Monotonicity of implied volatility for perpetual put options

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April 13, 2023

Abstract

We define and study properties of implied volatility for American perpetual put options. In particular, we show that if the market prices are derived from a local volatility model with a monotone volatility function, then also the corresponding implied volatility is monotone as a function of the strike price.

1 Introduction

In Black-Scholes theory, the standard text book approach is to specify a model for an underlying asset, and then derive an arbitrage-free value of derivative securities. One reason for the great success of Black-Scholes theory, however, is that the above procedure can be inverted. In fact, if a model based on geometric Brownian motion is assumed, and if a market price of a European call option is given, then one may determine the unique volatility of the underlying asset which would give that particular price. This particular value is referred to as the *implied volatility*, and it provides an efficient way of quoting option prices.

The standard notion of implied volatility refers to the inversion of the Black-Scholes formula for the pricing of call options. If the implied volatility is inferred for several different options on the same underlying and maturity but with different strike prices, the notion 'volatility smile' refers to the implied volatility being decreasing for small strikes and increasing for larger strikes; similarly, an implied volatility that is monotone as a function of the strike is referred to as a 'volatility skew'. The occurrence of volatility smiles and skews shows that the use of a constant volatility is too simplistic for modeling purposes, and that more advanced models are needed. In [5], a discussion of what models give rise to volatility skews is provided. In particular, it is argued that decreasing volatility skews may be obtained in (i) spectrally negative jump models, (ii) stochastic volatility models with negative correlation, and (iii) local volatility models with decreasing volatility functions. However, these are not necessarily precise mathematical results, but should rather be viewed as rules-of-thumb based on numerical evidence. In fact, it is shown in [8] that spectrally negative jump models exhibit the *opposite* monotonicity (i.e. *increasing* implied volatility) for short-time implied volatility. Similarly, numerical plots of implied volatility in stochastic volatility models often exhibit a non-monotone dependence of the implied volatility in the strike price, cf. e.g. [3]. For an interesting study of the dependence between local volatility and implied volatility, see [2] where a non-linear partial differential equation for the implied volatility is derived; a closer inspection of this equation strongly indicates that models of type (iii) above always give rise to decreasing implied volatilities, but a full study of this case is still missing.

While the notion of implied volatility, as discussed above, is usually defined for European options, it makes sense to quote also prices of other types of financial products in terms of volatility. In the current paper, we discuss implied volatility inferred from prices of perpetual

American put options, and we study its properties. In comparison with the definition of implied volatility for European options, the inversion of the pricing formula for perpetual options is easier, and therefore allows for more explicit calculations. In particular, we show that if the market uses a local volatility model with a decreasing volatility function, then the implied volatility is also decreasing, thus verifying claim (iii) above within our set-up. Similarly, if the market uses a local volatility model with an increasing volatility function, then the corresponding implied volatility is increasing in the strike price.

2 Set-up and main result

On a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{0\leq t<\infty})$, let X be the solution of a stochastic differential equation

$$dX_t = rX_t\,dt + \sigma(X_t)X_t\,dW_t \tag{1}$$

with initial condition $x_0 \in (0,\infty)$. Here $r > 0$ is the constant interest rate and $\sigma(\cdot) : (0,\infty) \to (0,\infty)$ is a given function. We will assume that the local volatility function $\sigma(\cdot)$ is continuous and bounded at infinity so that (1) has a unique strong solution up to the first hitting time of 0; we also assume that X is regular so that any point $y \in (0,\infty)$ can be reached in finite time, and we let 0 be an absorbing state so that if X reaches 0 in finite time, then it remains at 0 at all times.

For any given strike price $K \geq 0$, consider the price

$$P(K) := \sup_\tau \mathbb{E}\left[e^{-r\tau}(K - X_\tau)^+\right]$$

of a perpetual American put option with strike K. Note here that the discounting rate r is the same as in the drift of X in (1), so by standard arbitrage-theory, $P(K)$ is the unique arbitrage-free price of the put option. The following result is immediate (and well-known, cf. [7]).

