MOMENTUM LIQUIDATION UNDER PARTIAL INFORMATION

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Abstract. Momentum is the notion that an asset that has performed well in the past will continue to do so for some period. We study the optimal liquidation strategy for a momentum trade in a setting where the drift of the asset drops from a high value to a smaller one at some random change-point. This change-point is not directly observable for the trader, but it is partially observable in the sense that it coincides with one of the jump times of some exogenous Poisson process representing external shocks, and these jump times are assumed to be observable. Comparisons with existing results for momentum trading under incomplete information show that the assumption that the disappearance of the momentum effect is triggered by observable external shocks significantly improves the optimal strategy.

1. Introduction

A momentum trade is a strategy of buying assets that performed well in the past and selling assets that performed poorly. Such a strategy is motivated by statistical studies showing that stocks with a certain trend in the past often continue to drift in the same direction for some time. In the early reference [10] the authors showed that momentum effects often remain for periods up to a year. These findings were reinforced in [11], where the momentum phenomenon was tested on asset prices from another time period in order to reduce the risk of anomalies in the chosen data. In [11] the authors noted that momentum effects exist in the sense that stocks that performed well in the past typically continue to yield positive returns on a relatively short time horizon, and, somewhat surprisingly, they also found that the performance on longer time horizons is typically negative. For a study of momentum in international markets, see [15], and for further references about the existence of momentum effects, see [7].

An easy model for momentum is a model in which the drift of an asset drops from one value to a smaller one at some random time. In a setting with complete information where this random time is fully observable, the

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liquidation problem becomes trivial. For applications, however, it is more natural to assume incomplete information, i.e. that this change point is unobservable for the investor. In such a case, even if the change point is unobservable in itself, observations of the asset price trajectory can be used to infer a good guess of the current drift. In fact, classical filtering theory [14] provides the dynamics for the belief process, i.e. for the conditional probability that the change point has already occurred given observations of the asset price trajectory.

In [19], the optimal timing for the liquidation of a momentum trade is studied. In that paper, the asset price is modeled as a linear Brownian motion with a drift that drops from one constant to a smaller constant at an unobservable time θ. The optimal liquidation problem for a momentum trade modeled by a *geometric* Brownian motion was solved in [7]. Using a change of measure technique, it was shown that the two-dimensional optimal stopping problem associated with the liquidation problem can be reduced to an optimal stopping problem with only one underlying spatial variable, and thus be solved explicitly. For related studies of stock selling problems, see [4], [7], [8] and [20], and for studies of optimal stopping problems under incomplete information, see [6], [13] and [16].

In the present paper we consider a situation intermediate between the complete information case and the incomplete information setting studied in [7] and [19]. As in [7] we study a momentum trade modeled as a geometric Brownian motion with a drift that drops from one constant value to another at an unobservable time. However, the change point is assumed to coincide with one of the jump times of a Poisson process, and these jump times are observable. The Poisson process represents external shocks to the system (an unfavorable political decision, a financial crisis, the release of bad news, etc.) that possibly cause the momentum effect to disappear. We refer to this situation intermediate between the complete information case and the incomplete information setting as a model with *partial* information. It was shown in [16] that the conditional probability that the change point has already occurred then can be represented as a jump-diffusion, so the optimal liquidation problem under partial information can be reduced to a two-dimensional optimal stopping problem with complete information and jumps in the underlying. One of the main contributions of the current paper is to find an appropriate change of measure which transforms the two-dimensional problem to a one-dimensional optimal stopping problem. A second contribution is a thorough and rather technical treatment of this one-dimensional optimal stopping problem with a jump-diffusion as underlying process. To avoid the use of weak notions of solutions of integro-differential equations, we follow a scheme where the optimal stopping problem for a jump-diffusion is proven to be the limit of a recursively defined sequence of optimal stopping problems for a diffusion. This scheme appears for the first time in [9], see also [1], [5], and [16]. In comparison with these references, additional technical difficulties arise in our setting due to the unboundedness of the pay-off function. As a third contribution, we also provide comparative static properties of the value function and the optimal liquidation level. In particular, we show that the optimal liquidation level exhibits monotone dependencies on the volatility of the asset, on the intensity with which the momentum disappears, on the initial drift and on the probability that a given external shock in fact coincides with the change point of the drift.

In Section 2 we set up the model, we formulate the problem of optimal momentum liquidation under partial information, and we present Theorem 2.1 which describes the optimal strategy. In Section 3 we apply filtering techniques and a change of measure to show that the liquidation problem under partial information can be reduced to an auxiliary optimal stopping problem with complete information for a jump-diffusion. This auxiliary problem is treated in Sections 4-6, where we show that the optimal stopping problem for the jump-diffusion can be solved by iterating an optimal stopping problem for a diffusion process, thereby completing the proof of Theorem 2.1. In Section 7 we show that the rate of convergence in the iterative procedure is exponential, and, finally, in Section 8 we study parameter dependencies of the momentum value and of the optimal strategy.

2. The model and the optimal liquidation problem

Let N be a Poisson process with intensity λ/p, and let $T_1, T_2, ...$ be the sequence of jump times of N. Here $\lambda > 0$ and $p \in (0,1]$. Moreover, let $T_0 = 0$ and θ be a random variable taking values in the set $\{T_0, T_1, T_2, ...\}$ such that $\mathbb{P}(\theta = 0) = \pi$ and $\mathbb{P}(\theta = T_i | \theta > 0) = p(1-p)^{i-1}$, $i = 1, 2, ...$ for some constant $\pi \in [0,1)$. Next, let W be a Brownian motion which is independent of N and θ. The asset price process X is modeled by a geometric Brownian motion with a drift that drops from μ_2 to μ_1 at time θ. More precisely,

$$dX_t = \mu(t) X_t \, dt + \sigma X_t \, dW_t \tag{1}$$

and $X_0 = x > 0$, where

$$\mu(t) = \mu_2 - (\mu_2 - \mu_1) I(t \geq \theta).$$

Here μ_1, μ_2 and $\sigma > 0$ are constants with $\mu_1 < \mu_2$. Denote by $\mathcal{F} = (\mathcal{F}_t)_{t \in [0,\infty)}$ the filtration generated by X and N, and note that $\mathcal{F}$ is strictly contained in the filtration $\mathcal{G} = (\mathcal{G}_t)_{t \in [0,\infty)}$ generated by X, N and θ.

