SHORT-TIME IMPLIED VOLATILITY IN EXPONENTIAL LÉVY MODELS

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Abstract. We show that a necessary and sufficient condition for the explosion of implied volatility near expiry in exponential Lévy models is the existence of jumps towards the strike price in the underlying process. When such jumps do not exist, the implied volatility converges to the volatility of the Gaussian component of the underlying Lévy process as the time to maturity tends to zero. These results are proved by comparing the short-time asymptotics of the Black-Scholes price with explicit formulas for upper and lower bounds of option prices in exponential Lévy models.

1. Introduction

In financial markets, prices of European options are usually quoted in terms of their implied volatilities, which are computed through inversion of the classical Black-Scholes formula. An advantage of expressing prices in volatility is that it gives an easy comparison between options with different features. However, under a given modelling framework for the underlying asset price, there is little hope to obtain a closed-form expression of the implied volatility. Instead, a good deal of attention has been directed towards the asymptotic behavior of the volatility surface. Recently, the understanding of the short-maturity asymptotics of implied volatility has advanced, see [3] and [12] for results in local volatility models, [8], [9] and [10] for stochastic volatility models, and [1], [7], [14], [15] and [17] for exponential Lévy models.

In this paper we exclusively study the short-term behavior of implied volatility in exponential Lévy models. For a nice and thorough introduction to exponential Lévy processes we refer to [5]. The small-time behavior for at-the-money options is well-understood, see [14], [15] and [17]. These references show that the implied volatility converges to the volatility of the Gaussian component of the underlying Lévy process. Moreover, in [14] it is shown how to change variables in order to have a non-trivial limit when zooming in at-the-money and close to expiry. On the other hand, the precise asymptotic behavior for out-of-the-money and in-the-money options is not clear for general exponential Lévy models. In fact, existing literature typically focuses on cases in which the implied volatility explodes close to

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expiry, and then derives asymptotic expansions of implied volatility. In [7] and [17], it is assumed that the Lévy measure has full support, meaning that the stock price can jump to any positive value. The condition in [15] is slightly weaker, as it merely requires a positive probability that the stock price takes a single leap across the strike level. However, the exact condition on the model to have explosion of the short-time implied volatility seems to be lacking. Moreover, very little is known about the case in which the implied volatility does not blow up for not at-the-money options. A simple example is given in [15] though, where the implied volatility converges to zero for an in-the-money option in a spectrally positive exponential Lévy model with no Gaussian component and with additional restrictions on the jump activity.

The key contribution of the current paper is to give a complete picture of the short-time implied volatility behavior in a general exponential Lévy model. Our findings show that the implied volatility explodes in the short-time limit in all exponential Lévy models except in three cases, in which it converges to the volatility of the Gaussian component of the underlying Lévy process. Denoting the strike price by \( K \) and the stock price by \( s \), these three cases are:

(i) \( s < K \) in spectrally negative exponential Lévy models;
(ii) \( s > K \) in spectrally positive exponential Lévy models;
(iii) \( s = K \).

In particular, if the underlying Lévy process has no Gaussian part, the implied volatility in each of these three cases converges to zero. Our results imply that neither the Gaussian component nor the absolute value of the jump sizes in the underlying Lévy process decides whether the short-time implied volatility explodes or converges. In fact, the existence of jumps towards the strike price in the underlying process is the sole cause of an implied volatility explosion. Note that even if the jump size is bounded so that it takes more than one jump to reach the strike, implied volatility explosion occurs.

To prove the implied volatility explosion in cases when the underlying process have jumps towards the strike price, we use the convexity of option price in the underlying stock value to find a lower bound of the option price by removing the Brownian fluctuations as well as small jumps from the original Lévy process. For the resulting finite-activity model, the option price asymptotics can be derived explicitly, and the explosion of short-time implied volatility can thus be deduced easily by comparing the lower bound and the asymptotics of the Black-Scholes formula.

To prove that the short-term implied volatility converges to the volatility of the diffusion component in the remaining cases is technically more demanding, and we deal with it in several steps. First we consider the case \( s < K \) for models of spectrally negative Lévy type with finite activity. Another application of the convexity property shows that the call price is
bounded above by a call price in a jump-to-default model, which can be computed explicitly using the Black-Scholes formula. Using this explicit upper bound, the short-term implied volatility is shown to approach the volatility of the underlying. In the next step, the class of models is extended to models with an infinite jump intensity. Here we construct a new Lévy process by replacing the infinitely many small jumps with a Brownian motion with twice the variance. Using a maximum principle for parabolic integro-differential equations, we show that the call price with the new Lévy process as underlying is greater than the call price with the original Lévy process as underlying near expiry for \( s < K \), which, together with the result of the previous step, implies the result. Finally, in the last step we show that models of spectrally positive Lévy type can be reduced to the spectrally negative case by means of a measure change.