Lemma 1. *The function $P : [0,\infty) \to [0,\infty)$*

- *is non-decreasing and convex;*
- *satisfies $P(K) > 0$ for $K > 0$;*
- *satisfies $(K - x_0)^+ \leq P(K) \leq K$.*

2.1 Constant volatility and implied volatility

In the special case of constant volatility $\sigma(\cdot) \equiv \gamma > 0$, it is well-known that $P_\gamma(K) := P(K)$ is given by

$$P_\gamma(K) = \begin{cases} (K - z(K))\left(\frac{z_\gamma(K)}{x_0}\right)^{2r/\gamma^2} & K < \hat{K}_\gamma \\ K - x_0 & K \geq \hat{K}_\gamma, \end{cases} \tag{2}$$

where

$$z_\gamma(K) := \frac{2rK}{2r+\gamma^2},$$

and $\hat{K}_\gamma := \frac{(2r+\gamma^2)K}{2r}$. Moreover,

$$\tau_{z_\gamma(K)} := \inf\{t \geq 0 : X_t \leq z_\gamma(K)\}$$

is an optimal stopping time so that

$$P_\gamma(K) = \mathbb{E}\left[e^{-r\tau_{z_\gamma(K)}}(K - X_{\tau_{z_\gamma(K)}})^+\right].$$

Using the explicit formula (2), it is straightforward to check that the price is *monotonically increasing in volatility*: if $P_{\gamma_1}(K)$ is the price corresponding to a volatility $\gamma_1 > 0$ and $P_{\gamma_2}(K)$ is the price corresponding to a volatility $\gamma_2 > 0$, then

$$\gamma_1 \leq \gamma_2 \implies P_{\gamma_1}(\cdot) \leq P_{\gamma_2}(\cdot).$$

Furthermore, if $P_{\gamma_1}(K) > (K - x_0)^+$, then $P_{\gamma_2}(K) > P_{\gamma_1}(K)$ for $\gamma_2 > \gamma_1$, i.e. the price is strictly increasing in the volatility in the continuation region.

Remark 2. *Monotonicity of perpetual options with respect to volatility holds true for a much larger class of local volatility models, see [1] and [6].*

2.2 Main result

Now assume that a volatility function $\sigma(\cdot)$ is given, and let $P(\cdot)$ be the corresponding option price; recall that $P(\cdot)$ has the properties specified in Lemma 1. Denote by

$$\hat{K} := \inf\{K \geq 0 : P(K) = (K - x_0)^+\}.$$

Then $\hat{K} \in (x_0, \infty]$, and $P(K) > (K - x_0)^+$ for $K < \hat{K}$ and $P(K) = (K - x_0)^+$ for $K \geq \hat{K}$. By the strict monotonicity of the function $P_\gamma(K)$ with respect to γ (recall that $P_\gamma(K)$ is the price of a put option if the volatility is a constant γ), for each $K \in (0, \hat{K})$ there exists a unique $\gamma = \gamma(K) > 0$ such that $P(K) = P_\gamma(K)$. This value is referred to as the *implied volatility.*

Remark 3. *In contrast to the standard notion of implied volatility (defined by inverting the Black-Scholes formula for options with a finite horizon), which is well-defined for all strikes, the implied volatility $\gamma(\cdot)$ is uniquely determined only for $K < \hat{K}$.*

We now present our main result.

Theorem 4. *Let a volatility function $\sigma(\cdot)$ be given, let $P(\cdot)$ be the corresponding option value, and let $\hat{K} = \inf\{K \geq 0 : P(K) = (K - x_0)^+\}$.*

(i) If $\sigma(\cdot) : (0, \infty) \to (0, \infty)$ is non-increasing, then the implied volatility $\gamma(\cdot)$ is non-increasing on $(0, \hat{K})$.

(ii) If $\sigma(\cdot) : (0, \infty) \to (0, \infty)$ is non-decreasing, then the implied volatility $\gamma(\cdot)$ is non-decreasing on $(0, \hat{K})$.

The proof of Theorem 4 is given in Section 4 below.

3 A geometric approach to option pricing and implied volatility

While the classical method to determine the price P (along with an optimal stopping time) would be to make a Markovian embedding by varying the initial point x_0 and then study an associated free-boundary problem, it is more convenient for our purposes to follow a geometric approach presented in [7]. As we will see below, this geometric approach is well aligned with the inversion of the pricing function in (2) that is needed to calculate the implied volatility.

