The 22nd Nordic Mathematical Contest

31 March 2008
Solutions

Time allowed is 4 hours. Each problem is worth 5 points. The only permitted aids are writing and drawing materials.

Problem 1

Determine all real numbers A, B and C such that there exists a real function f that satisfies

$$f(x + f(y)) = Ax + By + C$$

for all real x and y.

Solution. Let A, B and C be real numbers and f a function such that $f(x + f(y)) = Ax + By + C$ for all x and y.

Let z be a real number and set $x = z - f(0)$ and $y = 0$. Then

$$f(z) = f(z - f(0) + f(0)) = A(z - f(0)) + B \cdot 0 + C = Az - Af(0) + C,$$

so there are numbers a and b such that $f(z) = az + b$ for all z. It follows that $f(x + f(y)) = ax + a^2y + (a + 1)b$. Thus $(A, B, C) = (a, a^2, c)$, where a and c are arbitrary real numbers, except for the case $a = -1$, in which case $c = 0$.

Problem 2

Assume that $n \geq 3$ people with different names sit around a round table. We call any unordered pair of them, say M and N, *dominating*, if

(i) M and N do not sit on adjacent seats, and

(ii) on one (or both) of the arcs connecting M and N along the table edge, all people have names that come alphabetically after the names of M and N.

Determine the minimal number of dominating pairs.

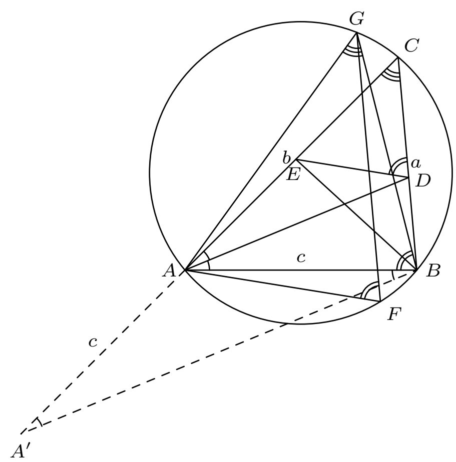
Solution. We will show by induction that the number of dominating pairs (hence also the minimal number of dominating pairs) is $n-3$ for $n \geq 3$.

If $n = 3$, all pairs of people sit on adjacent seats, so there are no dominating pairs. Assume that the number of dominating pairs is $n-3$ for some $n \geq 3$. If there are $n+1$ people around the table, let the person whose name is alphabetically last leave the table. The two people sitting next to that person, who formed a dominating pair, no longer do. On the other hand, any other dominating pair remains a dominating pair in the new configuration of n people, and any dominating pair in the new configuration was also a dominating pair in the old. The number of dominating pairs in the new configuration is $n-3$, so the number in the old was $(n+1)-3$.

Problem 3

Let ABC be a triangle and let D and E be points on BC and CA, respectively, such that AD and BE are angle bisectors of ABC. Let F and G be points on the circumcircle of ABC such that AF and DE are parallel and FG and BC are parallel. Show that

$$\frac{AG}{BG} = \frac{AB+AC}{AB+BC}.$$



Solution. Let $AB = c$, $BC = a$ and $CA = b$. Then it follows from the angle bisector theorem that $CD = ab/(b+c)$.

(The angle bisector theorem can be proved by letting A' be the intersection point of the line CA and the line through B parallel with the bisector AD (dashed lines). Then the angles BAD, ABA', CAD and $CA'B$ are equal, so the triangle $A'AB$ is isosceles, and the equality follows from the similarity of the triangles ACD and $A'CB$.)

Similarly, $CE = ab/(a+c)$, so $CE/CD = (b+c)/(a+c)$.

In the triangles AED and ABD $\angle BAD = \angle EAD$, the side AD is common and $AE = cb/(a+b) < b = AB$. Then $\angle EDA < \angle BDA = \pi - \beta - \frac{1}{2}\alpha = \gamma + \frac{1}{2}\alpha$, where $\angle CAB = \alpha$, $\angle ABC = \beta$, and $\angle BCA = \gamma$.

Let AT be a tangent line to the circle through A – we choose T such that T and B are on the same side of the line AC. Then $\angle TAD = \angle TAB +$

$\angle BAD = \gamma + \frac{1}{2}\alpha > \angle EDA$. Thus the ray $\overrightarrow{AF}$ has the same direction as $\overrightarrow{ED}$ and $\angle CAF = \angle CED$.

Let K be the intersection of the circle and the ray $\overrightarrow{AD}$ (it is the center of the arc BC). We showed that F is on the same side of AK as B. Moreover, $\angle KCF = \angle CAF - \frac{1}{2}\alpha = \angle CED - \frac{1}{2}\alpha < \angle CEB - \frac{1}{2}\alpha = \frac{1}{2}(\alpha + \beta) < \pi/2$. Finally, $\overrightarrow{FG}$ has the same direction as $\overrightarrow{BC}$. Thus $\angle AFG = \angle EDC$.

It follows from the arguments above that the points on the circle have either the order A, F, B, C, G or A, B, F, G, C. For each case $\angle ABG = \angle AFG$ and $\angle AGB = \angle ACB$. Thus the triangles CED and GAB are similar. The conclusion follows.

(If, by adding an additional assumption, we make $ABCG$ a convex quadrilateral, we can use Ptolemy's theorem to get the more interesting result that $GA = GB + GC$.)

Problem 4

The difference between the cubes of two consecutive positive integers is a square n^2, where n is a positive integer. Show that n is the sum of two squares.

Solution. Assume that $(m+1)^3 - m^3 = n^2$. Rearranging we get $3(2m+1)^2 = (2n+1)(2n-1)$. Since $2n+1$ and $2n-1$ are relatively prime (if they had a common divisor, it would have divided the difference, which is 2, but they are both odd), one of them is a square (of an odd integer, since it is odd) and the other divided by 3 is a square.

An odd number squared minus 1 is divisible by 4 since $(2t+1)^2 - 1 = 4(t^2 + t)$. From the first equation we see that n is odd, say $n = 2k+1$. Then $2n + 1 = 4k + 3$, so the square must be $2n - 1$, say $2n - 1 = (2t+1)^2$. Rearrangement yields $n = t^2 + (t+1)^2$.

An example: $8^3 - 7^3 = (2^2 + 3^2)^2$.