COMPARISON OF TWO METHODS FOR SUPERREPLICATION

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Abstract. We compare two methods for superreplication of options with convex pay-off functions. One method entails an overestimation of the unknown covariance matrix in the sense of quadratic forms. With this method the value of the superreplicating portfolio is given as the solution of a linear Black-Scholes type equation. In the second method the choice of quadratic form is made pointwise. This leads to a fully non-linear equation, the so-called Black-Scholes-Barenblatt equation, for the value of the superreplicating portfolio. In general, this value is smaller for the second method than for the first method. We derive estimates for the difference between the initial values of the superreplicating strategies obtained using the two methods.

1. Introduction

In this article we study and compare two methods for supperreplication of convex contracts on several underlying assets. In general future volatility is, of course, not known and the best the hedger of a contract can do is to give estimates of possible future volatilities. Exact replication of the contract is thus not possible. In this situation it is of interest for the writer of the option to find a self-financing portfolio that superreplicates the claim, meaning that if the volatility stays within the given estimated region, then the value of the hedging portfolio is with probability one at least the option pay-off at expiry. Of course, given the estimates of future volatility, the hedger wants the initial value of his portfolio to be as small as possible, but he also wants his strategy to be as simple as possible to find numerically or perhaps even explicitly.

The first method is based on finding a matrix overestimating the covariance matrix in the sense of quadratic forms and is described in Section 2, compare also for example Ekström et al [3] or El Karoui et al [4]. Once this matrix is found, one needs to solve a classical Black-Scholes equation. The advantage of this method lies in the simplicity of the equation, the drawback is the fact that this solution often is unnecessarily large.

Another method for superreplication is via the so-called Black-Scholes-Barenblatt equation, see for example Avellaneda et al [1], Lyons [9], Romagnoli and Vargiolu [12], Vargiolu [13] and Gozzi and Vargiolu [5], [6]. This method is also described in Section 2. The Black-Scholes-Barenblatt equation is a fully non-linear parabolic equation of Hamilton-Jacobi-Bellman type. It is of course associated with more numerical work than the linear method but on the other hand gives a less expensive superreplicating portfolio.

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In the case of one underlying asset and a convex contract function the Black-Scholes-Barenblatt equation reduces to a Black-Scholes equation and these two methods thus agree. The second method, in its full non-linearity, is thus only used for claims with mixed convexity. However, for several underlying assets the methods are in general different even for convex claims. The purpose of the present article is to compare these two methods precisely in this case. In particular we estimate the extra cost associated with the linear method, compare Corollaries 3.5, 3.9, 3.11, 3.14 below. We also provide explicit numerical examples for some standard contracts.

2. Two Methods for Superreplication

Consider a model for a market consisting of a bank account with price process

$$B(t) = B(0)\exp\{rt\},$$

where the interest rate r is a non-negative constant, and n risky assets, with the price X_i of the ith asset satisfying the stochastic differential equation

$$dX_i = rX_i(t)\,dt + \sum_{j=1}^{n} \sigma_{ij} X_i(t)\,dW_j \tag{1}$$

under some risk neutral probability measure, compare for example [7]. In this equation W is an n-dimensional Brownian motion on some probability space (we do not further specify this space since X and W are only used for calculating the pricing function F below), and the volatility matrix $\sigma = (\sigma_{ij})$ is a non-singular $n \times n$-matrix. The process X is generally refered to as an n-dimensional geometric Brownian motion. We will consider options on X with convex pay-off structures, i.e. the holder of the option receives at some pre-determined time T the amount $g(X(T))$ for some non-negative and convex contract function g of at most polynomial growth. Standard arbitrage theory yields that the price at time t of the option is calculated as $F(X(t), t)$, where

$$F(x,t) = \exp\{-r(T-t)\}E_{x,t}\,g(X(T)). \tag{2}$$

Here the expected value is taken with respect to a measure under which $X_t = x$. Moreover, this pricing function F solves the Black-Scholes parabolic differential equation

$$\frac{\partial F}{\partial t} + \sum_{i=1}^{n} r x_i \frac{\partial F}{\partial x_i} + \frac{1}{2}\sum_{i,j=1}^{n} a_{ij} x_i x_j \frac{\partial^2 F}{\partial x_i \partial x_j} = rF \tag{3}$$

with terminal condition $F(x,T) = g(x)$. In this equation the coefficients a_{ij} are the entries of the $n \times n$-matrix $\sigma\sigma^*$. We will refer to this matrix as the *covariance matrix.*

