DUPIRE'S EQUATION FOR BUBBLES

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Abstract. We study Dupire's equation for local volatility models with bubbles, i.e. for models in which the discounted underlying asset follows a strict local martingale. If option prices are given by risk-neutral valuation, then the discounted option price process is a true martingale, and we show that the Dupire equation for call options contains extra terms compared to the usual equation. However, the Dupire equation for put options takes the usual form. Moreover, uniqueness of solutions to the Dupire equation is lost in general, and we show how to single out the option price among all possible solutions. The Dupire equation for models in which the discounted derivative price process is merely a local martingale is also studied.

1. Introduction

Financial bubbles have been studied extensively over the last few years, see for instance [5], [7], [8], [10], [12] and [13]. It has been suggested to use models in which the underlying discounted price process is a *strict* local martingale under the pricing measure. Such models are known to exhibit several anomalies. For example, if option prices are given by risk-neutral valuation, then a call option price is not necessarily convex as a function of the spot price of the underlying, the put-call parity fails in its usual form, and the uniqueness of solutions to the corresponding Black-Scholes equation is lost.

The Dupire equation is a forward equation for the call option price C as a function of the strike price K and the time to maturity T. It is argued in [6] that if the underlying stock price process follows a local volatility model, then the call option price satisfies

$$\left\{ \begin{array}{ll} C_T(K,T) = \mathcal{L}C(K,T) & \text{for } (K,T) \in (0,\infty)^2 \\ C(K,0) = (x-K)^+, & \end{array} \right. \tag{1}$$

where $\mathcal{L}$ is the second order differential operator

$$\mathcal{L} = \frac{\sigma^2(K,T)}{2}\frac{\partial^2}{\partial K^2} - (r-q)K\frac{\partial}{\partial K} - q.$$

Here r is the interest rate, q is the continuous dividend yield, x is the current stock price and σ is a local volatility function that grows at most linearly in the spatial variable. Since call prices for different strikes and maturities are

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observable quantities, the Dupire equation is commonly used to recover the volatility from C and its derivatives with respect to K and T.

If no linear bound is imposed on the volatility at spatial infinity, then the discounted underlying asset is only guaranteed to be a local martingale, and, as we will see in Section 2, it is easy to check that the Dupire equation fails in its usual form described above. In the present paper we consider Dupire type equations for such local volatility models with bubbles. In our main result, Theorem 2.2, we show that if option prices are given by risk-neutral valuation, then the Dupire equation for call options contains extra terms. Surprisingly, the corresponding equation for put options does not contain these extra terms, and is therefore perhaps better suited for calibration issues. As is well-known for the corresponding Black-Scholes equation for bubbles, see [5], [8] or [10], special care is needed to ensure the uniqueness of solutions. We show that the option price is the unique classical solution of the Dupire equation with a bounded distance to the pay-off function.

Note that in our general result Theorem 2.2, option prices are assumed to be given by risk-neutral valuation so that discounted option prices processes are true martingales. However, several alternative definitions of the price of an option, for which the discounted option prices process is merely a local martingale, have been suggested in the literature. Examples of such alternative prices motivated by intermediate collateral requirements, compare [5], are treated in Theorem 2.3.

Even though the Dupire equation is of both theoretical interest and of practical use, the academic literature is somewhat sparse. In [14], the Dupire equation is derived for processes that cannot reach the boundary and with exponential Levy jumps. Although the main objective for us is to study the Dupire equation for volatilities with a super-linear growth at spatial infinity, we point out that our study also covers the case in which the underlying process may reach zero with positive probability. Since knowing the distribution of the stock price is equivalent to knowing all call option prices, this provides insight in what boundary conditions to impose on the forward equation for densities, compare [16].

In Section 2 we present the local volatility model, and we state our main results Theorems 2.2 and 2.3 about existence and uniqueness of solutions to the Dupire equation. We also discuss how to use Theorems 2.2 and 2.3 for calibration of models. Finally, Section 3 contains the proof of Theorem 2.2.