Now, let $\mathcal{T}$ be the set of (possibly infinite) $\mathcal{F}$-stopping times. We consider the optimal stopping problem

$$V = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-r\tau} X_\tau \right], \tag{2}$$

where the discount rate $r \geq 0$ is a given constant satisfying $\mu_1 < r < \mu_2$. Here (and in other similar situations below) we use the convention that $e^{-r\tau} X_\tau = 0$ on the event $\{\tau = \infty\}$.

Before proceeding, we first make a few comments on the model:

- Without the condition $\mu_1 < r < \mu_2$, the optimal stopping problem in (2) would degenerate. Indeed, if $\mu_1 < \mu_2 \leq r$, then $V = x$ and the supremum would be obtained for $\tau = 0$. If, on the other hand, $r \leq \mu_1 < \mu_2$, then the supremum would be obtained along the sequence $\tau_n = n$ of deterministic stopping times.
- The change point θ satisfies $\mathbb{P}(\theta = T_{j+i} | \theta > T_j) = \mathbb{P}(\theta = T_i | \theta > 0) = p(1-p)^{i-1}$. In particular, $\mathbb{P}(\theta = T_{j+1} | \theta > T_j) = \mathbb{P}(\theta = T_1 | \theta > 0) = p$. Moreover, the distribution of θ conditional on $\theta > 0$ is exponential with parameter λ.
- For $p = 1$ and $\pi = 0$ we have a case with complete information. The change-point θ then satisfies $\theta = T_1$, which is an $\mathcal{F}$-stopping time. Consequently, the value V can be calculated explicitly as

$$\begin{aligned} V &= \mathbb{E}\left[e^{-r\theta} X_\theta\right] = x \int_0^\infty e^{(\mu_2 - r)t} \lambda e^{-\lambda t}\, dt \\ &= \begin{cases} x\lambda/(\lambda + r - \mu_2) & \text{if } \mu_2 < r + \lambda \\ \infty & \text{if } \mu_2 \geq r + \lambda. \end{cases} \end{aligned}$$

  Henceforth we exclude this case and always assume $p \in (0,1)$.
- Consider the other extreme, that is the limit as $p \searrow 0$. Now the fact that θ coincides with a jump time of N provides no information, and we obtain a case where the change-point is completely unobservable and occurs with intensity λ. This is precisely the situation studied in [7].
- Even though the model is specified so that the change-point only can take values in the set $\{0, T_1, T_2, ...\}$, there is no such restriction on the stopping times in (2). In fact, the optimal stopping time will not take values in this set only, compare Theorem 2.1 below.
- One could also interpret X as the price of a bubble, which bursts at time θ.
- A straightforward extension of the above model would be to consider a situation in which the external shocks have different magnitudes, thus resulting in different probabilities that the change-point happens at a particular jump time. Mathematically, one could for example allow for a dependence on a label z in p, thus replacing the constant parameter p by a function $p : [0,1] \to (0,1)$, and then replace the Poisson process N_t with a Poisson random measure $N(dt, dz)$ on $[0, \infty) \times [0,1]$ with intensity measure $\frac{\lambda dt}{\int_0^1 p(y)\, dy} dz$. In this way, external shocks are labeled by $z \in [0,1]$, and the probability (given that the drift is still μ_2) that an external shock results in a drop of the drift is $p(z)$. However, for the sake of simplicity we refrain from this extension.

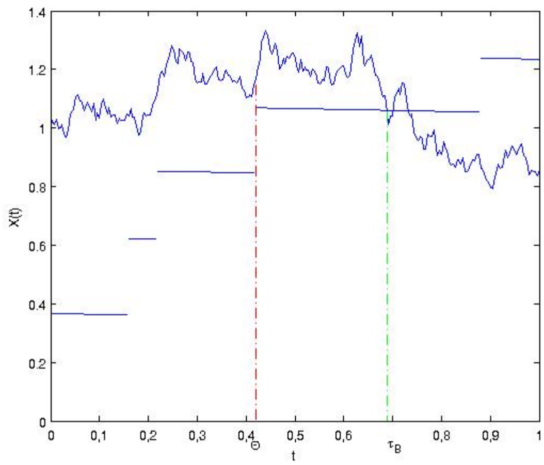

FIGURE 1. A simulated path of the stock price process X_t, along with the corresponding optimal exercise boundary as described in (3). In this simulation, the change-point coincides with the third jump of the Poisson process.