Remark Note that since σ is assumed to be non-singular, the matrix $A = \sigma\sigma^* = (a_{ij})$ is positive definite. Recall that there is a 1-1-correspondence between the set of quadratic forms on $\mathbb{R}^n$ and the set of symmetric $n \times n$-matrices. In this article we identify the set of quadratic forms with the set of symmetric matrices.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be a filtered probability space, and let $\tilde{W}$ be a Brownian motion with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. We emphasize that the filtration generated by $\tilde{W}$ may be strictly contained in $(\mathcal{F}_t)_{t\in[0,T]}$, thus allowing for an incomplete market. Assume that an option writer wants to hedge an option using a geometric Brownian motion model for the stock price as described above, whereas the true stock price vector $\tilde{X}$ evolves according to

$$d\tilde{X}_i = \tilde{\mu}_i(t)\tilde{X}_i(t)\,dt + \sum_{j=1}^{n} \tilde{\sigma}_{ij}(t)\tilde{X}_i(t)\tilde{X}_j(t)\,d\tilde{W}_j \tag{4}$$

for some $\mathcal{F}_t$-adapted processes $\tilde{\mu}_i$ and $\tilde{\sigma}_{ij}$. Also assume that

$$\int_0^T |\tilde{\mu}_i(t)|\,dt < \infty \quad \text{and} \quad \int_0^T \tilde{\sigma}_{ij}^2(t)\,dt < \infty$$

$\mathbb{P}$-almost surely (henceforth the measure $\mathbb{P}$ is suppressed). The hedger will then (incorrectly) price an option on the stocks according to (2), or equivalently according to (3). Moreover, if he tries to replicate the option with the hedging strategy suggested by his model, then he will form a self-financing portfolio with initial value $F(X(0), 0)$ and such that it at each instant t contains $\frac{\partial F}{\partial x_i}(\tilde{X}(t), t)$ numbers of shares of the ith asset and the remaining amount invested in the bank account. It is well-known, see Theorem 3.1 in [3] or Theorem 6.2 in [4], that the terminal value of the strategy described above almost surely exceeds the option pay-off $g(X(T))$, provided that $\sigma\sigma^* \geq \tilde{\sigma}(t)\tilde{\sigma}^*(t)$ in the sense of quadratic forms for all t almost surely. Thus, if the hedger over-estimates the covariance matrix, then he will superreplicate the option. This we refer to as the first method or the linear method. For clarity we formulate it as a theorem.

Theorem 2.1. *Assume that a hedger over-estimates the covariance matrix, i.e the volatility matrix σ used by the hedger satisfies $\sigma\sigma^* \geq \tilde{\sigma}(t)\tilde{\sigma}^*(t)$ for all t almost surely. Then the hedger will superreplicate any convex claim written on X.*

Remark For the first method to work it is important that the price of the option, calculated via the Black-Scholes equation, is convex as a function of the stock price vector at any time before maturity. It is the somewhat surprising result of [3], that within a rather large class of models, geometric Brownian motion is the only model in which the price of a convex claim necessarily is convex in the underlying stock price.

We now describe the second method. By some a priori estimate the hedger has determined that the covariance matrix lies in some set $\mathcal{A}$ of strictly positive quadratic forms. We call this set $\mathcal{A}$ the set of *admissible covariance matrices*, and we assume throughout the article the following hypothesis.

Hypothesis 2.2. *The set $\mathcal{A}$ of admissible covariance matrices is convex and compact.*

Using the set of admissible covariance matrices, the hedger calculates the price of the option according to the so-called Black-Scholes-Barenblatt equation

$$\frac{\partial H}{\partial t} + \sum_{i=1}^{n} r x_i \frac{\partial H}{\partial x_i} + \max_{A \in \mathcal{A}} \frac{1}{2} \sum_{i,j=1}^{n} a_{ij} x_i x_j \frac{\partial^2 H}{\partial x_i \partial x_j} = rH \tag{5}$$

with the same terminal condition $H(x, T) = g(x)$ as for the equation (3). For uniqueness, existence and regularity of solutions to this equation see [13]. Now, consider a hedger who forms a self-financing portfolio with initial value $H(X(0), 0)$ and such that it at each instant t contains $\frac{\partial H}{\partial x_i}(\tilde{X}(t), t)$ numbers of shares of the ith asset and the remaining amount invested in the bank account. Then it is known that he will superreplicate the claim $g(\tilde{X}(T))$ provided that the true covariance matrix $\tilde{\sigma}(t)\tilde{\sigma}^*(t)$ at each instant is in $\mathcal{A}$, compare Theorem 2 in [12].