Remark After the completion of the current article, we were informed about the preprint [2] by Bentata and Yor. In Theorem 2.2 of that article, the Dupire equation for put options written on local martingales is determined using stochastic methods. Furthermore, Bentata and Yor also allow for non-Markovian processes. On the other hand, our PDE-based approach includes a study of boundary conditions and uniqueness for the Dupire equation not performed in [2].

2. Pricing equations for bubbles

We let the risk free rate be a constant $r \geq 0$ and we assume that the stock pays a continuous dividend yield $q \geq 0$. Under the risk neutral measure, the stock price process X is modeled by

$$\left\{ \begin{array}{l} dX(t) = (r-q)X(t)\,dt + \sigma(X(t),t)\,dB(t) \\ X(0) = x, \end{array} \right. \tag{2}$$

where σ is a given local volatility function and B is a standard Brownian motion. The current stock price $x > 0$ denotes throughout the paper a given constant. If the boundary state zero can be reached in finite time, then we assume that zero is an absorbing barrier for the process X. By Ito's formula, the process $e^{-(r-q)t}X(t)$ is a local martingale, but not necessarily a martingale. Processes X for which $e^{-(r-q)t}X(t)$ is a strict local martingale have been suggested to model financial bubbles, compare [5] and [10].

Hypothesis 2.1. *The volatility function $\sigma : (0,\infty) \times [0,\infty) \to (0,\infty)$ is continuous. Moreover, it is locally Hölder(1/2) in the first variable, i.e. for any compact set $[D^{-1}, D] \times [0, D]$ there exists a constant C such that $|\sigma(x,t) - \sigma(y,t)| \leq C|x-y|^{1/2}$ for all $(x,y,t) \in [D^{-1}, D]^2 \times [0, D]$. Furthermore, there exists a constant A such that $\sigma(x,t) \leq A$ for $x \leq 1$.*

Remark Note that we do not require the volatility to be of at most linear growth at spatial infinity. Thus we allow for example models in which $\sigma(x,t)$ grows at least like $x^{1+\eta}$ for large x, where $\eta > 0$. In fact, in any such model, the process $e^{-(r-q)t}X(t)$ is a strict local martingale, see [3] and [8]. Also note that, regardless of the growth rate of σ at infinity, equation (2) has a unique solution that exists for all $t \geq 0$. Indeed, the linear bound at infinity is usually used to avoid exploding solutions; however, in the present context the process $e^{-(r-q)t}X(t)$ is automatically a supermartingale and hence does not explode.

In the case of bubbles in the underlying asset price X, several different definitions of the price of a derivative security written on X have been proposed in the literature. We mainly study the discounted expected values

$$C(K,T) := e^{-rT}E(X(T)-K)^+ \tag{3}$$

and

$$P(K,T) := e^{-rT}E(K-X(T))^+ \tag{4}$$

for different non-negative values of the strike price K and maturity dates T. By construction, defining option prices as discounted expected values implies that the corresponding discounted derivative price processes are martingales (and not merely local martingales). These discounted expected values coincide with the smallest initial fortune needed to superreplicate the corresponding option, compare Theorem 3.3 in [5]. We will refer to these expected values as the prices of call options and put options, respectively. However, at the end of the present section we return to the issue of other possible definitions of the price of a derivative.

As mentioned in the introduction, there are some subtleties for the Dupire equation in the presence of bubbles. For example, the call price C given by risk-neutral valuation does not satisfy the classical version (1) of the equation. Indeed, assume for simplicity that $r = q = 0$. It follows from equation (3) that C is convex in K, so $C_{KK} \geq 0$. On the other hand, it is well-known that C can be smaller than $(x-K)^+$ for some values of K and T, compare for example [8], so there exist points where $C_T < 0$. Therefore, equation (1) clearly fails. Another issue is the failure of uniqueness of solutions for the Dupire equation. A discussion of this is provided in the remark after Theorem 2.2 below.