The current paper focuses on giving a full picture of the limiting behavior of implied volatility in a general exponential Lévy model, rather than deriving asymptotic formulas in special models. However, our proofs build on various lower and upper bounds of the option price, and these bounds are of course helpful also in studies of asymptotic expansions of the implied volatility.

The paper is organized as follows. In Section 2 we introduce the model and state our main results. In Section 3 we present some well-known facts on option prices that are used later. In Section 4 we deal with the cases when explosion of the short-time implied volatility occurs. Finally, Section 5 treats the cases in which the short-term implied volatility converges to the volatility of the Gaussian component of the underlying Lévy process.

2. Exponential Lévy models and the main result

We consider the class of exponential Lévy processes for a stock price process

\[ S_t = se^{X_t}, \]

where \( s > 0 \) is a constant and \( X = (X_t)_{t \geq 0} \) is a Lévy process with Lévy-Khinchin representation \( \mathbb{E}[\exp\{izX_t\}] = \exp\{t\psi(z)\}, \ z \in \mathbb{R} \), where

\[ \psi(z) = -\frac{\sigma^2 z^2}{2} + i\gamma z + \int_\mathbb{R} (e^{izy} - 1 - izy 1_{\{|y| \leq 1\}})\nu(dy). \]

Here the triplet \((\gamma, \sigma^2, \nu)\) is referred to as the characteristic triplet of \( X \), \( \sigma \geq 0 \) is the volatility of the Gaussian component and \( \nu \) is the Lévy measure of the underlying Lévy process. We assume that the model is specified directly under the measure which is used for derivative pricing, and that the Lévy measure is such that the exponential moment \( \mathbb{E}[e^{X_t}] \) is finite. More precisely, we assume that

\[ \int_{|y| > 1} e^y \nu(dy) < \infty, \]
and that

\[ \gamma = \gamma(\sigma, \nu) = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y \mathbb{1}_{|y| \leq 1}) \nu(dy). \]

Under these assumptions, the stock price \((S_t)_{t \geq 0}\) is a martingale, and the price of a European call option is then defined by

\[ C(t, s, K) = \mathbb{E}[(S_t - K)^+] = \mathbb{E}[(se^{X_t} - K)^+], \]

where \(K\) is the strike price and \(t\) is the time to maturity.

**Remark 1.** We emphasize that the time parameter \(t\) in the option pricing formula \(C(t, s, K)\) denotes time left to maturity, whereas the Lévy process \(X\) always starts at 0 at time 0, so \(se^{X_t}\) denotes the position of the process after an elapsed time \(t\). By the time-homogeneity of \(X\), this is equivalent to starting \(X\) at 0 at time \(T - t\), and then letting it evolve until \(T\), where \(T\) denotes the (fixed) horizon.

The classical Black-Scholes formula for European call option prices in a geometric Brownian motion model is

\[ C^{BS}(t, s, K, \sigma) = s N \left( \frac{\ln \frac{s}{K} + \frac{1}{2} \sigma^2 t}{\sigma \sqrt{t}} \right) - K N \left( \frac{\ln \frac{s}{K} - \frac{1}{2} \sigma^2 t}{\sigma \sqrt{t}} \right) \]

for \((t, s, K, \sigma) \in (\mathbb{R}_+)^4\), where \(N(\cdot)\) is the standard normal cumulative distribution function. With all other parameters fixed, the Black-Scholes price is a strictly increasing function of the volatility \(\sigma\). Thus, for \(s > 0\) fixed, the Black-Scholes implied volatility function \(\bar{\sigma}(t, K) : \mathbb{R}^+ \times \mathbb{R}^+ \to [0, \infty)\) of an exponential Lévy model can be defined through

\[ C(t, s, K) = C^{BS}(t, s, K, \bar{\sigma}(t, K)). \]

The main result of the paper is stated in the following theorem.

**Theorem 1.** The implied volatility in an exponential Lévy model satisfies

\[ \lim_{t \downarrow 0} \bar{\sigma}(t, K) = \infty \]

unless

(i) \(s < K\) in spectrally negative exponential Lévy models,

(ii) \(s > K\) in spectrally positive exponential Lévy models,

or

(iii) \(s = K\).