Theorem 2.3. *Assume that an agent hedges a claim according to the Black-Scholes-Barenblatt equation for some set $\mathcal{A}$ of admissible covariance matrices. Then the agent will superreplicate the claim provided that the true covariance matrix $\tilde{\sigma}(t)\tilde{\sigma}^*(t)$ is in the admissible set $\mathcal{A}$ for all t almost surely. Moreover, the initial value of the superreplicating strategy calculated via the Black-Scholes-Barenblatt equation is the smallest possible initial value of a strategy which is superreplicating for any covariance matrix in $\mathcal{A}$.*

Remark Theorem 2.3 relies upon the non-degeneracy of the admissible covariances, which guarantees the regularity of the solution to the Black-Scholes-Barenblatt equation, see [13]. Non-degeneracy of the covariance matrix is also connected to absence of arbitrage in the market.

Remark Theorem 2.3 is valid not only for convex claims but also for any continuous contract function g of at most polynomial growth.

Remark For the existence of a smallest superhedging portfolio in more general contexts, see Kramkov [8] and Mykland [10]. The Black-Scholes-Barenblatt price, in the set-up of the present paper, coincides with Kramkov's superhedging price and with Mykland's conservative delta hedging price.

3. Estimates of Solutions of the Black-Scholes-Barenblatt Equation

The initial values of the superreplicating portfolios in Theorems 2.1 and 2.3 (or, equivalently, the solutions F and H to equations (3) and (5)) will be refered to as *the BS-price* and *the BSB-price*, respectively. We will use the notation $BS(A)$ for the BS-price computed with a volatility σ satisfying $A = \sigma\sigma^*$, and similarly we denote by $BSB(\mathcal{A})$ the BSB-price corresponding to a set $\mathcal{A}$ of admissible covariance matrices.

In this section we give bounds of the BSB-price in terms of the BS-price.

Definition 3.1. *We say that a quadratic form C dominates a set $\mathcal{A}$ of quadratic forms if $C \geq A$ for all $A \in \mathcal{A}$.*

The following result is a consequence of Theorem 2.1 and Theorem 2.3.

Theorem 3.2. *Assume that the contract function g is convex and that C is a quadratic form that dominates $\mathcal{A}$. Then*

$$BS(A) \leq BSB(\mathcal{A}) \leq BS(C)$$

for any $A \in \mathcal{A}$.

Proof. The first inequality follows since $BS(A) = BSB(A)$ and the BSB-price is increasing in $\mathcal{A}$, and the second one follows since the BSB-price is the smallest superreplicating price. □

Remark There is a similar result corresponding to subreplication. If D is an element that is dominated by all elements in $\mathcal{A}$, then $BS(D) \leq \underline{BSB}(\mathcal{A}) \leq BS(A)$ for any convex pay-off and any $A \in \mathcal{A}$, where $\underline{BSB}$ solves equation (5) but with taking the maximum over $\mathcal{A}$ replaced by taking the minimum. The results below therefore have counterparts for subreplicating portfolios.

Since the set $\mathcal{A}$ of admissible covariance matrices is compact, the eigenvalues of the elements in $\mathcal{A}$ are uniformly bounded by some constant. Thus it is easy to find an upper bound of $\mathcal{A}$. However, in general there is no *smallest* upper bound of $\mathcal{A}$. Different choices of the dominating covariance matrix yields different bounds of the BSB-price according to Theorem 3.2. Below we discuss some different choices of dominating strategy for $\mathcal{A}$. Recall that $\mathcal{A}$ is assumed to be convex and compact.