Let $\varphi : (0, \infty) \to (0, \infty)$ be a non-increasing solution of

$$\frac{\sigma^2(x)x^2}{2}\varphi''(x) + rx\varphi'(x) - r\varphi(x) = 0. \tag{3}$$

This function is unique up to multiplication with a positive constant, cf. [4, p.18-19]; we will for simplicity (and without loss of generality) assume that $\varphi(x_0) = 1$. It is straightforward to check that

$$\varphi(x) = Dx \int_x^\infty \frac{1}{y^2} \exp\left\{-\int_{x_0}^y \frac{2r}{z\sigma^2(z)}dz\right\} dy \tag{4}$$

where

$$D = \frac{1}{x_0 \int_{x_0}^{\infty} \frac{1}{y^2} \exp\left\{ -\int_{x_0}^{y} \frac{2r}{z\sigma^2(z)} dz \right\} dy}.$$

Moreover, straightforward differentiation yields

$$\varphi''(x) = \frac{2rD}{\sigma^2(x)x^2} \exp\left\{ -\int_{x_0}^{x} \frac{2r}{z\sigma^2(z)} dz \right\} > 0,$$

so φ is strictly convex. Furthermore, denoting by

$$\tau_z := \inf\{t \geq 0 : X_t \leq z\}$$

the first passage time below a level $z \in (0, \infty)$, we have for any $x \geq z$ that

$$\mathbb{E}_x \left[e^{-r\tau_z} \right] = \frac{\varphi(x)}{\varphi(z)}.$$

In particular, using $\varphi(x_0) = 1$,

$$\mathbb{E}_{x_0} \left[e^{-r\tau_z} \right] = \frac{1}{\varphi(z)} \tag{5}$$

for $z \in (0, x_0]$. Moreover, the above formulae also extend to $z = 0$ as $\varphi(0) = \infty$ in the case when 0 is unattainable.

The following result was provided in [7]; for completeness of the presentation, we include its proof.

Proposition 5. *The value function* $P : [0, \infty) \to [0, \infty)$ *satisfies*

$$P(K) = \sup_{z \leq K \wedge x_0} \frac{K - z}{\varphi(z)}. \tag{6}$$

Proof. Denote by

$$\hat{P}(K) := \sup_{z \leq K \wedge x_0} \frac{K - z}{\varphi(z)}$$

the right-hand side of (6). We clearly have

$$\begin{aligned} P(K) &= \sup_{\tau} \mathbb{E} \left[e^{-r\tau} (K - X_\tau)^+ \right] \geq \sup_{z \leq K \wedge x_0} \mathbb{E} \left[e^{-r\tau_z} (K - X_{\tau_z})^+ \right] \\ &= \sup_{z \leq K \wedge x_0} \frac{K - z}{\varphi(z)} = \hat{P}(K), \end{aligned}$$

where the second equality uses that $X_{\tau_z} = z$ on $\{\tau_z < \infty\}$, $\mathbb{P}$-a.s. and (5).

For the reverse inequality, first assume that $K \leq K' := -x_0/\varphi'(x_0)$. For such K, we have

$$\hat{P}(K) = \sup_{z} \frac{K - z}{\varphi(z)}$$

so that $\hat{P}(K)\varphi(z) \geq (K - z)^+$. Consequently,

$$P(K) = \sup_{\tau} \mathbb{E} \left[e^{-r\tau} (K - X_\tau)^+ \right] \leq \hat{P}(K) \sup_{\tau} \mathbb{E} \left[e^{-r\tau} \varphi(X_\tau) \right] \leq \hat{P}(K)\varphi(x_0) = \hat{P}(K),$$

where the second inequality uses that $\{e^{-rt}\varphi(X_t), 0 \leq t \leq \infty\}$ is a supermartingale.

Finally, the above argument gives that $P(K') = (K' - x_0)$; hence $P(K) = (K - x_0) \leq \hat{P}(K)$ for all $K \geq K'$ by Lemma 1, which completes the proof. □

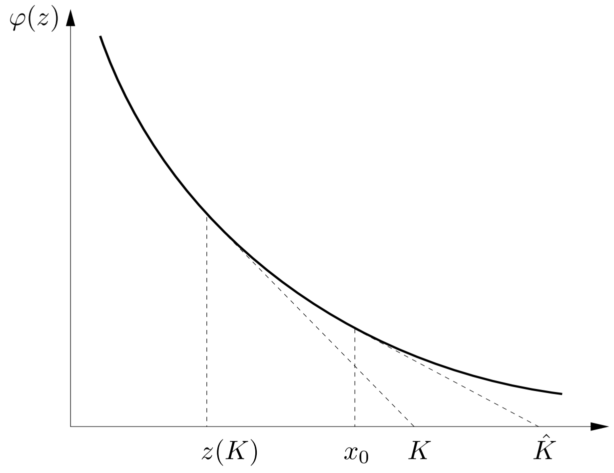

Figure 1: For a given $K \le \hat{K}$, the put price $P(K)$ equals the negative reciprocal of the slope of the tangent line to φ which passes through $(K, 0)$.

