Let a_t be the population at time t. The expected population at time $t+1$ is

$$a_{t+1} = n \sum_{k=0}^{\infty} p_k \, \phi(k)$$

number of sites

probability that there are k individuals at site

interaction fuction

Need formula for p_k?

When n large p_k is well-approximated by a Poisson distribution

$$p_k = \frac{(a_t/n)^k \, e^{-a_t/n}}{k!}$$

For scramble competition $\phi(k) = 0$ except for $b=1$.

$$p_1 = \frac{a_t/n \; e^{-a_t/n}}{1}$$

$$\Rightarrow \quad a_{t+1} = n \cdot \frac{a_t}{n} \cdot e^{-a_t/n} \cdot b$$

$$= b\, a_t\, e^{-a_t/n}$$

For contest competition:

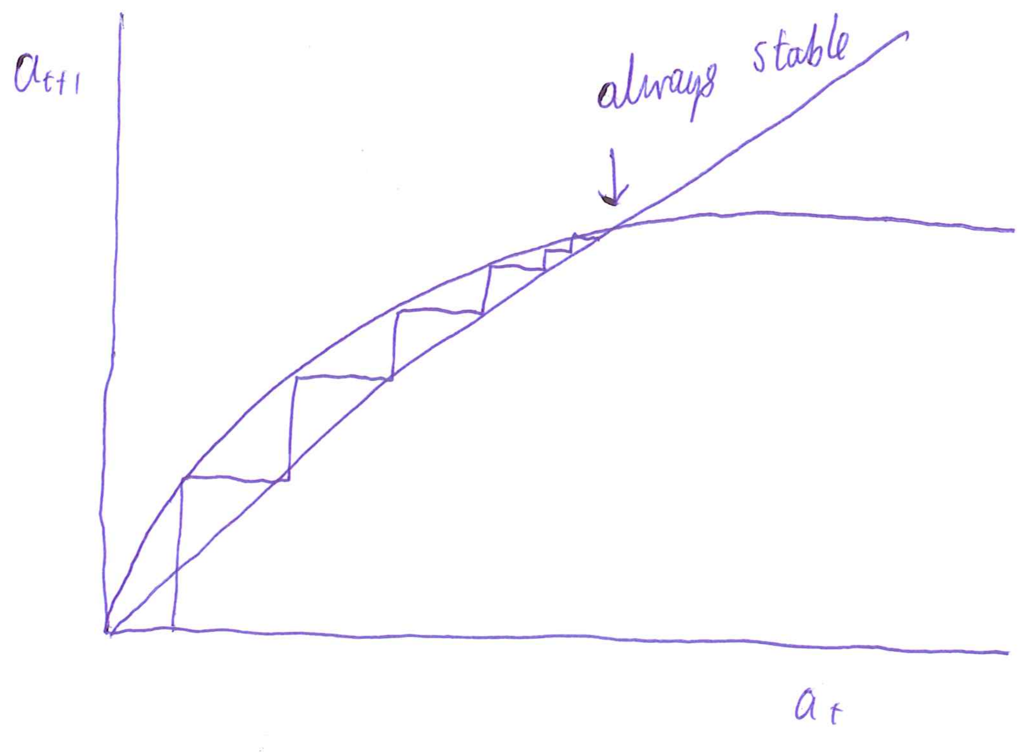
$$\phi(k) = b \quad \text{for} \quad k > 0$$

$$p_0 = 1 \cdot e^{-a_t/n}$$

$$1 - p_0 = 1 - e^{-a_t/n}$$

$$a_{t+1} = n(1-p_0) \cdot b$$

$$= bn\left(1 - e^{-a_t/n}\right)$$



<u>Entropy</u> is average "uncertainty" about the value of a random variable.

Let X be a r.v. with n possible outcomes. $p(x_i)$ is the probability of outcome i. The (Shannon) Entropy is,

$$H(X) = -\sum_{i=1}^{n} p(x_i)\log_2(p(x_i))$$

<u>Example 1</u>

$$X = \begin{cases} 1 & , \text{ with prob. } p \\ 0 & , \text{ with prob. } 1-p \end{cases}$$

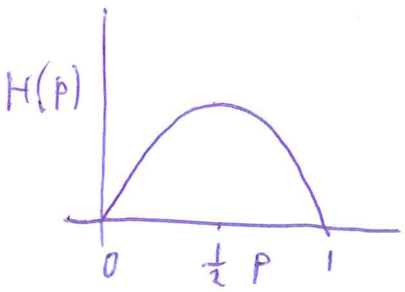
$$H(X) = -p\log_2(p) - (1-p)\log_2(1-p)$$

Expressing $H(p)$ as a function of p.