Theorem 3.3. *Let C be a quadratic form with largest possible determinant in $\mathcal{A}$. Then nC dominates $\mathcal{A}$.*

Proof. Note first that since $\mathcal{A}$ is compact, there exists C in $\mathcal{A}$ such that

$$\det C = \sup_{A \in \mathcal{A}} \det A.$$

Let A be an arbitrary quadratic form in $\mathcal{A}$. By Lemma 3.4 below we may assume, without loss of generality, that both A and C are diagonal. Since $\mathcal{A}$ is convex (convexity of $\mathcal{A}$ is preserved when diagonalizing A and C), the quadratic form $C(\lambda) := \lambda A + (1-\lambda)C$ is in $\mathcal{A}$. Using the fact that C has maximal determinant in $\mathcal{A}$ we know that

$$\frac{d}{d\lambda} \det C(\lambda)\bigg|_{\lambda=0} \le 0. \tag{6}$$

Denoting the diagonal elements of A and C by a_i and c_i, respectively, straightforward calculations yield that

$$\begin{aligned}
\frac{d}{d\lambda} \det C(\lambda)\bigg|_{\lambda=0} &= \frac{d}{d\lambda} \det \begin{pmatrix} \lambda a_1 + (1-\lambda)c_1 & & 0 \\ & \ddots & \\ 0 & & \lambda a_n + (1-\lambda)c_n \end{pmatrix}\Bigg|_{\lambda=0} \\
&= \frac{d}{d\lambda} \prod_{i=1}^{n} (\lambda a_i + (1-\lambda)c_i)\bigg|_{\lambda=0}.
\end{aligned}$$

The product in the above expression is a polynomial in λ of degree n. When differentiating it and plugging in $\lambda = 0$, only the coefficient of the first order term survives. Thus

$$\begin{aligned}
&\frac{d}{d\lambda} \prod_{i=1}^{n} (\lambda a_i + (1-\lambda)c_i)\bigg|_{\lambda=0} \\
&\qquad = c_1 \dots c_{n-1} a_n + c_1 \dots c_{n-2} a_{n-1} c_n + \dots + a_1 c_2 \dots c_n - n c_1 \dots c_n.
\end{aligned}$$

Combining this with (6), we find that

$$c_1 \dots c_{n-1} a_n + c_1 \dots c_{n-2} a_{n-1} c_n + \dots + a_1 c_2 \dots c_n \le n c_1 \dots c_n.$$

In particular, since a_i and c_i are all positive, it follows that $a_i \le n c_i$ for all $i = 1, \dots, n$. Consequently, nC dominates A, which finishes the proof. □

We include the following elementary result.

Lemma 3.4. *Two quadratic forms on $\mathbb{R}^n$, one of which is positive definite, can be diagonalized simultaneously. Moreover, this can be done without changing the determinants of the representing matrices.*

Proof. Let the two quadratic forms be represented by the matrices A and C in a given basis, where A is positive definite. Let T be a change of basis matrix. It is important to note that after performing the corresponding change of basis, the new representations of the quadratic forms are given by T^*AT and T^*CT, respectively. (If we regard A and C as linear mappings, the new representations are $T^{-1}AT$ and $T^{-1}CT$. This distinction disappears if T is an orthogonal transformation since then $T^* = T^{-1}$.) Now, we first perform an orthogonal transformation so that $T_1^*AT_1 = T_1^{-1}AT_1$ is a diagonal matrix with the entries $\lambda_1, ..., \lambda_n$. Next we want to transform this matrix to the identity matrix. This is accomplished using the diagonal transformation matrix

$$T_2 = \begin{pmatrix} \frac{1}{\sqrt{\lambda_1}} & & 0 \\ & \ddots & \\ 0 & & \frac{1}{\sqrt{\lambda_n}} \end{pmatrix}.$$

Then $T_2^*T_1^*AT_1T_2$ is the identity matrix. Note that in this step it is crucial that we use the transpose T_2^* of T_2 and not the inverse T_2^{-1}. After these two transformations we are left with the matrices $T_2^*T_1^*AT_1T_2$ and $T_2^*T_1^*CT_1T_2$, the first of which is the identity matrix and the second one is symmetric. Now, finally, we choose an orthogonal transformation T_3 that diagonalizes the second matrix, which can be done since that matrix is symmetric. Since $T_3^* = T_3^{-1}$, the identity matrix is unaffected by this change of basis. Consequently, the change of basis matrix $T_1T_2T_3$ diagonalizes the two quadratic forms simultaneously.

If, in addition, we want the determinants of the representing matrices to be preserved, T_2 should be replaced by $T_2 \det(A)^{1/2n}$. This completes the proof. □

As a consequence of Theorems 3.2 and 3.3 we have the following result.