In cases (i)-(iii), we have

\[ \lim_{t \downarrow 0} \bar{\sigma}(t, K) = \sigma, \]

i.e. the implied volatility converges to the volatility of the Gaussian component of the underlying Lévy process.

Note that the convergence result (3) for case (iii) has been proved in [15] (see also [17], where it is proved under the additional assumption \(\int x^2 \nu(dx) < \infty\).
3. Ordering and asymptotic properties

In this section we state some well-known results about the option price and implied volatility. Our first proposition gives an ordering result for models with similar characteristic triplets.

**Proposition 1.** Assume that $X^i$, $i = 1, 2$ are two Lévy processes with characteristic triplets $(\gamma(\sigma_1, \nu), \sigma_1, \nu)$ and $(\gamma(\sigma_2, \nu), \sigma_2, \nu)$, respectively, so that $e^{X^i}$, $i = 1, 2$ are both martingales with the same Lévy measure $\nu$, but with different volatilities of the Gaussian components. Let $\hat{\sigma}_i(t, K)$, $i = 1, 2$, be the corresponding implied volatilities. If $0 \leq \sigma_1 \leq \sigma_2$, then

$$\sigma_1 \leq \hat{\sigma}_1(t, K) \leq \hat{\sigma}_2(t, K).$$

**Proof.** Let $X^1$ be a Lévy process with a diffusion component with volatility $\sigma_1$, and let $W_t$ be a Brownian motion, independent of $X^1$. Then $S^2 = se^{X^2}$ can be represented as

$$S_t^2 = se^{X^1_t}M_t,$$

where $M$ is the exponential martingale

$$M_t = e^{-\sigma_2^2 - \sigma_1^2 t + \sqrt{\sigma_2^2 - \sigma_1^2}W_t}.$$

We then have

$$C_2(t, s, K) = \mathbb{E}[(se^{X^1_t}M_t - K)^+] = \mathbb{E}[\mathbb{E}[(se^{X^1_t}M_t - K)^+ | X^1_t]] \geq \mathbb{E}[(se^{X^1_t} - K)^+] = C_1(t, s, K),$$

where the inequality is obtained from Jensen’s inequality for conditional expectations. Consequently, $\hat{\sigma}_2(t, K) \geq \hat{\sigma}_1(t, K)$.

A similar argument, in which one decomposes $X^1$ in a Brownian part and a jump part, gives $\hat{\sigma}_1(t, K) \geq \sigma_1$. \hfill \square

Our next proposition describes the asymptotics of the option price for not at-the-money options. For two functions $f$ and $g$ we write $f \sim g$ if $f(t) / g(t) \to 1$ as $t \to 0$.

**Proposition 2.** As $t \to 0$, the Black-Scholes price defined in (2) has the asymptotics

$$C_{BS}(t, s, K, \sigma) - (s - K)^+ \sim C_1(\sigma^3 t^{3/2} \exp \left\{ -\frac{(\ln K - \ln s)^2}{2t\sigma^2} \right\})$$

for $s \neq K$, where

$$C_1 = \frac{\sqrt{sK}}{\sqrt{2\pi (\ln K - \ln s)^2}}.$$

Moreover, for a given exponential Lévy model, the option price $C$ satisfies

$$C(t, s, K) - (s - K)^+ \sim C_1(\sigma^3(t, K) t^{3/2} \exp \left\{ -\frac{(\ln K - \ln s)^2}{2t\sigma^2(t, K)} \right\})$$

for $s \neq K$. 
Proof. The asymptotics (5) are well-known, see for example [12].
To prove (6), note that
\[ C(t,s,K) = \mathbb{E}[(K - S_t)^+] + s - K \to (K - s)^+ + s - K = (s - K)^+ \]
as \( t \downarrow 0 \) by bounded convergence. Since \( C^{BS}(t,s,K,\sigma) \) only depends on \( t \) and \( \sigma \) through the quantity \( \sigma^2 t \), this implies that \( \hat{\sigma}^2(t,K)t \to 0 \) as \( t \downarrow 0 \). The asymptotics (6) therefore follow from (5).

\[ \square \]

4. IMPLIED VOLATILITY EXPLOSION

This section treats the case in which the implied volatility explodes as time to maturity vanishes. To prove the following theorem, we determine an explicit lower bound of the option price by removing the Brownian motion and certain jumps using the convexity of the payoff.