To describe the optimal strategy, define random variables $\{\Phi_{T_k}\}_{k=0}^{\infty}$ inductively by setting $\Phi_{T_0} = \pi/(1-\pi)$ and

$$\Phi_{T_{k+1}} = \frac{p}{1-p} + \frac{\Phi_{T_k}}{1-p}\left(\frac{X_{T_k}}{X_{T_{k+1}}} e^{\frac{-\sigma^2+\mu_1+\mu_2}{2}(T_{k+1}-T_k)}\right)^{\omega/\sigma},$$

$k \geq 0$. Note that Φ_{T_k} is $\mathcal{F}_{T_k}$-measurable. Therefore the random time

$$\begin{aligned} \tau_B \quad := \quad & \inf\{t \geq 0 : X_t \leq X_{T_k}(\Phi_{T_k}/B)^{\sigma/\omega} e^{\frac{\mu_1+\mu_2-\sigma^2}{2}(t-T_k)} \\ & \text{and } t \in [T_k, T_{k+1}) \text{ for some } k \geq 0\} \end{aligned} \tag{3}$$

is an $\mathcal{F}$-stopping time for each fixed constant $B > 0$.

Theorem 2.1. *There exists a $B > 0$ such that τ_B defined in* (3) *is an optimal stopping time in* (2).

Theorem 2.1 is proven in Sections 3-6 below. We also show how to determine B.

3. Filtering techniques and a reduction of dimensions

In this section we apply filtering techniques from [16] to write the one-dimensional optimal stopping problem under partial information as a two-dimensional optimal stopping problem with complete information. Moreover, we show that the Girsanov theorem then can be used to reduce the number of spatial dimensions so that we arrive at a one-dimensional auxiliary optimal stopping problem with complete information.

Define the probability process Π by

$$\Pi_t = \mathbb{P}(\theta \leq t | \mathcal{F}_t), \tag{4}$$

and note that $\Pi_0 = \pi$. Then

$$d\Pi_t = p(1-\Pi_t)\,dN_t - \omega\Pi_t(1-\Pi_t)\,d\bar{W}_t, \tag{5}$$

where

$$\omega := (\mu_2 - \mu_1)/\sigma$$

and the innovations process $\bar{W}_t$ is a $\mathbb{P}$-Brownian motion with respect to $\mathcal{F}_t$ given by

$$\begin{aligned} d\bar{W}_t &= \frac{dX_t}{\sigma X_t} - \frac{1}{\sigma}(\Pi_t\mu_1 + (1-\Pi_t)\mu_2)\,dt \\ &= \frac{1}{\sigma}(\mu(t) - \Pi_t\mu_1 - (1-\Pi_t)\mu_2)\,dt + dW_t, \end{aligned} \tag{6}$$

see [16]. In terms of $\bar{W}$ we have

$$dX_t = (\Pi_t\mu_1 + (1-\Pi_t)\mu_2)X_t\,dt + \sigma X_t\,d\bar{W}_t. \tag{7}$$

Furthermore, if the likelihood ratio Φ is defined as

$$\Phi_t := \frac{\Pi_t}{1-\Pi_t},$$

then $\Phi_0 = \phi := \pi/(1-\pi)$ and Ito's formula gives that

$$d\Phi_t = \frac{p}{(1-p)}(1+\Phi_t)\,dN_t + \omega^2\Pi_t\Phi_t\,dt - \omega\Phi_t\,d\bar{W}_t. \tag{8}$$

We let $\mathbb{Q}$ be the unique probability measure such that $d\mathbb{Q}/d\mathbb{P}$ coincides with

$$M_t := \exp\left\{-\frac{1}{2}\int_0^t (\sigma + \omega\Pi_s)^2\,ds + \int_0^t (\sigma + \omega\Pi_s)\,d\bar{W}_s\right\} \exp\{\lambda t\}\,(1-p)^{N_t}$$

on $\mathcal{G}_t$ for $0 \leq t < \infty$ (such a measure can be found provided the Brownian motion W is defined as the coordinate mapping process on $C([0,\infty))$ using Wiener measure, see [12, Corollary 3.5.2 and the discussion preceding it]; for a similar construction under incomplete information, see Section 3.2 in [6]). Then the process Z defined by

$$dZ_t = (\sigma + \omega\Pi_t)\,dt - d\bar{W}$$

is a $\{\mathbb{Q}, \mathcal{F}\}$-Brownian motion. Furthermore, the Poisson process N is independent of Z under $\mathbb{Q}$, it has $\mathbb{Q}$-intensity $\lambda(1-p)/p$, and

$$d\Phi_t = \frac{p}{1-p}(1+\Phi_t)\,dN_t - \omega\sigma\Phi_t\,dt + \omega\Phi_t\,dZ_t.$$

Proposition 3.1. *Let τ be an $\mathcal{F}$-stopping time. Then*

$$\mathbb{E}\left[e^{-r\tau}X_\tau\right] = \frac{x}{1+\phi}\mathbb{E}^{\mathbb{Q}}\left[e^{(\mu_2-\lambda-r)\tau}(1+\Phi_\tau)\right].$$

Proof. Define the process η by $\eta_t = 1/M_t$. Then, by Ito's formula,

$$\frac{d\eta_t}{\eta_t} = -\lambda\, dt + (\sigma + \omega\Pi_t)\, dZ_t + \frac{p}{1-p}\, dN_t.$$

Next, consider the process

$$K_t := \frac{x(1+\Phi_t)e^{(\mu_2-\lambda)t}}{(1+\phi)X_t}.$$

Then $K_0 = \eta_0 = 1$, and another application of Ito's formula yields

$$\frac{dK_t}{K_t} = -\lambda\, dt + (\sigma + \omega\Pi_t)\, dZ_t + \frac{p}{1-p}\, dN_t = \frac{d\eta_t}{\eta}.$$

Consequently, $\eta_t = K_t$, and the result follows. □

It follows from Proposition 3.1 that

$$V = \frac{x}{1+\phi}U,$$

where

$$U = \sup_{\tau\in\mathcal{T}} \mathbb{E}^{\mathbb{Q}}\left[e^{(\mu_2-\lambda-r)\tau}(1+\Phi_\tau)\right]. \tag{9}$$

Moreover, a stopping time is optimal in (9) if and only if it is optimal in (2). In Sections 4-6 below we show that there exists an optimal stopping time in (9) (hence also in (2)) of the form

$$\tau_B = \inf\{t \geq 0 : \Phi_t \geq B\} \tag{10}$$

for some constant B.