Our main result Theorem 2.2 shows that the Dupire equation remains valid for put options, but the uniqueness of solutions is lost in general. We also show how to single out the put option price among all possible solutions. For call options, extra terms involving the discounted expected value

$$m(T) := e^{-rT} EX(T)$$

appear in the equation, and the option price C is the unique bounded solution. Note that if $e^{-(r-q)t}X(t)$ is a martingale, then $m(T) = xe^{-qT}$ and $m_T(T) = -qm(T)$. Consequently, the partial differential equation in (5) then reduces to the usual Dupire equation (1). However, if $e^{-(r-q)t}X(t)$ is merely a *local* martingale, then it is a supermartingale being bounded from below, and $e^{qT}m(T)$ decreases with T. In that case, $qm(T) + m_T(T)$ gives a negative contribution in (5).

Theorem 2.2. *Assume that Hypothesis 2.1 holds. Then the call price $C(K,T)$ is the unique bounded classical solution of the equation*

$$\left\{ \begin{array}{ll} C_T = \mathcal{L}C + qm + m_T & \text{for } (K,T) \in (0,\infty)^2 \\ C(K,0) = (x-K)^+ & \\ C(0,T) = m(T). & \end{array} \right. \tag{5}$$

The put price $P(K,T)$ is a classical solution of

$$\left\{ \begin{array}{ll} P_T = \mathcal{L}P & \text{for } (K,T) \in (0,\infty)^2 \\ P(K,0) = (K-x)^+ & \\ P(0,T) = 0. & \end{array} \right. \tag{6}$$

Moreover, P is the unique classical solution of (6) *satisfying*

$$(e^{-rT}K - e^{-qT}x)^+ \leq P(K,T) \leq e^{-rT}K \tag{7}$$

for all $(K,T) \in (0,\infty)^2$.

Remark Equation (6) can formally be viewed as a pricing (Black-Scholes) equation for a *call* option if we regard K as the spot price of an underlying asset. Consequently, one solution of (6) is given by the stochastic representation

$$\tilde{P}(K,T) = e^{-qT} E(k(T) - x)^+,$$

where k is the diffusion process

$$\left\{ \begin{array}{l} dk(t) = -(r-q)k(t)\,dt + \sigma(k(t), T-t)\,dB(t) \\ k(0) = K \end{array} \right. \tag{8}$$

absorbed at 0. Indeed, it follows from [8] that $\tilde{P}$ is a classical solution to (6). In fact, it is the smallest non-negative solution, so

$$\tilde{P}(K,T) \leq P(K,T). \tag{9}$$

In the case of bubbles, $\tilde{P}$ is typically not convex in the spatial variable, compare [5] and [8]. However, it follows directly from (3) and (4) that the functions C and P are convex in the strike price K. Accordingly, the inequality (9) may be strict, and $\tilde{P}$ does not necessarily coincide with P for models with bubbles. Thus there is no uniqueness of solutions to equation (6) in the class of functions of at most linear growth. (If the volatility σ satisfies a linear bound at infinity, then P and $\tilde{P}$ coincide.)

As mentioned above, Theorem 2.2 relies on the assumption that option prices are given by risk-neutral valuation, compare (3) and (4). In [5] it is suggested that the set of admissible portfolios is restricted so that the hedging portfolio satisfies a collateral requirement at all times before maturity. In the present setting, this requirement means that the hedging portfolio for a call option should be worth at least $\alpha(e^{-q(T-t)}X(t) - K)^+$ at each instant $t \in [0,T]$, where $\alpha \in [0,1]$ is some given constant. Note that this condition is automatically satisfied in the absence of bubbles, but not in the present case. This illustrates that the notion of price of an option on a bubble is more sensitive to natural restrictions on the set of admissible hedging portfolios than is the case in the standard setting. According to [5], the smallest initial value of a superreplicating portfolio satisfying this collateral requirement is given by

$$C^\alpha(K,T) = C(K,T) + \alpha(xe^{-qT} - m(T)). \tag{10}$$

Since the pay-off of a put option is bounded, it is not natural to impose collateral conditions on the hedging portfolio in this case. We therefore refrain from considering alternative prices for put options, as do the authors of [5]. The proof of the next result directly follows from Theorem 2.2 and (10).