**Theorem 2.** The implied volatility tends to infinity as time to expiry tends to zero in all exponential Lévy models that do not belong to one of the cases (i), (ii) or (iii) of Theorem 1.

Proof. For \( s > K \), consider an exponential Lévy model which is not of spectrally positive type. We decompose the underlying Lévy process as the sum of two independent Lévy processes, where one of them has no Brownian fluctuations and only large negative jumps, and the other one contains the continuous fluctuations as well as positive jumps and small negative jumps. To be precise, given a Lévy process \( X \) which is not spectrally positive and with characteristic triplet \( (\gamma, \sigma^2, \nu) \), take \( \varepsilon > 0 \) such that
\[ \lambda := \int_{y<\varepsilon} \nu(dy) > 0, \]
and decompose \( X \) as
\[ X_t = X_t^1 + X_t^2, \]
where
\[ X_t^1 = -t \int_{y<\varepsilon} (e^y - 1)\nu(dy) + \int_{y<\varepsilon} yN(t,dy) \]
and
\[ X_t^2 = -\frac{\sigma^2}{2} t + \sigma W_t - t \int_{y\geq\varepsilon} (e^y - 1 - y 1_{|y|\leq 1})\nu(dy) \]
\[ + \int_{-\varepsilon \leq y \leq 1} y\check{N}(t,dy) + \int_{y>1} yN(t,dy). \]
Here \( W \) is a standard Brownian motion, \( N \) is a Poisson random measure with intensity \( \nu(dy)dt \), \( \check{N} \) is the compensated version of \( N \). Note that \( X^1 \) is a compound Poisson process with drift and with jump intensity \( \lambda \), and that \( e^{X^1} \) and \( e^{X^2} \) are independent martingales. Let
\[ \beta = \frac{1}{\lambda} \int_{y<\varepsilon} e^y\nu(dy) \]
so that $1 - \beta$ is the expected relative jump size of $e^{X_t}$, let

$$\mu = \lambda (1 - \beta)$$

and let

$$N_t^1 = \int_{y < -\epsilon} N(t, dy)$$

count the number of jumps of $X^1$. Then

\begin{equation}
\mathbb{E}[(se^{X_t} - K)^+] = \mathbb{E}[\mathbb{E}[(se^{X_t^1 + X_t^2} - K)^+ | N_t^1]] \\
\geq \mathbb{E}[(s \mathbb{E}[e^{X_t^1} | N_t^1] - K)^+] \\
= \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^n}{n!} (se^{\mu t} \beta^n - K)^+ \\
\geq e^{-\lambda t} \frac{(\lambda t)^n}{n!} (K - se^{\mu t} \beta^n) + s - K,
\end{equation}

where the first inequality follows from Jensen’s inequality for conditional expectations, and in the last inequality we chose the unique integer $n \geq 1$ such that $s \beta^n < K$ and $s \beta^{n-1} \geq K$.

By (7) and Proposition 2, we have

\begin{equation}
\lim_{t \downarrow 0} \sup e^{-\lambda t} \frac{(\lambda t)^n}{n!} (K - se^{\mu t} \beta^n) \\
\leq \lim_{t \downarrow 0} \sup \frac{\mathbb{E}[(se^{X_t} - K)^+ - (s - K)]]}{C_1 n! \hat{\sigma}^3(t, K) t^{3/2} \exp \left\{ -\frac{\ln K - \ln s}{2\hat{\sigma}^2(t, K)} \right\}} = 1,
\end{equation}

which implies that $\hat{\sigma}(t, K) \to \infty$ as $t \downarrow 0$.

Finally, a change of measure technique (see e.g. [13]) can be used to reduce the case when $s < K$ and $X$ is a Lévy model which is not spectrally negative to the previous case. Indeed, assume that $X$ has positive jumps and satisfies $\mathbb{E}[e^{X_t}] = 1$. Define a probability measure $Q$ by

$$\frac{dQ}{dP} := e^{X_t}.$$

Note that $-X$ is a Lévy process which has negative jumps, and $(e^{-X_t})_{s \in [0, t]}$ is a $Q$-martingale. Moreover,

$$\mathbb{E}[(se^{X_t} - K)^+] = \mathbb{E}^Q[(s - Ke^{-X_t})^+] = \mathbb{E}^Q[(Ke^{-X_t} - s)^+] + K - s,$$