Corollary 3.5. *Let C be a quadratic form with largest possible determinant in $\mathcal{A}$. Then*

$$BS(C) \leq BSB(\mathcal{A}) \leq BS(nC).$$

Remark The assumption about $\mathcal{A}$ being convex is essential in Theorem 3.3. For example, it is easy check that there is no element C in

$$\mathcal{A} := \left\{ \begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in [1/3, 3] \right\}$$

such that $2C$ dominates $\mathcal{A}$.

Example 3.6. Corollary 3.5 states that the BSB-price for some given set $\mathcal{A}$ always can be estimated by a BS-price for a multiple of some quadratic form in $\mathcal{A}$. Further, the constant n cannot be improved upon. To see this, consider the two-dimensional case, let $a > 0$ and

$$\mathcal{A} := \left\{ \begin{pmatrix} \gamma & 0 \\ 0 & a-\gamma \end{pmatrix} : \gamma \in [\epsilon, a-\epsilon] \right\}$$

for some small $\epsilon > 0$. By symmetry, the best choice $A \in \mathcal{A}$ is the mid-point, i.e. the quadratic form

$$A = \begin{pmatrix} a/2 & 0 \\ 0 & a/2 \end{pmatrix},$$

and it is easy to see that $\lambda = 2 - 2\epsilon/a$ is the smallest constant such that λA majorizes $\mathcal{A}$. Since $\epsilon > 0$ is arbitrary, the best constant in general is $n = 2$.

Also note that a similar construction works in higher dimensions.

One natural way of expressing an estimate of future volatility is simply to give upper and lower bounds for each of the entries of the covariance matrix. In this case Theorem 3.3 can be significantly improved.

Definition 3.7. *Let Q be the set of positive quadratic forms in n dimensions. We say that $\mathcal{A}$* is a box *if*

$$\mathcal{A} = \left\{ A = (a_{ij})_{i,j=1}^n : a_{ij} = a_{ji} \in I_{ij} \right\} \subseteq Q$$

for some closed intervals $I_{ij} = I_{ji} = [l_{ij}, r_{ij}]$, $1 \le i, j \le n$.

Next we present a method which shows that if all quadratic forms in $\mathcal{A}$ are close to a diagonal quadratic form, i.e. if the correlations between the assets are small, then the solution to the Black-Scholes-Barenblatt equation can be approximated very well with solutions to the Black-Scholes equation with a diagonal quadratic form.

Lemma 3.8. *Assume that $\mathcal{A}$ is a box. For each position (i, j), let $\hat{a}_{ij} := \sup\{|a_{ij}| : a_{ij} \in I_{ij}\}$. Next, let $C = (c_{ij})$ be the diagonal positive quadratic form defined by $c_{ii} = \sum_{j=1}^n \hat{a}_{ij}$. Then C majorizes $\mathcal{A}$.*

Proof. Let $A = (a_{ij})$ be a quadratic form in $\mathcal{A}$ and let $\xi \in \mathbb{R}^n$. Then

$$\begin{aligned}
\sum_{i,j}^n a_{ij}\xi_i\xi_j &= \sum_{i=1}^n a_{ii}\xi_i^2 + \sum_{i \ne j} a_{ij}\xi_i\xi_j \\
&\le \sum_{i=1}^n \hat{a}_{ii}\xi_i^2 + \sum_{i \ne j} \hat{a}_{ij}\frac{1}{2}(\xi_i^2 + \xi_j^2) \\
&= \sum_{i=1}^n \hat{a}_{ii}\xi_i^2 + \sum_{i \ne j} \hat{a}_{ij}\xi_i^2 = \sum_{i=1}^n c_{ii}\xi_i^2,
\end{aligned}$$

finishing the proof. □

Corollary 3.9. *With the notation of Lemma 3.8 we have*

$$BS(A) \le BSB(\mathcal{A}) \le BS(C),$$

where A is any quadratic form in $\mathcal{A}$.

Remark Lemma 3.8 can be strengthened by using the inequality

$$2\xi_i\xi_j \le \lambda\xi_i^2 + \frac{1}{\lambda}\xi_j^2$$

for $\lambda > 0$ (this inequality is used in the proof with $\lambda = 1$). It follows that given any matrix (λ_{ij}) satisfying $\lambda_{ij} = 1/\lambda_{ji} > 0$, the quadratic form C can be defined as the diagonal matrix where the ith diagonal entry equals $\sum_{j=1}^n \lambda_{ij}\hat{a}_{ij}$.