Theorem 2.3. *The smallest superreplicating price $C^\alpha(K,T)$, in the presence of collateral requirements as described above, is the unique bounded classical solution of the equation*

$$\begin{cases} C^\alpha_T = \mathcal{L}C^\alpha + (1-\alpha)(qm + m_T) & \text{for } (K,T) \in (0,\infty)^2 \\ C^\alpha(K,0) = (x-K)^+ \\ C^\alpha(0,T) = (1-\alpha)m(T) + \alpha x e^{-qT}. \end{cases} \tag{11}$$

Remark The price corresponding to $\alpha = 1$ is the one that behaves most like the price when there is no bubble in the underlying. For example, note that in this case the usual Dupire equation is obtained.

Remark The above results may be used for calibration of models from given option prices. The existence of a local volatility consistent with observed option data is closely related to the problem of finding a Markov process with the same distributional properties as a given stochastic process, compare [9] and [1]. Assume that prices $\tilde{C}(K,T)$ of call options are given (or more realistically, that $\tilde{C}(K,T)$ is constructed from a discrete set of observed prices using some suitable method of interpolation). If α is specified to be 1, then define

$$\sigma(K,T) = \sqrt{\frac{2(\tilde{C}_T(K,T) + (r-q)K\tilde{C}_K(K,T) + q\tilde{C}(K,T))}{\tilde{C}_{KK}(K,T)}}. \tag{12}$$

Assuming that σ satisfies Hypothesis 2.1, the corresponding call option prices $C^1(K,T)$ can be calculated according to (10), or equivalently by solving (11). If $\tilde{C}(K,T)$ is bounded and satisfies the boundary conditions

$\tilde{C}(K,0) = (x-K)^+$ and $\tilde{C}(0,T) = xe^{-qT}$, then by uniqueness of bounded solutions to (11) we have $\tilde{C} \equiv C^1$. Thus we have found a local volatility model which is consistent with the given market data and with the given collateral requirement corresponding to $\alpha = 1$. The case of a general $\alpha \neq 1$ can be treated similarly by inserting observed option prices (and their derivatives) in (11), and then solving for σ.

3. Proof of Theorem 2.2

Below we prove Theorem 2.2 in several steps. First, however, we briefly discuss why the proof of Dupire's equation in a standard setting where the underlying is a martingale is not directly applicable in the strict local martingale setting. To do this, assume for simplicity that $r = q = 0$, and let $p(y,t) = P(X_t \in dy)/dy$ denote the density of X (assuming that this density exists). Then

$$\begin{aligned} C(K,T) &:= E(X(T)-K)^+ = \int_K^\infty (y-K)p(y,T)\,dy \\ &= \int_K^\infty \int_y^\infty p(z,T)\,dzdy, \end{aligned} \tag{13}$$

where the last equality is justified by integration by parts since X_T has a finite mean. Differentiating (13) with respect to T and using the forward equation for p, we get

$$\begin{aligned} C_T(K,T) &= \int_K^\infty \int_y^\infty \frac{1}{2}(\sigma^2(z,T)p(z,T))_{zz}\,dzdy \\ &= \frac{1}{2}\sigma^2(K,T)p(K,T) = \frac{1}{2}\sigma^2(K,T)C_{KK}(K,T), \end{aligned} \tag{14}$$

provided that the out-integrated terms vanish. However, these terms *do not* vanish if X is a strict local martingale since the density in such a case does not decay rapidly at infinity, and thus the standard argument fails to generalise. It turns out, however, that a similar argument involving put options instead of call options *does* generalise since in this case there are no out-integrated terms at infinity. This motivates the line of proof we follow below starting with put options rather than with call options.