where the second equality is the put-call-parity. Now, since

$$\mathbb{E}^Q[(Ke^{-X_t} - s)^+]$$

is a call price in an exponential Lévy model with negative jumps, with stock price $K$ and strike price $s < K$, and since the Black-Scholes price satisfies

$$C^{BS}(t, K, s, \sigma) + K - s = C^{BS}(t, s, K, \sigma),$$

implied volatility explosion follows from the first part of the proof. \qed
Remark 2. (Explosion rates.) To conclude that the short-term implied volatility explodes and to derive its explosion rate, existing literature uses the small-time asymptotics

$$\liminf_{t \to 0} \frac{C(t,s,K) - (s-K)^+}{t} \geq \eta > 0$$

for the option price under certain conditions (e.g. full support of Lévy measure). The proof of Theorem 2 utilizes the fact that if \( s > K \) and if the underlying Lévy measure supports jumps towards the strike price, then the option price satisfies the weaker asymptotics

$$\liminf_{t \to 0} \frac{C(t,s,K) - (s-K)^+}{t^n} \geq \eta$$

for some integer \( n \geq 1 \) and some constant \( \eta > 0 \). However, as is shown in the proof above, (10) also leads to implied volatility explosion. Moreover, it is straightforward to find a lower bound for the explosion rate. Indeed, following the proof of [17, Proposition 4] gives

$$\liminf_{t \to 0} \frac{2n\hat{\sigma}^2(t,K)t \ln \frac{1}{t}}{(\ln \frac{s}{K})^2} \geq 1.$$

Remark 3. (Spectrally negative models do not necessarily lead to skews.) Consider the spectrally negative exponential Lévy model in which

$$S_t = s e^{\lambda t \beta N_t},$$

where \( N \) is a Poisson process with intensity \( \lambda \) and the relative jump size \( 1 - \beta \in (0, 1) \) is fixed. In this model, as \( t \downarrow 0 \), the implied volatility satisfies

$$2n\hat{\sigma}^2(t,K)t \ln \frac{1}{t} \sim (\ln \frac{s}{K})^2$$

for \( s > K \), where \( n \geq 1 \) is the unique integer such that

$$s\beta^n < K \text{ and } s\beta^{n-1} \geq K.$$

Thus the rate of the implied volatility explosion for \( K \) just below \( \hat{K} := s\beta \) (for which \( n = 2 \)) is strictly smaller than the explosion rate for \( K \) just above \( \hat{K} \) (for which \( n = 1 \)). An interesting consequence of this is that spectrally negative models not necessarily lead to volatility skews (compare the statement that "the presence of a skew is attributed to the fear of large negative jumps", [5, page 10]).

5. Convergence of implied volatility

In this section we show that the implied volatility converges to the volatility of the Gaussian component of the underlying Lévy process as the time to maturity goes to zero in cases (i) and (ii) of Theorem 1 (case (iii) is proved in [15]). The convergence results are proved in three steps. The first step treats models of spectrally negative exponential Lévy type with finite activity for \( s < K \). In the second step we examine the case of spectrally
negative exponential Lévy processes with infinite activity for \(s < K\). Finally, in the third step we study the case of spectrally positive exponential Lévy processes for \(s > K\).

**Step 1.** This step treats case (i) of Theorem 1 under the additional assumption that the underlying Lévy process has finite jump intensity.

**Theorem 3.** In spectrally negative exponential Lévy models with finite activity, i.e. when the Lévy measure satisfies \(\nu(0, \infty) = 0\) and \(\nu(-\infty, 0) < \infty\), the implied volatility satisfies
\[
\lim_{t \downarrow 0} \hat{\sigma}(t, K) = \sigma
\]
for \(s < K\).