Remark 6. *For a given K, convexity of φ yields the existence of a unique straight line through the point $(K, 0)$ that is tangent to the graph $\{(z, \varphi(z)), z \le K \wedge x_0\}$. Denoting by $(z(K), \varphi(z(K)))$ the tangent point (which is unique by strict convexity of φ), the price $P(K)$ is given by*

$$P(K) = (K - z(K))/\varphi(z(K)).$$

Thus $P(K)$ equals the negative reciprocal of the slope of this tangent line, cf. Figure 3.

Moreover, $z(K)$ is non-decreasing in K, and it follows from the proof of Proposition 5 that $\tau_{z(K)}$ is an optimal stopping time.

Remark 7. *In contrast with the standard embedding approach, where an option price is produced for a fixed strike K but for any initial stock price x, the geometric approach presented in Proposition 5 produces prices for all K, but for a fixed initial stock price x_0. This feature will turn out useful for us when comparing implied volatility for different strike prices.*

We now return to the problem of determining the implied volatility, assuming that market prices are calculated using a local volatility function $\sigma(\cdot)$ with corresponding function $\varphi(\cdot)$ as above. For a given $K > 0$, let l_K be the unique straight line through $(K, 0)$ that is tangent to $\{(z, \varphi(z)), z \le K \wedge x_0\}$. Denoting the tangent point by $(z(K), \varphi(z(K)))$, we have

$$l_K(z) = -\frac{\varphi(z(K))}{K - z(K)}(z - K).$$

To describe how to determine the implied volatility in the local volatility model $\sigma(\cdot)$, note that for a given constant volatility $\gamma > 0$, the decreasing positive solution of

$$\frac{\gamma^2 x^2}{2} f''(x) + rxf'(x) - rf(x) = 0$$

on $(0, \infty)$, imposing also the normalizing condition $f(x_0) = 1$, is given by

$$f(x) := f_\gamma(x) := \left(\frac{x}{x_0}\right)^{-2r/\gamma^2}. \tag{7}$$

Clearly, on $(0, x_0)$, f_γ is strictly decreasing in γ. Consequently, given a strike price $K > 0$ and the corresponding tangent line $l_K(\cdot)$ (constructed from $\sigma(\cdot)$ and $\varphi(\cdot)$ as above) with $l_K(x_0) < 1$ (equivalently, $z(K) < x_0$), there exists a unique $\gamma = \gamma(K)$ such that $l(\cdot)$ is a tangent line also to $f_\gamma(\cdot)$. By Proposition 5, $\gamma(K)$ is the implied volatility of the option.

4 Monotone local volatility functions

In this section we provide a result regarding the function φ corresponding to non-increasing local volatility functions, see Proposition 10. For its proof we will use a monotonicity result (Lemma 8) with respect to the local volatility functions; its proof is adapted from a similar situation in [6]. Using Proposition 10, the proof of Theorem 4 then follows.

Lemma 8. *Assume that $\sigma_i : (0,\infty) \to (0,\infty)$, $i = 1,2$ satisfy $\sigma_1(\cdot) \le \sigma_2(\cdot)$. Let h_i, $i = 1,2$ be the solutions of*

$$\begin{cases} \frac{\sigma_i^2(x)x^2}{2} h_i''(x) + rxh_i'(x) - rh_i(x) = 0 & x \in (a,b) \\ h_i(a) = A \\ h_i(b) = B \end{cases}$$

for $0 < a < b < \infty$ and $A, B \ge 0$. If $\frac{A}{a} \ge \frac{B}{b}$, then $h_1(\cdot) \le h_2(\cdot)$ on $[a,b]$.