We end this section by showing how the likelihood ratio Φ and the stock price process X are related. As a consequence, the optimal stopping time τ_B can be expressed in terms of X.

Proposition 3.2. *We have* $\Phi_0 = \Phi_{T_0} = \pi/(1-\pi)$ *and*

$$\Phi_t = \begin{cases} \Phi_{T_k}\left(\frac{X_{T_k}}{X_t}e^{\frac{\mu_1+\mu_2-\sigma^2}{2}(t-T_k)}\right)^{\omega/\sigma} & t \in [T_k, T_{k+1}) \\ \frac{p}{1-p} + \frac{\Phi_{T_k}}{1-p}\left(\frac{X_{T_k}}{X_{T_{k+1}}}e^{\frac{\mu_1+\mu_2-\sigma^2}{2}(T_{k+1}-T_k)}\right)^{\omega/\sigma} & t = T_{k+1}. \end{cases}$$

Proof. Between two jump times T_k and T_{k+1} we have

$$X_t = X_{T_k}\exp\left\{\int_{T_k}^t (\mu_2 - \sigma\omega\Pi_s - \sigma^2/2)\, ds + \sigma(\bar{W}_t - \bar{W}_{T_k})\right\}$$

and

$$\Phi_t = \Phi_{T_k}\exp\left\{\int_{T_k}^t (\omega^2\Pi_s - \omega^2/2)\, ds - \omega(\bar{W}_t - \bar{W}_{T_k})\right\}$$

by (7) and (8). From this it is straightforward to check the result for $t \in [T_k, T_{k+1})$.

Next, note that at a jump time $t = T_{k+1}$ we have by (8) that

$$\Phi_{T_{k+1}} = \Phi_{T_{k+1}-} + \frac{p}{1-p}(1 + \Phi_{T_{k+1}-}).$$

Consequently, the result for $t = T_{k+1}$ also holds. □

Remark. It follows from Proposition 3.2 that τ_B in (3) and τ_B in (10) coincide.

4. The optimal stopping problem with jumps

In this section we study the optimal stopping problem (9). To embed it into a Markovian framework, let

$$U(\phi) = \sup_{\tau} \mathbb{E}^{\mathbb{Q}}_{\phi}\left[e^{(\mu_2-\lambda-r)\tau}(1+\Phi_\tau)\right], \tag{11}$$

where

$$d\Phi_t = \frac{p}{1-p}(1+\Phi_t)\,dN_t - \omega\sigma\Phi_t\,dt + \omega\Phi_t\,dZ_t \tag{12}$$

and where the subindex of the expectation operator indicates that $\Phi_0 = \phi$, and N is a Poisson process with $\mathbb{Q}$-intensity $\lambda(1-p)/p$. If $\mu_2-\lambda-r \geq 0$, then $U = \infty$ (the supremum is attained along a sequence $\tau_n := \inf\{t : \Phi_t \geq n\}$ of hitting times of the level n). In the remainder of this article we treat the case $\mu_2 - \lambda - r < 0$.

Associated with the optimal stopping problem (11) is the integro-differential operator $\mathcal{L}$ defined by

$$\begin{aligned}
\mathcal{L}g(\phi) &= \frac{\omega^2\phi^2}{2}g''(\phi) - \omega\sigma\phi g'(\phi) + (\mu_2 - r - \lambda)g(\phi) \\
&\quad + \lambda\frac{1-p}{p}(g(S(\phi)) - g(\phi)) \\
&= \frac{\omega^2\phi^2}{2}g''(\phi) - \omega\sigma\phi g'(\phi) + (\mu_2 - r - \lambda/p)g(\phi) + \lambda\frac{1-p}{p}g(S(\phi)),
\end{aligned}$$

where

$$S(\phi) = \frac{p+\phi}{1-p}.$$

Note that

$$\mathcal{L}(1+\phi) = \mu_2 - r - (r-\mu_1)\phi$$

which is positive if $\phi < \hat{B} := (\mu_2 - r)/(r-\mu_1)$ and negative if $\phi > \hat{B}$. This observation, together with the time-homogeneity of the optimal stopping problem (11), suggests the existence of some barrier $B \geq \hat{B}$ such that stopping the first instant the process Φ exceeds B is optimal. In fact, we have the following verification theorem. The proof follows along standard lines and is therefore somewhat sketchy.

Theorem 4.1. (Verification result for the problem with jumps.) *Assume that $F \in C([0,\infty)) \cap C^1((0,\infty)) \cap C^2((0,\infty)\setminus\{B\})$ for some point $B \in (\hat{B},\infty)$. Further assume that*

i) $\phi F'(\phi)$ *is bounded on* $(0, B)$*;*
ii) $F \geq 1 + \phi$*, with equality if* $\phi \geq B$*;*
iii) $\mathcal{L}F = 0$ *on* $(0, B)$*.*

Then $F = U$*, where* U *is defined in* (11)*. Furthermore,* $\tau_B := \inf\{t \geq 0 : \Phi_t \geq B\}$ *is an optimal stopping time.*