**Proof.** First note that it follows from Proposition 1 that it suffices to consider the case \(\sigma > 0\). Since the underlying spectrally negative Lévy process has finite activity, its discontinuous part is just a compound Poisson process with negative jumps. Let \(\lambda := \nu(-\infty, 0)\) be the intensity of the compound Poisson process, denote by \(F(dx) = \frac{1}{\lambda} \nu(dx)\) the distribution of the jump size, let \(\beta = \int_{x < 0} e^x F(dx)\) so that \(1 - \beta\) is the expected relative jump size, and let \(\mu = \lambda(1 - \beta)\). Write \(C(t, s)\) for \(C(t, s, K)\), and note that since \(\sigma > 0\), \(C\) is a classical solution of
\[
C_t = \frac{1}{2} \sigma^2 s^2 C_{ss} + \mu s C_s + \lambda \int_{x < 0} C(t, e^x s) F(dx) - \lambda C,
\]
see [6]. By convexity of \(s \mapsto C(t, s)\) and since \(C(t, 0) = 0\), we have
\[
\int_{x < 0} C(t, e^x s) F(dx) \leq \int_{x < 0} e^x C F(dx) = \beta C,
\]
so
\[
(11) \quad C_t \leq \frac{1}{2} \sigma^2 s^2 C_{ss} + \mu s C_s + \lambda \beta C - \lambda C.
\]
Let \(\overline{C}(t, s)\) be the Black-Scholes option price with interest rate \(\mu\), i.e.
\[
\overline{C}(t, s) = C^{BS}(t, s, Ke^{-\mu t}, \sigma),
\]
so that
\[
(12) \quad \overline{C}_t = \frac{1}{2} \sigma^2 s^2 \overline{C}_{ss} + \mu s \overline{C}_s - \mu \overline{C}.
\]
Set \(V(t, s) = \overline{C}(t, s) - C(t, s)\), and note that
\[
V_t \geq \frac{1}{2} \sigma^2 s^2 V_{ss} + \mu s V_s - \mu V
\]
for \(t > 0\) and \(s > 0\) by (11) and (12). Since \(V(0, s) = 0\), \(V(t, 0) = 0\) and \(-K \leq V(t, s) \leq K\) for all \(t \geq 0\) and \(s \geq 0\), it follows from the maximum
principle (see [11, Theorem 9, Chapter 2]) that $C \leq \bar{C}$. By the definition of implied volatility we get that

$$\limsup_{t \downarrow 0} C_{BS}(t,s,K,\hat{\sigma}(t,K)) \leq \limsup_{t \downarrow 0} \frac{C_{BS}(t,s,K,\hat{\sigma}(t,K))}{C(t,s)} = 1. \quad (13)$$

On the other hand, it follows from (6) and the Black-Scholes asymptotics with positive interest rate (see [12, Lemma 2.5]) that

$$C_{BS}(t,s,K,\hat{\sigma}(t,K)) \sim \frac{\hat{\sigma}^3(t,K)}{\sigma^3} \exp \left\{ -\frac{\mu \ln K}{\sigma^2} + \frac{(\ln K)^2}{2t} \left( \frac{1}{\sigma^2} - \frac{1}{\hat{\sigma}^2(t,K)} \right) \right\} \quad \text{as } t \to 0. \quad (14)$$

Combining (13), (14) and $\hat{\sigma}(t,K) \geq \sigma$ from Proposition 1 we see that $\hat{\sigma}(t,K) \to \sigma$ as $t \downarrow 0$, which finishes the proof. \hfill \square

**Step 2.** In this step we show convergence of the short-time implied volatility when $s < K$ in spectrally negative exponential Lévy models with infinite jump activity. Given such a model, we construct a new model by replacing the small jumps by a Brownian motion with twice the variance. This replacement technique is often used for simulation of Lévy processes, see [2], [16] and [17].

To be precise, let $X$ be a spectrally negative Lévy process $X$ with Lévy triplet $(\gamma, \sigma^2, \nu)$. As in Step 1, Proposition 1 yields that it suffices to consider the case $\sigma > 0$. Also, we assume that $\nu((-\varepsilon, 0)) > 0$ for any $\varepsilon > 0$ (otherwise we are done by Step 1). Given $\varepsilon > 0$, let $X^\varepsilon$ be a Lévy process characterized by the triplet $(\gamma^\varepsilon, \sigma^2, \nu^\varepsilon)$, where

$$\sigma^2 = \sigma^2 + 2 \int_{-\varepsilon < y < 0} y^2 \nu(dy),$$

$$\nu^\varepsilon(dy) = 1_{y < -\varepsilon} \nu(dy)$$

and

$$\gamma^\varepsilon = -\frac{\sigma^2}{2} - \int_{\mathbb{R}} (e^y - 1 - y 1_{|y| \leq 1}) \nu(dy).$$

Then $e^{X^\varepsilon}$ is a martingale, and $\sigma^\varepsilon > \sigma$. For fixed $K > 0$, the price

$$C^\varepsilon(t,s) = \mathbb{E}[(s e^{X^\varepsilon_t} - K)^+]$$

of a call option with $X^\varepsilon$ as underlying is then a classical solution of

$$C^\varepsilon_t = \frac{1}{2} \sigma^2 \sigma^2 t C^\varepsilon_{ss} + \int_{y < -\varepsilon} (C^\varepsilon(t, se^y) - C^\varepsilon - s(e^y - 1)C^\varepsilon_s) \nu(dy), \quad (15)$$

see [6]. Set

$$\Omega := \{(t,s) : 0 < t \leq \frac{\ln K/s}{d^\varepsilon}, \ 0 < s < K\},$$
where
\[ d_\varepsilon = \frac{3}{2} \sigma_\varepsilon^2 - \int_{y < -\varepsilon} (e^y - 1) \nu(dy). \]

Our first result states that \( C_\varepsilon \) is \( S \)-superharmonic in \( \Omega \).

