separate the two lamps of the first pair. However, we cannot avoid the separation if we add one more combination. Indeed, for every pair of lamps, we see that if we turn on a combination of lamps $2^{2009} + 1$ times, there must be at least one setup where exactly one of the lamps is turned on and the other is turned off. Thus, the answer is $2^{2009} + 1$.

b) For every new step with a combination of lamps turned on, we get a partition of the set of lamps into smaller and smaller subsets where elements belonging to the same subset cannot be separated. In each step every subset is either unchanged or divided into two smaller parts, i.e. the total number of subsets after k steps will be at most 2^k. We are finished when the number of subsets is equal to 2010, so the answer is at least

$\lceil \log_2 2010 \rceil = 11$. But it is easy to see that Laura certainly can choose buttons in every step in such a way that there are at most 2^{11-k} lamps in every part of the partition after k steps. Thus, the answer is 11.

4. *A positive integer is called simple if its ordinary decimal representation consists entirely of zeroes and ones. Find the least positive integer k such that each positive integer n can be written as $n = a_1 \pm a_2 \pm a_3 \pm \ldots \pm a_k$ where $a_1, \ldots, a_k$ are simple.*

Solution: We write $n = a_1 + a_2 + \ldots + a_9$, where a_j has 1's on the places where n has digits greater or equal to j and 0's on the other places. Then $n = a_1 + a_2 + \ldots + a_9$ and we get $k \leq 9$.

On the other hand, consider $n = 10203040506070809$. Suppose $n = a_1 + a_2 + \ldots + a_j - a_{j+1} - a_{j+2} - \ldots - a_k$ where $a_1, \ldots a_k$ are simple, $k < 9$. Then all digits of $b_1 = a_1 + \ldots + a_j$ are not greater than j and all digits of $b_2 = a_{j+1} + \ldots + a_k$ are not greater than $k - j$. We have $n + b_2 = b_1$. We perform column addition of n and b_2 and consider digit $j + 1$ in the number n. There will be no summand coming from lower decimal places, since the sum there is less that $10\ldots0 + 88\ldots8 = 98\ldots8$. So we get the sum of $j + 1$ and the corresponding digit in b_2, the resulting digit should be less than $j + 1$ thus in b_2 we have at least $9 - j \leq k - j$, implying $k \geq 9$.

Hence, we have proved that $k = 9$.