$$H(0) = 0 \quad , \quad H(1) = 0$$

$$H(\tfrac{1}{2}) = (-p - (1-p)) \cdot \log_2(\tfrac{1}{2})$$

$$= 1$$



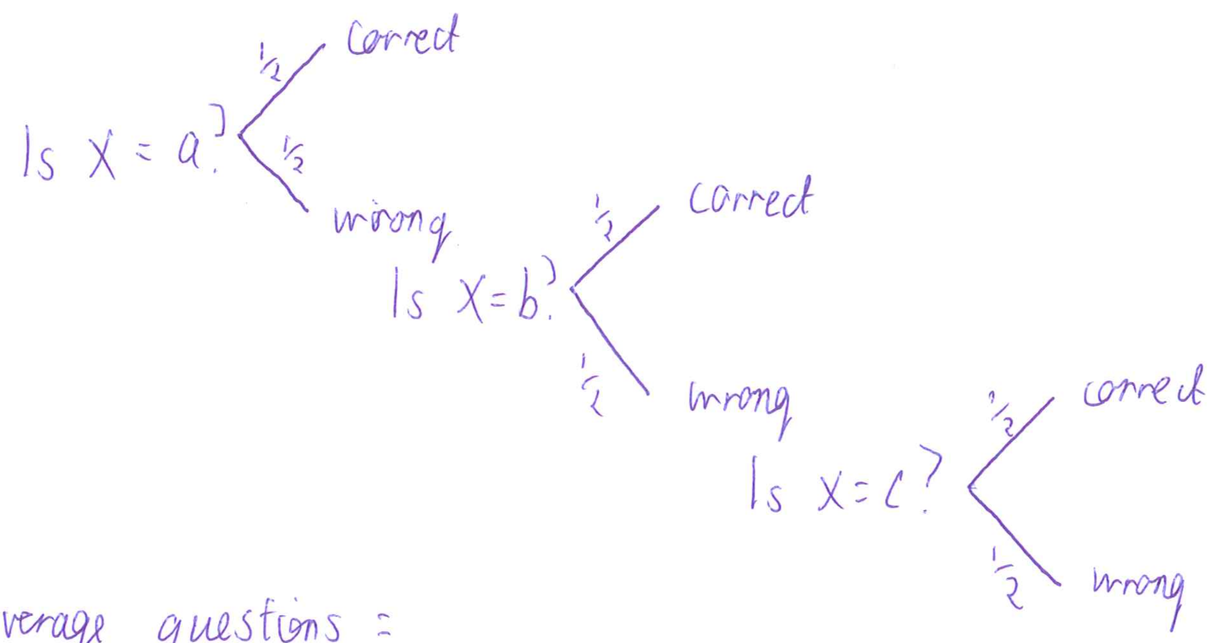
$\Rightarrow$ Entropy maximised when uncertainty highest.

Example 2

$$X = \begin{cases} a & \text{with prob. } 1/2 \\ b & \text{" \quad " \quad } 1/4 \\ c & \text{" \quad " \quad } 1/8 \\ d & \text{" \quad " \quad } 1/8 \end{cases}$$

$$H(X) = -\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{4}\log_2\frac{1}{4} - \frac{1}{8}\log_2\frac{1}{8} - \frac{1}{8}\log_2\frac{1}{8} = \frac{7}{4} = 1.75$$

Lets find X by asking questions.



Average questions =

$$\frac{1}{2}\cdot 1 + 2\cdot\left(\frac{1}{2}\right)^2 + 3\cdot\left(\frac{1}{2}\right)^3 = \frac{1}{2} + \frac{1}{2} + \frac{3}{\cancel{4}\,8}$$

$$= 1.75 = H(X)$$

In general, minimum expected number of binary questions to determine ~~H(X)~~ X is between

$H(X)$ and $H(X) + 1$

Lyapunov Exponent gives the rate at which nearby starting values of a model diverge.

Imagine two starting positions

$$x_A(0) \quad \text{and} \quad x_B(0)$$

Let $\Delta x(t) = x_A(t) - x_B(t)$

Then the Lyapunov Exponent λ is defined as

$$|\Delta x(t)| = |\Delta x(0)| \, e^{\lambda t}$$

$\lambda < 0$ then nearby trajectories converge

$\lambda > 0$ " " " diverge

Re-arrange,

$$\lambda = \frac{1}{t} \ln \left| \frac{\Delta x(t)}{\Delta x(0)} \right|$$

Note that,

$$\frac{\Delta x(t)}{\Delta x(0)} = \frac{\Delta x(1)}{\Delta x(0)} \cdot \frac{\Delta x(2)}{\Delta x(1)} \cdots \frac{\Delta x(t)}{\Delta x(t-1)}$$

Therefore,

$$\lambda = \frac{1}{t} \sum_{i=0}^{t-1} \ln \left| \frac{\Delta x(i+1)}{\Delta x(i)} \right|$$

Computing this for large t gives estimate of Lyapunov exponent

To measure Lyapunov exponent well one should start with different starting conditions and repeatedly calculate λ. These starting conditions should be sampled from the attractor.

i.e

$$\lambda = \sum_{i=1}^{n} p(x_i) \cdot \ln \left| \frac{F(x_i+\varepsilon) - F(x_i)}{\cancel{x_i}\,\varepsilon} \right|$$

proportion of time simulation is in state x_i

F – is one step of simulation.