Proof. Let τ be a stopping time. By Ito's formula for jump-diffusions,

$$e^{(\mu_2-\lambda-r)(t\wedge\tau)}F(\Phi_{t\wedge\tau}) = F(\phi) + \int_0^{t\wedge\tau} e^{(\mu_2-\lambda-r)s}\mathcal{L}F(\Phi_s)1_{\{\Phi_s \neq B\}}\,ds + M_{t\wedge\tau},$$

where M is a martingale by i). Using ii) and iii) we find that

$$\mathbb{E}_\phi^{\mathbb{Q}}\left[e^{(\mu_2-\lambda-r)(t\wedge\tau)}(1+\Phi_{t\wedge\tau})\right] \leq \mathbb{E}_\phi^{\mathbb{Q}}\left[e^{(\mu_2-\lambda-r)(t\wedge\tau)}F(\Phi_{t\wedge\tau})\right] \leq F(\phi).$$

It then follows from Fatou's lemma and the arbitrariness of τ that

$$U(\phi) = \sup_\tau \mathbb{E}_\phi^{\mathbb{Q}}\left[e^{(\mu_2-\lambda-r)\tau}(1+\Phi_\tau)\right] \leq F(\phi).$$

To prove the reverse inequality, note that the inequalities in the analysis above reduce to equalities if $\tau = \tau_B$ (instead of Fatou's lemma one applies the bounded convergence theorem), which shows that

$$F(\phi) = \mathbb{E}_\phi^{\mathbb{Q}}\left[e^{(\mu_2-\lambda-r)\tau_B}(1+\Phi_{\tau_B})\right] \leq U(\phi).$$

It follows that $U = F$, and τ_B is an optimal stopping time. □

In view of Theorem 4.1, the optimal stopping problem (11) is reduced to solving a free boundary problem for an integro-differential equation. Due to the integral term in the equation, there seems to be little hope of finding an explicit solution. Below we instead find a solution of the free boundary problem using a fixed point approach.

5. The building block: An optimal stopping problem without jumps

To treat the optimal stopping problem (11), involving a jump-diffusion Φ, we use the technique of writing the value function U as the limit of a sequence of value functions for optimal stopping problems involving a diffusion process ([9], see also [1], [5] and [16]). In this section we provide a detailed study of the basic building block in this iterative procedure.

Let Y be a geometric Brownian motion satisfying

$$dY_t = -\omega\sigma Y_t\,dt + \omega Y_t\,dZ_t.$$

Given a non-negative function $f : [0, \infty) \to [0, \infty)$, define $Jf : [0, \infty) \to [0, \infty)$ by

(13)
$$Jf(y) = \sup_\tau \mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2-r-\lambda/p)\tau}(1+Y_\tau) + \frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)s} f(S(Y_s))\,ds\right],$$

where $S(y) = \frac{p+y}{1-p}$, and the supremum is taken over stopping times with respect to the filtration generated by the $\mathbb{Q}$-Brownian motion Z.

Remark. According to standard optimal stopping theory, one expects the value function $u(y) := Jf(y)$ to satisfy the variational inequality

$$0 = \min\{-\mathcal{A}u - \frac{\lambda(1-p)}{p} f(S(y)), u - (1+y)\},$$

where

$$\mathcal{A}u := \frac{\omega^2 y^2}{2} u'' - \omega\sigma y u' + (\mu_2 - r - \lambda/p)u.$$

Note that if u is a fixed point of J, i.e. if $Ju = u$, then

$$\begin{aligned} 0 &= \min\{-\mathcal{A}u - \frac{\lambda(1-p)}{p} u(S(y)), u - (1+y)\} \\ &= \min\{-\mathcal{L}u, u - (1+y)\}, \end{aligned}$$

which is the variational inequality corresponding to the problem with jumps of the previous section, compare Theorem 4.1. This observation suggests a connection between a fixed point of the operator J and the optimal stopping problem (11).

Definition 5.1. *Let $\mathbb{F}$ be the set of convex and non-decreasing functions $f : [0,\infty) \to [0,\infty)$ such that $1+y \le f(y) \le C+y$, where $C = \lambda/(\lambda+r-\mu_2)$.*

Below we study the operator J acting on functions f belonging to the class $\mathbb{F}$. First note that if $f \in \mathbb{F}$, then the function

$$\begin{aligned} l(y) &:= \mathcal{A}(1+y) + \frac{\lambda(1-p)}{p} f(S(y)) \\ &= (\mu_1 - r - \lambda/p)y + \mu_2 - r - \lambda/p + \frac{\lambda(1-p)}{p} f(S(y)) \end{aligned} \tag{14}$$

is convex and satisfies

$$l(0+) = \mu_2 - r - \lambda/p + \frac{\lambda(1-p)}{p} f(p/(1-p)) \ge \mu_2 - r > 0$$

and

$$\lim_{y\to\infty} \frac{l(y)}{y} = \mu_1 - r < 0.$$

Consequently, $l(y) > 0$ if and only if $y < \hat{b}$, where $\hat{b}$ is the unique positive solution of $l(y) = 0$. This indicates the existence of a barrier $b \ge \hat{b}$ such that the first passage time

$$\tau_b := \inf\{t \ge 0 : Y_t \ge b\}$$

is an optimal stopping time in (13). The following verification result is a standard result in optimal stopping theory, and it can be proved along the same lines as Theorem 4.1. We therefore omit the details.

Theorem 5.2. (Verification result for the problem without jumps.) *Let $f \in \mathbb{F}$, and assume that $H \in C([0,\infty)) \cap C^1((0,\infty)) \cap C^2((0,\infty) \setminus \{b\})$ for some point $b \in (\hat{b}, \infty)$. Further assume that*

i) *yH_y is bounded on $(0,b)$;*
ii) *$H \geq 1+y$, with equality if $y \geq b$;*
iii) *$\mathcal{A}H + \frac{\lambda(1-p)}{p} f(S(y)) = 0$ on $(0,b)$.*

Then $H = Jf$, where Jf is defined in (13). Furthermore, $\tau_b = \inf\{t \geq 0 : Y_t \geq b\}$ is an optimal stopping time.