Proof. Denoting by φ_1 the decreasing fundamental solution corresponding to σ_1, the general solution of the ODE for h_1 is

$$h_1(x) = Cx + D\varphi_1(x),$$

where C and D are constants. Imposing the boundary conditions $h_1(a) = A$ and $h_2(b) = B$ yields

$$D = \frac{bA - aB}{b\varphi_1(a) - a\varphi_1(b)},$$

which is positive by assumption. Consequently, the function h_1 is convex.

Next, for $x \in (a,b)$, let X^2 be the solution of

$$dX_t^2 = rX_t^2\,dt + \sigma_2(X_t^2)X_t^2\,dW_t$$

with $X_0^2 = x$. By Ito's formula, the process $Y_t := e^{-rt}h_1(X_t^2)$ satisfies

$$\begin{aligned} dY_t &= e^{-rt}\left(\frac{\sigma_2^2(X_t^2)(X_t^2)^2}{2}h_1''(X_t^2) + rX_t^2h_1'(X_t^2) - rh_1(X_t^2)\right)dt + e^{-rt}\sigma^2(X_t^2)X_t^2h_1'(X_t^2)\,dW_t \\ &= e^{-rt}\frac{\sigma_2^2(X_t^2) - \sigma_1^2(X_t^2)}{2}(X_t^2)^2h_1''(X_t^2)\,dt + e^{-rt}\sigma^2(X_t^2)X_t^2h_1'(X_t^2)\,dW_t \end{aligned}$$

for $t \le \tau_{ab} := \inf\{s \ge 0 : X_s^2 \notin (a,b)\}$. Since $0 < \sigma_1 \le \sigma_2$ and h_1 is convex, Y_t is a submartingale. Similarly, the process $e^{-rt}h_2(X_t^2)$ is a martingale. Consequently, optional sampling gives

$$h_1(x) \le \mathbb{E}_x\left[e^{-r\tau_{ab}}h_1(X_{\tau_{ab}}^2)\right] = \mathbb{E}_x\left[e^{-r\tau_{ab}}h_2(X_{\tau_{ab}}^2)\right] = h_2(x),$$

which completes the proof. □

Remark 9. *The monotonicity result in Proposition 8 also extends to the case of $b = \infty$ and $B = 0$. In that case, the function h_i coincides with the decreasing fundamental solution φ_i (possibly scaled with a positive constant).*

Proposition 10. *Assume that $\sigma(\cdot) : (0,\infty) \to (0,\infty)$ is a given volatility function, and let $\varphi(\cdot)$ be the corresponding decreasing solution of (3). For a constant volatility $\gamma > 0$, let f_γ be as in (7). Moreover, assume that $f_\gamma(a) = \varphi(a)$ for some $a \in (0, x_0)$. If*

(i) $\sigma(\cdot)$ is non-increasing, then $f_\gamma(\cdot) \le \varphi(\cdot)$ on $[a, x_0]$;

(ii) $\sigma(\cdot)$ is non-decreasing, then $f_\gamma(\cdot) \ge \varphi(\cdot)$ on $[a, x_0]$.

Proof. Assume that $\sigma(\cdot)$ is non-increasing. Then we have $\sigma(x_0) \leq \sigma(a)$; we consider the three cases $\gamma \leq \sigma(x_0)$, $\sigma(a) \leq \gamma$ and $\sigma(x_0) < \gamma < \sigma(a)$ separately.

First, assume that $\gamma \leq \sigma(x_0)$. Then $\gamma \leq \sigma(x)$ for all $x \in [a, x_0]$, and Lemma 8 yields immediately $f_\gamma(\cdot) \leq \varphi(\cdot)$ on $[a, x_0]$.

Second, assume that $\sigma(a) \leq \gamma$. If $\gamma = \sigma(x_0)$, then $\sigma(\cdot) \equiv \gamma$ on $[a, x_0]$, and $\varphi(\cdot) \equiv f_\gamma(\cdot)$ on $[a, x_0]$. Therefore we may assume that $\gamma > \sigma(x_0)$. Note that $\sigma(\cdot) \leq \gamma$ on $[a, \infty)$, so Lemma 8 and Remark 9 give that $\varphi(\cdot) \leq f_\gamma(\cdot)$ on $[a, \infty)$. Since $f_\gamma(x_0) = 1 = \varphi(x_0)$ and $f_\gamma(\cdot) \geq \varphi(\cdot)$ on $[a, \infty)$, we then must have $f'_\gamma(x_0) = \varphi'(x_0)$ and $f''_\gamma(x_0) \geq \varphi''(x_0)$. Therefore,