**Lemma 1.** The function \( C_\varepsilon \) satisfies
\[ C_\varepsilon \geq \frac{1}{2} \sigma^2 s^2 C_{ss}^\varepsilon + \int_{y < 0} (C_\varepsilon(t, se^y) - C_\varepsilon - s(e^y - 1)C_s^\varepsilon) \nu(dy) \]
on \( \Omega \).

**Proof.** Since \( X_\varepsilon \) has finite jump intensity, it can be decomposed as
\[ X_\varepsilon t = \sigma_\varepsilon W_t - Y_t + \tilde{\gamma} t, \]
where \( Y \geq 0 \) is a compound Poisson process with only positive jumps and
\[ \tilde{\gamma} = -\frac{\sigma_\varepsilon^2}{2} - \int_{y < -\varepsilon} (e^y - 1) \nu(dy). \]

Denoting the density of \( e^{X_\varepsilon} \) by \( f(x) \), we have
\[ f(x) = \frac{\partial}{\partial x} \mathbb{E} \left[ \mathbb{P} \left( W_t \leq \frac{\ln x + Y_t - \tilde{\gamma} t}{\sigma_\varepsilon} \middle| Y_t \right) \right] \]
\[ = \frac{\partial}{\partial x} \mathbb{E} \left[ \int_{-\infty}^{\ln x + Y_t - \tilde{\gamma} t} \frac{\varphi(y)}{\sigma_\varepsilon \sqrt{t}} dy \right] \]
\[ = \mathbb{E} \left[ \frac{1}{x \sigma_\varepsilon \sqrt{t}} \varphi \left( \frac{\ln x + Y_t - \tilde{\gamma} t}{\sigma_\varepsilon \sqrt{t}} \right) \right], \]
where \( \varphi \) denotes the density of the standard normal distribution. Thus differentiating \( C_\varepsilon(t, s) = \int_{\mathbb{R}} (sx - K)^+ f(x) \, dx \) three times with respect to \( s \) gives
\[ C_{sss}^\varepsilon(t, s) = \frac{K}{s^3 \sigma_\varepsilon \sqrt{t}} \mathbb{E} \left[ \varphi \left( \frac{\ln(K/s) - \tilde{\gamma} t + Y_t}{\sigma_\varepsilon \sqrt{t}} \right) \left( \frac{\ln(K/s) - \tilde{\gamma} t + Y_t}{\sigma_\varepsilon \sqrt{t}} - 2 \right) \right] \]
\[ \geq \frac{K}{s^3 \sigma_\varepsilon \sqrt{t}} \mathbb{E} \left[ \varphi \left( \frac{\ln(K/s) - \tilde{\gamma} t + Y_t}{\sigma_\varepsilon \sqrt{t}} \right) \left( \frac{\ln(K/s) - \tilde{\gamma} t}{\sigma_\varepsilon \sqrt{t}} - 2 \right) \right]. \]
Consequently, \( C_{sss}^\varepsilon \geq 0 \) in \( \Omega \).
We then have
\[
\int_{-\varepsilon \leq y < 0} (C^e(t, se^y) - C^e - s(e^y - 1)C^e_s) \nu(dy)
\]
\[
\leq \int_{-\varepsilon \leq y < 0} s(1 - e^y)(C^e_s - C^e_s(t, se^y)) \nu(dy)
\]
\[
\leq \int_{-\varepsilon \leq y < 0} s^2(1 - e^y)^2 C^e_{ss} \nu(dy)
\]
\[
\leq s^2 C^e_{ss} \int_{-\varepsilon \leq y < 0} y^2 \nu(dy)
\]
\[
= \frac{\sigma^2 - \sigma^2}{2} s^2 C^e_{ss}
\]
for \((t, s) \in \Omega\), where the first inequality follows from \(C^e_{ss} \geq 0\) and the second one from \(C^e_{sss} \geq 0\) in \(\Omega\). Combining (15) and (16) we obtain the lemma. \(\Box\)

Next we show that \(C^e(t, s) \geq C(t, s)\) in \(\Omega \tau := \Omega \cap \{t \leq \tau\}\) for some \(\tau > 0\\tag{17}\)

**Lemma 2.** For a fixed \(K\), there exists \(\tau > 0\) such that the option prices satisfy
\[
C^e(t, s) \geq C(t, s),
\]
for \((t, s) \in \Omega \tau\).