We now aim at constructing a function H satisfying the conditions of Theorem 5.2 for a given $f \in \mathbb{F}$. To do this, let $\gamma_1 < 0$ and $\gamma_2 > 1$ be the negative and the positive root of the quadratic equation

$$\gamma^2 - (1 + 2\sigma/\omega)\gamma + 2(\mu_2 - r - \lambda/p)/\omega^2 = 0,$$

respectively, and define

$$\varphi(y) = y^{\gamma_1} \text{ and } \psi(y) = y^{\gamma_2}.$$

Then ψ and φ are solutions of $\mathcal{A}u = 0$. Moreover, ψ (φ) is, up to multiplication with positive constants, the unique increasing (decreasing) and positive solution, compare pages 18-19 in [3]. For a fixed $b \in (0,\infty)$, define H_b by

$$\begin{aligned} (15)\quad H_b(y) &= \varphi(y) \int_0^y \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\varphi(z)}\,dz \\ &+\psi(y)\left(\frac{1+b}{\psi(b)} + \int_y^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\psi(z)}\,dz - \frac{\varphi(b)}{\psi(b)}\int_0^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\varphi(z)}\,dz\right) \end{aligned}$$

for $y < b$ and $H_b(y) = 1+y$ for $y \geq b$. It is straightforward to check that $H_b : (0,\infty) \to (0,\infty)$ is continuous, that it has a finite limit at 0, and that $\mathcal{A}H_b + \frac{\lambda(1-p)}{p} f(S(y)) = 0$ on $(0,b)$. To find b so that the smooth fit condition $H_b \in C^1$ is satisfied, note that the left derivative at $y = b$ is given by

$$\begin{aligned} (16)\qquad H_b'(b) &= \varphi_y(b)\int_0^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\varphi(z)}\,dz \\ &\quad +\psi_y(b)\left(\frac{1+b}{\psi(b)} - \frac{\varphi(b)}{\psi(b)}\int_0^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\varphi(z)}\,dz\right) \\ &= \gamma_2\frac{1+b}{b} - \frac{\varphi(b)}{b}\int_0^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2 z\varphi(z)}\,dz, \end{aligned}$$

where we in the second equality used $\varphi\psi_y - \varphi_y\psi = (\gamma_2-\gamma_1)\varphi\psi/y$. Consequently, H_b satisfies the smooth fit relation $H_b' = 1$ at $y = b$ precisely if

$$\gamma_2 + (\gamma_2 - 1)b - \varphi(b)\int_0^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2 z\varphi(z)}\,dz = 0,$$

or, equivalently, if

$$\int_0^b \frac{l(z)}{z\varphi(z)}\,dz = 0, \tag{17}$$

where l is defined in (14).

Lemma 5.3. *Let $f \in \mathbb{F}$. Then the free boundary equation* (17) *has a unique solution $b \in (0,\infty)$. Moreover, this solution satisfies $b > \hat{b} \geq \hat{B}$, where $\hat{b}$ is the unique positive solution of $l(\hat{b}) = 0$.*

Proof. First note that

$$\begin{aligned} l(\hat{B}) &= \frac{\lambda(1-p)}{p} f(S(\frac{\mu_2 - r}{r-\mu_1})) - (r - \mu_1 + \lambda/p)\frac{\mu_2 - r}{r-\mu_1} \\ &\quad -(r-\mu_2+\lambda/p) \geq 0, \end{aligned}$$

where the inequality follows using $f(z) \geq 1 + z$. This proves that $\hat{b} \geq \hat{B}$.

Next, consider the function $h : (0,\infty) \to \mathbb{R}$ defined by

$$h(y) = \int_0^y \frac{l(z)}{z\varphi(z)}\, dz,$$

compare (17). Then $h(0+) = 0$, and h is strictly increasing on $(0,\hat{b})$ and strictly decreasing on $(\hat{b},\infty)$. Moreover,

$$h'(y) = \frac{l(y)}{y\varphi(y)} \sim \frac{-(r-\mu_0)}{\varphi(y)}$$

for large y, so the slope of h tends to $-\infty$. This proves the existence and uniqueness of a solution of (17). Moreover, this solution is larger than $\hat{b}$. $\square$

Remark. Note that the function h is increasing in f, and therefore also b is increasing in f. Consequently, lower and upper bounds for b are obtained by plugging in the the lower and upper bounds $1 + y$ and $C + y$ for f. Straightforward calculations show that h satisfies

$$\begin{aligned} h(y) &\geq \int_0^y (\mu_2 - r - (r-\mu_1)z)\, \frac{1}{z\varphi(z)}\, dz \\ &= \left(\frac{\mu_2 - r}{-\gamma_1} - \frac{r-\mu_1}{1-\gamma_1} y \right) \frac{1}{\varphi(y)} \end{aligned}$$

and

$$\begin{aligned} h(y) &\leq \int_0^y (\mu_2 - r + (C-1)\lambda(1-p)/p - (r-\mu_1)z)\, \frac{1}{z\varphi(z)}\, dz \\ &= \left(\frac{(\mu_2 - r)(r+\lambda/p-\mu_2)}{-\gamma_1(r+\lambda-\mu_2)} - \frac{r-\mu_1}{1-\gamma_1} y \right) \frac{1}{\varphi(y)}. \end{aligned}$$

Hence b satisfies the inequalities

$$\frac{(1-\gamma_1)(\mu_2 - r)}{-\gamma_1(r-\mu_1)} \leq b \leq \frac{(1-\gamma_1)(\mu_2 - r)(r+\lambda/p-\mu_2)}{-\gamma_1(r-\mu_1)(r+\lambda-\mu_2)}.$$

Theorem 5.4. *Let $f \in \mathbb{F}$ be given, let b be the unique solution of* (17)*, and let $H := H_b$. Then $Jf \equiv H$.*

Proof. To apply the verification result Theorem 5.2 we need to check that the conditions of that theorem are fulfilled. Note that condition iii) clearly holds, and that b is chosen so that the C^1 assumption is satisfied. It remains to verify conditions i) and ii).