Theorem 3.10. *Assume that $\mathcal{A}$ is a box. Then there is an element $C \in \mathcal{A}$ such that $2C$ dominates $\mathcal{A}$.*

Proof. Let C be the element in $\mathcal{A}$ with off-diagonal elements defined as the mid-point of the interval and diagonal elements as large as possible. More precisely, in the notation of Definition 3.7, $C = (c_{ij})$ is given by $c_{ii} = r_{ii}$ and $c_{ij} = (l_{ij} + r_{ij})/2$ for $i \ne j$. It suffices to check that $2C \ge A$ for matrices A in $\mathcal{A}$ with maximal diagonal entries. This, however, follows from the fact that $2C - A \in \mathcal{A}$ for every such A. □

Corollary 3.11. *If $\mathcal{A}$ is a box, then there exists $C \in \mathcal{A}$ such that*

$$BS(C) \leq BSB(\mathcal{A}) \leq BS(2C).$$

Next we derive estimates for the case when $\mathcal{A}$ is a thin box.

Theorem 3.12. *Assume that $\mathcal{A}$ is a box. Let C be the "upper center" of the box (mid-point in the off-diagonal intervals, largest element on the diagonal), and let $\epsilon_{\max}$ be the length of the longest off-diagonal side in the box. Further, let $\lambda_{\min}$ be the smallest eigenvalue of C. Then $(1+\delta)C$, where $\delta = \frac{(n-1)\epsilon_{\max}}{2\lambda_{\min}}$, dominates $\mathcal{A}$.*

Proof. It suffices to check that $(1+\delta)C$ majorizes all elements A in $\mathcal{A}$ having maximal diagonal entries. Thus, let A be such an element. Writing $D = A - C$, we find that D is a quadratic form $D = (d_{ij})$ with $d_{ii} = 0$ and $d_{ij} \in \frac{1}{2}[-\epsilon_{\max}, \epsilon_{\max}]$ for $i \neq j$. Using for example Lemma 3.8 it can be checked that $D \leq \frac{(n-1)\epsilon_{\max}}{2} I$, where I is the identity matrix. Now we need to check that $(1+\delta)C \geq A = C + D$, so the inequalities

$$\delta C \geq \delta\lambda_{\min} I = \frac{(n-1)\epsilon_{\max}}{2} I \geq D$$

finish the proof. □

The estimate in Theorem 3.12 is not sharp. Indeed, the factor $1 + \frac{(n-1)\epsilon_{\max}}{2\lambda_{\min}}$ is in general not the best (smallest) possible factor. In two dimensions we have the following sharp estimate.

Theorem 3.13. *Assume that $\mathcal{A}$ is a box, that $n = 2$ and let ϵ be the length of the off-diagonal interval. Then there exists a quadratic form $C \in \mathcal{A}$ such that $(1+\delta)C$, where $\delta = \frac{\epsilon}{2\sqrt{r_{11}r_{22}}}$, dominates $\mathcal{A}$. Moreover, $\gamma = \delta$ is the smallest number such that $(1+\gamma)D$ dominates $\mathcal{A}$ for some $D \in \mathcal{A}$.*

Proof. Define

$$c := \frac{\sqrt{r_{11}r_{22}}(l_{12} + r_{12})}{r_{12} - l_{12} + 2\sqrt{r_{11}r_{22}}}.$$

We claim that $C = \begin{pmatrix} r_{11} & c \\ c & r_{22} \end{pmatrix}$ will do. Indeed, to show that $C \in \mathcal{A}$ we need to check that

$$l_{12} \leq c \leq r_{12},$$

and this is readily verified using the inequalities $r_{11}r_{22} \geq l_{12}^2$ and $r_{11}r_{22} \geq r_{12}^2$. Moreover, to check that $(1+\delta)C$ dominates $\mathcal{A}$, it suffices to show that $(1+\delta)C$ dominates the elements in $\mathcal{A}$ of the form $A = \begin{pmatrix} r_{11} & z \\ z & r_{22} \end{pmatrix}$ for $z = l_{12}$ and $z = r_{12}$. Since the diagonal elements of $(1+\delta)C - A$ are clearly positive, we only need to check that the determinant of $(1+\delta)C - A$ is non-negative. Thus

$$\det((1+\delta)C - A) = \det \begin{pmatrix} \delta r_{11} & \pm\frac{\epsilon}{2} \\ \pm\frac{\epsilon}{2} & \delta r_{22} \end{pmatrix} = 0 \tag{7}$$

which finishes the first part of the theorem.