Step 1. First assume that σ satisfies the bounds

$$D^{-1}x \leq \sigma(x,t) \leq Dx \tag{15}$$

for some constant $D > 0$ and has bounded derivatives of all orders. By Ito's formula, the process $Y(t) := \ln X(t)$ satisfies

$$dY(t) = \beta_Y(Y(t),t)\,dt + \sigma_Y(Y(t),t)\,dB(t),$$

where

$$\beta_Y(y,t) := -\frac{\sigma^2(e^y,t)}{2e^{2y}} + r - q$$

and

$$\sigma_Y(y,t) := \frac{\sigma(e^y,t)}{e^y}.$$

The process Y is a diffusion on the real line with the drift and the volatility possessing bounded derivatives of all orders, and the volatility is bounded from below. Consequently, Y has a smooth transition density

$$p_Y(z,T) := P(Y(T) \in dz)/dz$$

which satisfies the forward equation

$$(p_Y)_T = (\frac{\sigma_Y^2}{2} p_Y)_{zz} - (\beta_Y p_Y)_z,$$

compare for example [17], and $p_Y(y,T)$ and its derivatives decay like $o(e^{-|y|})$ for large $|y|$. It follows that also the process X has a smooth density $p(y,T) = P(X(T) \in dy)/dy$ which satisfies

$$p_T = (\frac{\sigma^2}{2} p)_{yy} - ((r-q)yp)_y.$$

Now, since

$$\begin{aligned} P(K,T) &= e^{-rT} E(K - X(T))^+ = e^{-rT} \int_0^K (K-y) p(y,T)\,dy \\ &= e^{-rT} \int_0^K \int_0^y p(z,T)\,dzdy \end{aligned}$$

by integration by parts, the put price $P(K,T)$ is smooth on $(0,\infty)^2$. Straightforward differentiation shows that

$$\begin{aligned} P_T(K,T) &= e^{-rT} \int_0^K \int_0^y p_T(z,T)\,dzdy - rP(K,T) \\ &= e^{-rT} \int_0^K \int_0^y (\frac{\sigma^2(z,T)}{2} p(z,T))_{zz} - (r-q)(zp(z,T))_z\,dzdy \\ &\quad -rP(K,T) \\ &= \frac{\sigma^2(K,T)}{2} e^{-rT} p(K,T) - e^{-rT} \int_0^K (r-q) y p(y,T)\,dy - rP(K,T) \\ &= \frac{\sigma^2(K,T)}{2} e^{-rT} p(K,T) - (r-q) K e^{-rT} \int_0^K p(y,T)\,dy \\ &\quad +(r-q) e^{-rT} \int_0^K \int_0^y p(z,T)\,dz - rP(K,T) \\ &= \frac{\sigma^2(K,T)}{2} e^{-rT} p(K,T) - (r-q) K e^{-rT} \int_0^K p(y,T)\,dy \\ &\quad -qP(K,T). \end{aligned}$$

Since $P_K(K,T) = e^{-rT} \int_0^K p(y,T)\,dy$ and $P_{KK}(K,T) = e^{-rT} p(K,T)$, we find that

$$P_T(K,T) = \frac{\sigma^2(K,T)}{2} P_{KK}(K,T) - (r-q) K P_K(K,T) - qP(K,T). \tag{16}$$

Step 2. Next we carry out an approximation argument to remove the bound (15) for small values of the underlying. Thus we assume that σ, in addition to Hypothesis 2.1, satisfies

$$0 < \sigma(x,t) \le D(1+x) \tag{17}$$

for all $(x, t) \in (0, \infty) \times [0, \infty)$, and we assume that zero is an absorbing boundary for the corresponding solution X of (2). Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of volatilities such that

- $\sigma_n(x, t) \to \sigma(x, t)$ as $n \to \infty$ for all (x, t),
- each σ_n satisfies the bound (15) for some constant $D_n > 0$ and has bounded derivatives of all orders,
- σ_n satisfies the upper bound in (17) uniformly in n, and has a Hölder norm (in the spatial variable) which is bounded on compact subsets of $(0, \infty)^2$ uniformly in n.