We have

$$\begin{aligned} H'(y) &= \varphi'(y)\int_0^y \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\varphi(z)}\,dz \\ +\psi'(y)&\left(\frac{1+b}{\psi(b)}+\int_y^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\psi(z)}\,dz - \frac{\varphi(b)}{\psi(b)}\int_0^b \frac{2\lambda(1-p)f(S(z))}{p\omega^2(\gamma_2-\gamma_1)z\varphi(z)}\,dz\right), \end{aligned}$$

and it is straightforward to check that

$$|H'(y)| \le D/y \tag{18}$$

for small y and some constant D, thus verifying condition i).

We next claim that ii) holds, i.e. $H = H_b$ satisfies $H(y) \ge 1+y$. To see this, note that $H'(b) = 1$ and

$$\begin{aligned} \frac{\omega^2 b^2}{2}H''(b-) &= \omega\sigma b H'(b) - (\mu_2 - r - \lambda/p)H(b) - \frac{\lambda(1-p)}{p}f(S(b)) \\ &= -(\mu_1 - r - \lambda/p)b - (\mu_2 - r - \lambda/p) - \frac{\lambda(1-p)}{p}f(S(b)) \\ &= -l(b) > 0 \end{aligned}$$

since $b > \hat{b}$ by Lemma 5.3. Consequently, $H(y) > 1+y$ in some left neighborhood $(b-\epsilon, b)$ of b. Now, let $a \in (0,b)$ be the largest point below b such that $H(a) = 1+a$ and $H(y) > 1+y$ for $y \in (a,b)$. Then H coincides with the function H_a on $(0,a]$. Therefore $H'(a) = H_a'(a-) < 1$ since $a < b$, which contradicts $H(y) > 1+y$ for $y \in (a,b)$. Consequently, the point a does not exist, so $H(y) > 1+y$ for $y \in (0,b)$. Thus $H(y) \ge 1+y$. □

Remark. Alternatively, to prove that $H(y) > 1+y$ for $y \in (0,b)$ one could try to verify that H is convex by direct computations (it is indeed convex, compare below). However, this seems technically more demanding.

Proposition 5.5. *Assume that $f \in \mathbb{F}$. Then also Jf belongs to $\mathbb{F}$.*

Proof. It follows by choosing $\tau = 0$ in (13) that $Jf(y) \ge 1+y$. Next, define $\hat{f}(y) = C + y$, where C is as in Definition 5.1, and let

$$\hat{u}(y) = \sup_\tau \mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2 - r - \lambda/p)\tau}\hat{f}(Y_\tau) + \frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)s}\hat{f}(S(Y_s))\,ds\right].$$

Since $C \ge 1$ and $f \in \mathbb{F}$, $Jf \le \hat{u}$. On the other hand,

$$\mathcal{A}\hat{f} + \frac{\lambda(1-p)}{p}\hat{f}(S(y)) = (\mu_1 - r)y + (\mu_2 - r - \lambda)C + \lambda \le 0,$$

so it is straightforward to show, using similar arguments as when proving the verification results above, that $\hat{u}(y) = \hat{f}(y)$. Consequently, $Jf(y) \le C + y$.

Finally, for a fixed stopping time τ, the expression

$$\mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2-r-\lambda/p)\tau}(1+Y_\tau)+\frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)s}f(S(Y_s))\,ds\right]$$

is convex and non-decreasing in y since Y_t has a linear dependence on the initial point. Being the supremum of such functions, H is also convex and non-decreasing. □

6. The fixed point approach

We have shown above (see Theorem 5.4) that, for a given $f\in\mathbb{F}$, Jf can be determined on a semi-explicit form. Moreover, the operator J maps $\mathbb{F}$ into $\mathbb{F}$, i.e. if $f\in\mathbb{F}$, then $Jf\in\mathbb{F}$ (Proposition 5.5). This allows us to define f_n recursively by

$$\begin{cases} f_0(y)=1+y \\ f_{n+1}(y)=Jf_n(y) \quad n\geq 0. \end{cases}$$

We also let b_n, $n\geq 1$, be the solution of the free boundary equation (17) with $f=f_{n-1}$. First note that the sequence $\{f_n\}_{n=0}^\infty$ is increasing in n. Indeed,

$$f_1(y)=Jf_0(y)\geq 1+y=f_0(y)$$

by choosing $\tau=0$ in the definition of Jf_0. Moreover, if $f_n\geq f_{n-1}$, then

$$\begin{aligned} f_{n+1}(y) &= Jf_n(y)\\ &= \sup_\tau \mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2-r-\lambda/p)\tau}(1+Y_\tau)+\frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)s}f_n(S(Y_s))\,ds\right]\\ &\geq \sup_\tau \mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2-r-\lambda/p)\tau}(1+Y_\tau)+\frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)s}f_{n-1}(S(Y_s))\,ds\right]\\ &= Jf_{n-1}(y)=f_n(y). \end{aligned}$$

Consequently, f_n is increasing in n by induction. It follows that also b_n is increasing in n.

We denote by f_∞ the point-wise limit of the sequence f_n, and by b_∞ the point-wise limit of b_n. Being the supremum of a sequence of convex functions, f_∞ is also convex. Also note that $f_\infty\in\mathbb{F}$.