$$\begin{aligned} 0 &= \frac{\sigma^2(x_0)x_0^2}{2}\varphi''(x_0) + rx_0\varphi'(x_0) - r\varphi(x_0) \\ &\leq \frac{\sigma^2(x_0)x_0^2}{2} f''_\gamma(x_0) + rx_0 f'_\gamma(x_0) - rf_\gamma(x_0) \\ &< \frac{\gamma^2 x_0^2}{2} f''_\gamma(x_0) + rx_0 f'_\gamma(x_0) - rf_\gamma(x_0) = 0, \end{aligned}$$

which shows that the point $a \in (0, x_0)$ with $f_\gamma(a) = \varphi(a)$ cannot exist.

Thirdly, assume that $\sigma(x_0) < \gamma < \sigma(a)$. We first show that $f_\gamma(\cdot) < \varphi(\cdot)$ in a left-neighborhood $(x_0 - \epsilon, x_0)$ of x_0 (for some $\epsilon > 0$). To see this, note that $\sigma(\cdot) \leq \gamma$ on (x_0, ∞), so Lemma 8 yields $\varphi(\cdot) \leq f_\gamma(\cdot)$ on (x_0, ∞). Since $\varphi(x_0) = 1 = f_\gamma(x_0)$, we have $\varphi'(x_0) \leq f'_\gamma(x_0)$; consequently,

$$\begin{aligned} 0 &= \frac{\sigma^2(x_0)x_0^2}{2}\varphi''(x_0) + rx_0\varphi'(x_0) - r\varphi(x_0) \\ &< \frac{\gamma^2 x_0^2}{2}\varphi''(x_0) + rx_0 f'_\gamma(x_0) - rf_\gamma(x_0) \\ &= \frac{\gamma^2 x_0^2}{2}(\varphi''(x_0) - f''_\gamma(x_0)), \end{aligned}$$

so $\varphi''(x_0) > f''_\gamma(x_0)$. This inequality, together with $\varphi(x_0) = f_\gamma(x_0)$ and $\varphi'(x_0) \leq f'_\gamma(x_0)$, imply that $f_\gamma(\cdot) < \varphi(\cdot)$ in a left-neighborhood of x_0.

If the set $\{x \in (a, x_0) : f_\gamma(x) > \varphi(x)\}$ is empty, then $f_\gamma(\cdot) \leq \varphi(\cdot)$ on $[a, x_0]$ and there is nothing to prove. Instead, assume that $f_\gamma(x) > \varphi(x)$ for some $x \in (a, x_0)$, and let

$$c := \sup\{x \in (a, x_0) : f_\gamma(x) > \varphi(x)\}.$$

Since $f_\gamma(\cdot) < \varphi(\cdot)$ in a left-neighborhood of x_0, we then have $c \in (a, x_0)$; also note that, by continuity, $f_\gamma(c) = \varphi(c)$.

Now, if $\gamma \geq \sigma(c)$, then $\gamma \geq \sigma(\cdot)$ on (c, x_0). Thus Lemma 8 gives $\varphi(\cdot) \leq f_\gamma(\cdot)$ on (c, x_0), which contradicts that $f_\gamma(\cdot) < \varphi(\cdot)$ in a left-neighborhood of x_0. On the other hand, if $\gamma < \sigma(c)$, then $\gamma \leq \sigma(\cdot)$ on (a, c), and Lemma 8 yields $f_\gamma(\cdot) \leq \varphi(\cdot)$ on (a, c). Since we also have $f_\gamma(\cdot) \leq \varphi(\cdot)$ on (c, x_0) by the definition of c, this shows that $f_\gamma(\cdot) \leq \varphi(\cdot)$ on (a, x_0).

Next, assume that $\sigma(\cdot)$ is non-decreasing. Again, three cases are considered, namely $\sigma(x_0) \leq \gamma$, $\gamma \leq \sigma(a)$, and $\sigma(a) < \gamma < \sigma(x_0)$.

First, if $\sigma(x_0) \leq \gamma$, then $\gamma \geq \sigma(\cdot)$ on $[a, x_0]$, and Lemma 8 yields $f_\gamma(\cdot) \geq \varphi$ on $[a, x_0]$.