Let X^n be the solution of (2) with σ replaced by σ_n, and let P^n be defined by

$$P^n(K, T) = e^{-rT} E(K - X^n(T))^+.$$

By Theorem 6 in [11], it follows that $P^n(K, T) \to P(K, T)$ as $n \to \infty$ for each $(K, T) \in [0, \infty) \times [0, T]$. By Step 1 above, each P_n satisfies

$$P^n_T(K, T) = \frac{\sigma_n^2(K, T)}{2} P^n_{KK}(K, T) - (r - q)KP^n_K(K, T) - qP^n(K, T)$$

on $(0, \infty)^2$. Since the functions $P^n(K, T)$ are locally bounded uniformly in n, interior Schauder estimates, see [4] or [15], imply that P^n has derivatives P^n_K, P^n_{KK} and P^n_T that are locally bounded, uniformly in n. Moreover, these derivatives are locally Hölder(1/2) continuous (with respect to the parabolic distance) with Hölder norms that are bounded uniformly in n. By the Arzela-Ascoli theorem, the sequence $\{P^n\}_{n=1}^{\infty}$ has a subsequence $\{P^{n_k}\}_{k=1}^{\infty}$ such that P^n and its derivatives P^n_K, P^n_{KK} and P^n_T converge locally uniformly to a function $\tilde{P}$ and its corresponding derivatives. Clearly, by uniqueness of limits we have $\tilde{P} = P$. Since σ_n converges to σ, the limit function P satisfies (16).

Step 3. Now we consider the general case of a volatility σ that merely satisfies the requirements in Hypothesis 2.1. Let $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of volatilities satisfying Hypothesis 2.1 with a Hölder norm that is bounded on compacts uniformly in n. Moreover, we assume that $\sigma_n(x, t) = \sigma(x, t)$ for $x \leq n$ and that the growth assumption (17) holds for constants D_n. Let X^n be the corresponding stock price process. Since σ_n coincides with σ on $(0, n) \times [0, \infty)$, the random variables $X^n(T)$ converge almost surely to $X(T)$. Thus $P^n(K, T)$ converges to $P(K, T)$ by bounded convergence. Another application of the interior Schauder estimates shows that P solves (16).

Remark The boundedness of the pay-off function $y \mapsto (K - y)^+$ of a put option is essential in the argument for the convergence of P^n to P used in Step 3. Note that the corresponding call prices C^n do not converge to C in general. Indeed, $C^n(K, T) \geq (x - K)^+$, whereas $C(K, T)$ may be strictly smaller than $(x - K)^+$ for certain values of K and T, compare [8]. Also note that dominated convergence cannot be applied to prove $C^n \to C$ since the random variable $X^*_T := \sup_n X^n(T)$ is not necessarily integrable.

Step 4. Since $e^{-(r-q)t}X(t)$ is a supermartingale, it follows from Jensen's inequality that

$$\begin{aligned} P(K,T) &= e^{-rT}E(K-X(T))^+ \geq e^{-rT}(K-EX(T))^+ \\ &\geq (e^{-rT}K - xe^{-qT})^+. \end{aligned}$$

On the other hand, we clearly have $P(K,T) \leq e^{-rT}K$. It follows that P is continuous up to the boundary $K=0$ and that $P(0,T)=0$. Moreover, since the paths of X are continuous, we have that $X(T) \to x$ as $T \downarrow 0$. Therefore, another application of bounded convergence shows that $P(K,T)$ is continuous up to the initial boundary $T=0$, and $P(K,0)=(K-x)^+$. This finishes the proof that the put option price P is a classical solution of (6) that satisfies (7).