To see that $\delta = \frac{\epsilon}{2\sqrt{r_{11}r_{22}}}$ is the best constant, let $D \in \mathcal{A}$. Without loss of generality we may assume that the diagonal elements are r_{11} and r_{22}, respectively.

Now, if the off-diagonal element of D is different from c, then the determinant $\det((1+\gamma)D - A)$, compare (7), is negative for either $z = l_{12}$ or $z = r_{12}$ unless γ is chosen strictly larger than δ. This finishes the proof. □

Corollary 3.14. *Assume that $\mathcal{A}$ is a box. Then there exists $C \in \mathcal{A}$ such that, with the notation of Theorem 3.12 (Theorem 3.13, respectively),*

$$BS(C) \leq BSB(\mathcal{A}) \leq BS((1+\delta)C).$$

Remark If the interest rate $r = 0$, then the notions "volatility" and "time to maturity" are equivalent. More precisely, $BS((1+\delta)C, \tau) = BS(C, (1+\delta)\tau)$, where $BS(C, \tau)$ is the Black-Scholes price when using a covariance structure C and the time to maturity is τ. (This can be seen for example by inserting $r = 0$ in equation (3).) Thus the bounds above can be re-expressed as

$$BS(C, \tau) \leq BSB(\mathcal{A}, \tau) \leq BS(C, (1+\delta)\tau).$$

Consequently, the bounds are expressed in terms of Black-Scholes prices with the same volatility structures but with different times to maturity.

4. Numerical examples

In this section we illustrate our results by means of three explicit examples. In a market consisting of two different risky assets we consider the spread call option, the call option on the sum of the two assets, and a call option on the maximum of the two assets. The value of the model parameters are chosen in line with the results of many statistical studies, see for example pages 912-914 in [2] and Table 3 in [11].

We assume that the volatility matrix σ is uncertain, but that the covariance matrix $\sigma\sigma^*$ stays within a box

$$\mathcal{A} = \left\{ A = (a_{ij})_{i,j=1}^2 : a_{ij} = a_{ji} \in I_{ij} \right\}$$

for some closed intervals $I_{11} = [l_{11}, r_{11}]$, $I_{22} = [l_{22}, r_{22}]$ and

$$I_{12} = I_{21} = [(\rho - \epsilon)\sqrt{r_{11}r_{22}}, (\rho + \epsilon)\sqrt{r_{11}r_{22}}].$$

Here $\rho \in (-1, 1)$ and $\epsilon > 0$ are constants satisfying $\epsilon < \sqrt{\frac{r_{11}r_{22}}{l_{11}l_{22}}} - |\rho|$ (this inequality guarantees that the box $\mathcal{A}$ consists of positive quadratic forms).

Consider an option with a convex pay-off function g. According to Theorem 3.13 and Corollary 3.14, the BS-price corresponding to the covariance matrix

$$C = \begin{pmatrix} r_{11} & \rho\sqrt{r_{11}r_{22}}/(1+\epsilon) \\ \rho\sqrt{r_{11}r_{22}}/(1+\epsilon) & r_{22} \end{pmatrix}$$

and the Black-Scholes-Barenblatt price satisfy

$$BS(C) \leq BSB(\mathcal{A}) \leq BS((1+\epsilon)C).$$

In Table 1 below we provide the BSB-price and the corresponding price bounds of a spread call option for some different values of the model parameters. The pay-off of a spread call option is $g(X_1(T), X_2(T)) = (X_1(T) - X_2(T) - K)^+$.

Similarly, in Tables 2 and 3 we give the prices and bounds for call options on the sum and on the maximum, respectively. The corresponding pay-off functions are $g(X_1(T), X_2(T)) = (X_1(T) + X_2(T) - K)^+$ and $g(X_1(T), X_2(T)) = (\max\{X_1(T), X_2(T)\} - K)^+$.