**Proof.** We have
\[
\frac{C^e(t, Ke^{-d\varepsilon t})}{C(t, Ke^{-d\varepsilon t})} = \frac{\mathbb{E}[(\exp(X^e_t - d\varepsilon t)K - K)^+]}{\mathbb{E}[(\exp(X_t - d\varepsilon t)K - K)^+]} \geq \frac{\mathbb{E}[(\exp(X^e_t - d\varepsilon t) - 1)^+]}{\mathbb{E}[(\exp(\sigma W_t - \frac{1}{2}\sigma^2 t) - 1)^+]} \cdot \frac{\mathbb{E}[(\exp(\sigma W_t - \frac{1}{2}\sigma^2 t) - 1)^+]}{\mathbb{E}[(\exp(X_t - 1)^+)].
\]
The first factor on the right hand side converges to 1 as \(t \downarrow 0\) by [15, Lemma 2.7]. Using the asymptotics for at-the-money options from [15, Theorem 3.1], the second factor converges to \(\sigma^e/\sigma > 1\) as \(t \downarrow 0\). Hence there exists \(\tau > 0\) such that

\[
C^e(t, Ke^{-d\varepsilon t}) \geq C(t, Ke^{-d\varepsilon t})
\]
for \(t \leq \tau\). Let \(V(t, s) = C^e(t, s) - C(t, s)\). Lemma 1 above and [6, Proposition 2] yield that
\[
V_t \geq \frac{1}{2} \sigma^2 s^2 V_{ss} + \int_{y < 0} (V(t, se^y) - V + (1 - e^y)sV_s) \nu(dy)
\]
in \(\Omega \tau\). On the boundaries \(V\) satisfies \(V(t, 0) = V(0, s) = 0\) and \(V(t, Ke^{-d\varepsilon t}) \geq 0\) for \(t \leq \tau\). Since \(X\) is spectrally negative, the non-local term in (17) only depends on the value of \(V\) within \(\Omega \tau\). Consequently, a maximum principle similar to the one in [4] can be applied, which yields \(V(t, s) \geq 0\) for \((t, s) \in \Omega \tau\). \(\Box\)
We are now ready to conclude that the implied volatility converges to $\sigma$ in case (i) in Theorem 1.

**Theorem 4.** In spectrally negative exponential Lévy models, the implied volatility satisfies

\[ \lim_{t \uparrow 0} \hat{\sigma}(t, K) = \sigma \]  

for $s < K$.

*Proof.* Denote by $\hat{\sigma}_\varepsilon(t, K)$ and $\hat{\sigma}(t, K)$ the implied volatilities in the Lévy models $X^\varepsilon$ and $X$, respectively. Proposition 1, Lemma 2 and Theorem 3 imply that

\[ \sigma \leq \liminf_{t \downarrow 0} \hat{\sigma}(t, K) \leq \limsup_{t \downarrow 0} \hat{\sigma}(t, K) \leq \limsup_{t \downarrow 0} \hat{\sigma}_\varepsilon(t, K) = \sigma_\varepsilon. \]

Since $\varepsilon > 0$ was arbitrary, and since $\lim_{\varepsilon \downarrow 0} \sigma^2_\varepsilon = \sigma^2$, the theorem follows. \( \square \)

**Step 3.** In this step we show that the convergence of the short-term implied volatility in case (ii) of Theorem 1 follows from Theorem 4.

**Theorem 5.** In spectrally positive exponential Lévy models, the short-term implied volatility satisfies

\[ \lim_{t \downarrow 0} \hat{\sigma}(t, K) = \sigma \quad \text{for} \quad s > K. \]  

*Proof.* Assume that $s > K$, and let $X$ be a spectrally positive Lévy process such that $S_t = se^{X_t}$ is a martingale. Defining a measure $Q$ by

\[ \frac{dQ}{dP} := e^{X_t}, \]

similar arguments as in the last part of the proof of Theorem 2 show that equation (19) follows from Theorem 4. \( \square \)

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**References**


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