Step 5. Next we apply maximum principle techniques to prove that P is the unique classical solution of (6) that satisfies (7). To do that, assume that P^1 and P^2 both satisfy (6) and (7). Then $F(K,T) := P^1(K,T) - P^2(K,T)$ is a bounded classical solution of

$$\begin{cases} F_T(K,T) = \mathcal{L}F(K,T) & \text{for } (K,T) \in (0,\infty)^2 \\ F(0,T) = 0 \\ F(K,0) = 0. \end{cases}$$

Define

$$h(K) = 1 + K,$$

and note that

$$h_T - \mathcal{L}h = rK + q > 0.$$

For $\epsilon > 0$, define

$$F^\epsilon(K,T) = F(K,T) + \epsilon h(K),$$

let $\Gamma := \{(K,T) \in [0,\infty) \times [0,\overline{T}] : F^\epsilon < 0\}$ for some $\overline{T} > 0$, and assume that $\Gamma \neq \emptyset$. Since F is bounded and $F(0,T)=0$, the set Γ is contained in $(D^{-1}, D) \times [0,\overline{T}]$ for some constant $D > 0$. Thus, by compactness, the infimum

$$T_0 := \inf\{T \geq 0 : (K,T) \in \overline{\Gamma} \text{ for some } K \in (0,\infty)\}$$

is attained at some point (K_0,T_0), and $F^\epsilon(K_0,T_0) = 0$ by continuity. Since $F^\epsilon(K,0) = \epsilon h(K) > 0$, we have $T_0 > 0$. Therefore, at the point (K_0,T_0) we have

$$F^\epsilon_T(K_0,T_0) - \mathcal{L}F^\epsilon(K_0,T_0) = \epsilon(h_T - \mathcal{L}h)(K_0,T_0) > 0.$$

On the other hand, by the definition of T_0 and K_0, the function $K \mapsto F^\epsilon(K,T_0)$ has a local minimum at $K = K_0$. Consequently, the function F^ϵ satisfies $F^\epsilon = 0$, $F^\epsilon_K = 0$, $F^\epsilon_{KK} \geq 0$ and $F^\epsilon_T \leq 0$ at the point (K_0,T_0). Consequently,

$$F^\epsilon_T(K_0,T_0) - \mathcal{L}F^\epsilon(K_0,T_0) \leq 0.$$

This contradiction shows that $\Gamma = \emptyset$, so $F^\epsilon \geq 0$ on $(0,\infty) \times [0,\overline{T}]$. Since $\epsilon > 0$ and $\overline{T}$ are arbitrary, it follows that $0 \leq F = P^1 - P^2$. Interchanging the role of P^1 and P^2 yields the reverse inequality, i.e. $P^1 = P^2$.

Step 6. Finally, we treat the call option price C using a put-call parity relation. Taking expected values in the equality

$$(X(T)-K)^+ = (K-X(T))^+ - K + X(T)$$

we find that

$$C(K,T) = P(K,T) - e^{-rT}K + m(T). \tag{18}$$

Therefore,

$$\begin{aligned} C_T(K,T) &= P_T(K,T) + re^{-rT}K + m_T(T) \\ &= \frac{\sigma^2(K,T)}{2}P_{KK}(K,T) - (r-q)KP_K(K,T) - qP(K,T) \\ &\quad + re^{-rT}K + m_T(T) \\ &= \frac{\sigma^2(K,T)}{2}C_{KK}(K,T) - (r-q)KC_K(K,T) - qC(K,T) \\ &\quad + qm(T) + m_T(T), \end{aligned}$$

where we used (18), $P_K = C_K + e^{-rT}$ and $P_{KK} = C_{KK}$. The fact that C satisfies the given boundary conditions also follows from the put-call parity (18) and the boundary behaviour of P. Finally, the proof of the uniqueness of the solutions to equation (6) within the given class also shows uniqueness of solutions with a bounded difference to $e^{-rT}K$. This translates directly to uniqueness for equation (5) for bounded functions. This finishes the proof of Theorem 2.2.

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