Second, if $\gamma \leq \sigma(a)$, then $\gamma \leq \sigma(\cdot)$ on $[a, \infty)$, so Lemma 8 implies that $f_\gamma(\cdot) \leq \varphi(\cdot)$ on $[a, \infty)$. If $\gamma = \sigma(a) = \sigma(x_0)$, then $f_\gamma(\cdot) = \varphi(\cdot)$ on $[a, x_0]$, and there is nothing to prove. Thus we may assume that $\gamma \leq \sigma(a) < \sigma(x_0)$; in this case, however, $f_\gamma(x_0) = 1 = \varphi(x_0)$ and $f_\gamma(\cdot) \leq \varphi(\cdot)$ imply that

$$\begin{aligned} 0 &= \frac{\sigma^2(x_0)x_0^2}{2}\varphi''(x_0) + rx_0\varphi'(x_0) - r\varphi(x_0) \\ &\geq \frac{\sigma^2(x_0)x_0^2}{2} f''_\gamma(x_0) + rx_0 f'_\gamma(x_0) - rf_\gamma(x_0) \\ &> \frac{\gamma^2 x_0^2}{2} f''_\gamma(x_0) + rx_0 f'_\gamma(x_0) - rf_\gamma(x_0) = 0, \end{aligned}$$

which is impossible.

Thirdly, assume that $\sigma(a) < \gamma < \sigma(x_0)$. We then have $\gamma \le \sigma(\cdot)$ on $[x_0, \infty)$, so $f_\gamma(\cdot) \le \varphi(\cdot)$ on $[x_0, \infty)$. Since $f_\gamma(x_0) = 1 = \varphi(x_0)$, this implies that $f'_\gamma(x_0) \le \varphi'(x_0)$, so

$$\begin{aligned} 0 &= \frac{\sigma^2(x_0)x_0^2}{2}\varphi''(x_0) + rx_0\varphi'(x_0) - r\varphi(x_0) \\ &> \frac{\gamma^2 x_0^2}{2}\varphi''(x_0) + rx_0 f'_\gamma(x_0) - rf_\gamma(x_0) \\ &= \frac{\gamma^2 x_0^2}{2}(\varphi''(x_0) - f''_\gamma(x_0)). \end{aligned}$$

Therefore, $f''_\gamma(x_0) > \varphi''(x_0)$, and $f_\gamma(\cdot) > \varphi(\cdot)$ in a left-neighborhood $(x_0 - \epsilon, x_0)$ of x_0.

Now, assume that there exists at least one point $x \in (a, x_0)$ with $f_\gamma(x) < \varphi(x)$, and define

$$c := \sup\{x \in (a, x_0) : f_\gamma(x) < \varphi(x)\}.$$

Since $f_\gamma(\cdot) > \varphi(\cdot)$ in a left-neighborhood of x_0, we then have $c \in (a, x_0)$. If $\gamma \le \sigma(c)$, then $\gamma \le \sigma(\cdot)$ on $[c, x_0]$. Lemma 8 then yields $f_\gamma(\cdot) \le \varphi(\cdot)$ on $[c, x_0]$, which contradicts $f_\gamma(\cdot) > \varphi(\cdot)$ in a left-neighborhood of x_0. On the other hand, if $\gamma > \sigma(c)$, then Lemma 8 yields $f_\gamma(\cdot) \ge \varphi(\cdot)$ on $[a, c]$, and by definition of c we find that $f_\gamma(\cdot) \ge \varphi(\cdot)$ also on $[a, x_0]$.

□

Proposition 11. *Assume that $K < \hat{K}$, and denote by $z(K)$ the optimal stopping boundary for the volatility $\sigma(\cdot)$. Let $\gamma = \gamma(K)$ be the corresponding implied volatility, and let $z_\gamma(K)$ be the corresponding optimal stopping boundary. Then $z(K) \le z_\gamma(K)$ ($z(K) \ge z_\gamma(K)$) provided $\sigma(\cdot)$ is non-increasing (non-decreasing).*

Proof. First assume that $\sigma(\cdot)$ is non-increasing, and (for a contradiction) that $z(K) > z_\gamma(K)$. Then, by convexity of $\varphi(\cdot)$ and $f_\gamma(\cdot)$, there exists a point $a \in (z_\gamma(K), z(K))$ such that $\varphi(a) = f_\gamma(a)$. By Proposition 10, $f_\gamma \le \varphi$ on $[a, x_0]$, which contradicts that $f_\gamma(z(K)) > l_K(z(K)) = \varphi(z(K))$. Consequently, $z(K) \le z_\gamma(K)$.