	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.2$
$\rho = 0$	[1.48, 1.50, 1.50]	[1.48, 1.51, 1.51]	[1.48, 1.55, 1.55]
$\rho = 0.3$	[1.38, 1.39, 1.39]	[1.38, 1.41, 1.41]	[1.39, 1.45, 1.45]

TABLE 1. Price bounds for the BSB-price of a spread call option. The first entry is the BS(C)-price, the middle entry is the BSB($\mathcal{A}$)-price, and the third one is the BS($(1+\epsilon)C$)-price. The parameters used are $T = 0.5$, $r = 0.06$, $x_1 = 11$, $x_2 = 6$, $K = 4$, $r_{11} = 0.04$ and $r_{22} = 0.09$.

	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.2$
$\rho = 0$	[1.50, 1.52, 1.53]	[1.50, 1.55, 1.56]	[1.50, 1.59, 1.63]
$\rho = 0.3$	[1.64, 1.67, 1.67]	[1.63, 1.69, 1.70]	[1.62, 1.73, 1.76]

TABLE 2. Price bounds for the BSB-price of a call option on the sum of the two assets. The first entry is the BS(C)-price, the middle entry is the BSB($\mathcal{A}$)-price, and the third one is the BS($(1+\epsilon)C$)-price. The parameters used are $T = 1$, $r = 0.03$, $x_1 = 4$, $x_2 = 6$, $K = 10$, $r_{11} = 0.16$ and $r_{22} = 0.25$.

	$\epsilon = 0.05$	$\epsilon = 0.1$	$\epsilon = 0.2$
$\rho = 0$	[2.40, 2.42, 2.46]	[1.23, 1.25, 1.29]	[1.23, 1.27, 1.36]
$\rho = 0.3$	[2.25, 2.27, 2.31]	[2.26, 2.30, 2.38]	[2.27, 2.35, 2.51]

TABLE 3. Price bounds for the BSB-price of a call option on the maximum of the two assets. The first entry is the BS(C)-price, the middle entry is the BSB($\mathcal{A}$)-price, and the third one is the BS($(1+\epsilon)C$)-price. The parameters used are $T = 0.5$, $r = 0.03$, $x_1 = 10$, $x_2 = 9$, $K = 10$, $r_{11} = 0.25$ and $r_{22} = 0.36$.

For the spread call and for the call on the sum, the BSB-price is very close to the upper bound $BS((1+\epsilon)C)$. This suggests that the simple model with a constant covariance matrix $(1+\epsilon)C$ can be used for super-replication at a small additional cost compared to the BSB-price. However, for the call on the maximum of two assets, the BSB-price is closer to the lower bound. It remains a delicate open problem to determine for what contracts the upper bound gives a good approximation of the BSB-price.

5. Concluding Remarks

In this paper we compare two different approaches to superreplication of convex claims in the presence of model uncertainty. One method is based on overestimation of the covariance matrix, and it boils down to solving a linear partial differential equation. The second method gives a less expensive superreplicating portfolio, but is more complex since a fully non-linear differential equation needs to be solved. We provide different bounds for the solution to this non-linear BSB-equation in terms of solutions to the simpler BS-equation.

It is reasonable to assume that the set $\mathcal{A}$ of admissible covariance matrices is convex. In Theorem 3.3 we present a general method for finding a matrix $C \in$

$\mathcal{A}$ such that nC dominates $\mathcal{A}$. As Example 3.6 shows, the factor n cannot be improved in general. Consequently, the bounds for the solution to the Black-Scholes-Barenblatt equation provided in Theorem 3.3 and Corollary 3.5 are typically rather coarse. Especially in high dimensions, the bounds have no practical applications.

On the other hand, if the set $\mathcal{A}$ of admissible covariance matrices is small, then the bounds can be improved significantly. Theorems 3.12 and 3.13 provide the bounds

$$BS(C) \leq BSB(\mathcal{A}) \leq BS((1+\delta)C)$$

for some $C \in \mathcal{A}$ and some δ which is small if $\mathcal{A}$ is a small box. Moreover, for some options, the BSB-price appears to be close to the upper bound. In such cases, the method in which the covariance matrix is over-estimated could be used, at a small additional cost, to avoid the numerical complexity of the BSB-equation. However, as Table 3 shows, this does not hold for all contracts. To determine for which options the upper bound is a close approximation of the BSB-price is a challenging open question.

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