Next, assume that $\sigma(\cdot)$ is non-decreasing, and that $z(K) < z_\gamma(K)$. Then, by convexity, there exists a unique $a \in (z(K), z_\gamma(K))$ with $\varphi(a) = f_\gamma(a)$ (where $\gamma := \gamma(K)$ again is the implied volatility). Proposition 10 then gives $f_\gamma \ge \varphi$ on $[a, x_0]$, which is a contradiction to $f_\gamma(z_\gamma(K)) = l_K(z(K)) < \varphi(z_\gamma(K))$. □

Proof of Theorem 4. For $i = 1, 2$, let K_i be given strike prices with $K_1 < K_2$, denote by $(z(K_i), \varphi(z(K_i)))$ the tangent point of the tangent line l_{K_i} of φ through the point $(K_i, 0)$, and let $\gamma_i := \gamma(K_i)$ be the corresponding implied volatilities. Then the lines

$$l_{K_i}(z) = -\frac{\varphi(z(K_i))}{K_i - z(K_i)}(z - K_i), \quad i = 1, 2,$$

are tangent lines to the functions

$$f_{\gamma_i}(z) = \left(\frac{z}{x_0}\right)^{-2r/\gamma_i^2},$$

respectively, with unique tangent points with z-coordinates $z'_i := z_{\gamma_i}(K_i)$, for which $f_{\gamma_i}(z'_i) = l_{K_i}(z'_i)$. We denote by $\hat{z}$ the z-coordinate of the intersection point between l_{K_1} and l_{K_2}. For a graphical illustration, see Figure 4.

Now assume that $\sigma(\cdot)$ is non-increasing. By Proposition 11, we then have $z(K_i) \le z'_i$, $i = 1, 2$. If $z'_1 > \hat{z}$, then automatically $\gamma_1 \ge \gamma_2$ since f_γ is decreasing in γ on $(0, x_0)$. On the other hand, if $z'_1 \in [z(K_1), \hat{z}]$, then by convexity there is a unique point $a \in [z(K_1), z'_1]$ such that $f_{\gamma_1}(a) = \varphi(a)$. By Proposition 10, we then have $f_{\gamma_1}(\cdot) \le \varphi(\cdot)$ on $[a, x_0]$. Consequently, at $z(K_2)$ we have

$$f_{\gamma_1}(z(K_2)) \le \varphi(z(K_2)) = l_{K_2}(z(K_2)).$$

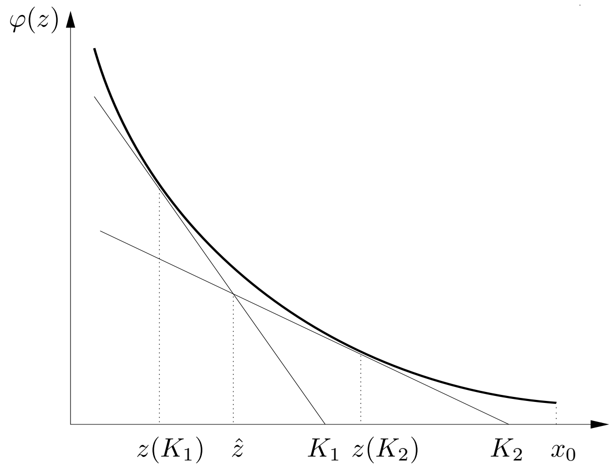

Figure 2: The function φ, together with two strike prices $K_1 < K_2$, the corresponding tangent lines l_{K_1} and l_{K_2}, and their intersection point $\hat{z}$.

Since f_γ is decreasing in γ on $(0, x_0)$, it follows that $\gamma_1 \geq \gamma_2$.

Next, assume that $\sigma(\cdot)$ is non-decreasing. Then $z_1' \leq z(K_1)$ by Proposition 11, so by convexity there exists a point $a \in [z_1', z(K_1)]$ with $f_1(a) = \varphi(a)$. By Proposition 10, we must have $f_{\gamma_1}(\cdot) \geq \varphi(\cdot) \geq l_{K_2}(\cdot)$ on $[a, x_0]$, and it follows that $\gamma_1 \leq \gamma_2$.

□

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