Proposition 6.1. *The function f_∞ is a fixed point for the operator J.*

Proof. We have

$$\begin{aligned} f_\infty(y) &= \sup_n f_n(y)\\ &= \sup_{\tau,n} \mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2-r-\lambda/p)\tau}(1+Y_\tau)+\frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)t}f_{n-1}(S(Y_t))\,dt\right]\\ &= \sup_\tau \mathbb{E}_y^{\mathbb{Q}}\left[e^{(\mu_2-r-\lambda/p)\tau}(1+Y_\tau)+\frac{\lambda(1-p)}{p}\int_0^\tau e^{(\mu_2-r-\lambda/p)t}f_\infty(S(Y_t))\,dt\right]\\ &= Jf_\infty(y), \end{aligned}$$

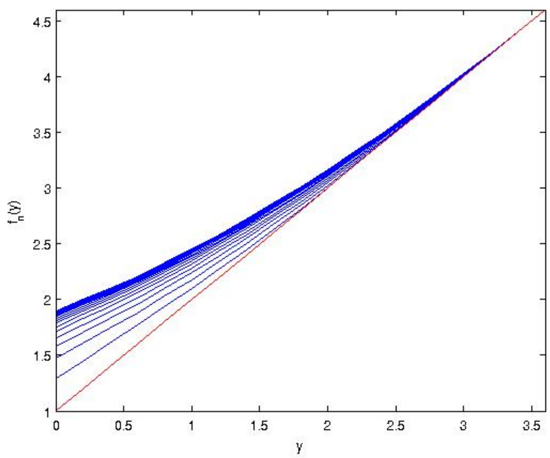

FIGURE 2. The convergence of the value functions $f_0 - f_{20}$, with parameters $\mu_1 = -0.8$, $\mu_2 = 0.8$, $r = 0.05$, $\lambda = 1$, $\sigma = 1$ and $p = 0.3$.

where the third equality follows by monotone convergence. Thus f_∞ is a fixed point for J. □

Lemma 6.2. *The limit $b_\infty = \lim_{n\to\infty} b_n$ is the unique solution of* (17) *for $f = f_\infty$.*

Proof. Let l_n be as in (14) for $f = f_{n-1}$. We have

$$\int_0^{b_\infty} \frac{l_n(z)}{z\varphi(z)}\,dz = \int_0^{b_n} \frac{l_n(z)}{z\varphi(z)}\,dz + \int_{b_n}^{b_\infty} \frac{l_n(z)}{z\varphi(z)}\,dz = \int_{b_n}^{b_\infty} \frac{l_n(z)}{z\varphi(z)}\,dz \to 0$$

as $n \to \infty$ since $0 \le b_\infty - b_n \to 0$ and l_n is bounded on compacts uniformly in n. On the other hand, by monotone convergence,

$$\int_0^{b_\infty} \frac{l_n(z)}{z\varphi(z)}\,dz \to \int_0^{b_\infty} \frac{l_\infty(z)}{z\varphi(z)}\,dz,$$

where l_∞ is defined as in (14) with $f = f_\infty$. Consequently,

$$\int_0^{b_\infty} \frac{l_\infty(z)}{z\varphi(z)}\,dz = 0,$$

which finishes the proof. □

Corollary 6.3. *The value function U defined in* (11) *satisfies $U = f_\infty$. Moreover, let $B := b_\infty$. Then τ_B is an optimal stopping time in* (11).

Proof. This follows from the verification result Theorem 4.1. □

Remark. In view of Proposition 3.1, τ_B is an optimal stopping time for the liquidation problem (2). Moreover, the value V satisfies $V = \frac{x}{1+\phi} f_\infty(\phi)$.

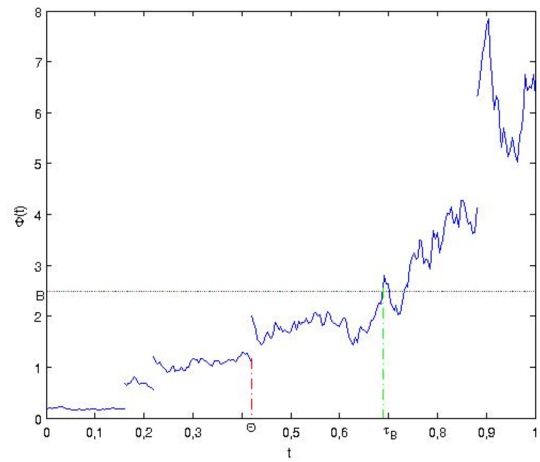

FIGURE 3. The process Φ_t corresponding to the simulated path in Figure 1. The parameters are $\mu_2 = 0.25$, $\mu_1 = -0.15$, $r = 0$, $\lambda = 1.5$, $p = 0.3$ and $\sigma = 0.4$, which yield $B = 2.48$.

Remark. It is straightforward to check that the operator J is a contraction on the space of continuous functions f such that $1 + y \leq f(y) \leq C + y$ equipped with the supremum distance. Consequently, the fixed-point f_∞ is unique by the Banach fixed point theorem, and we could start the iteration with any element $f_0 \in \mathbb{F}$. The choice $f_0 = 1 + y$ (or $f_0 = C + y$) seems convenient since f_n then forms an increasing (decreasing) sequence. Moreover, the choice $f_0 = 1 + y$ is natural since f_n in that case equals the value function in the optimal stopping problem if stopping is restricted to the time interval $[0, T_n]$.

7. Rate of convergence

In this section we study the rate of convergence for the iterative procedure above. We show that both the convergence $f_n \to f_\infty$ and $b_n \to b_\infty$ are exponential.

Theorem 7.1. *We have*

$$0 \leq f_{n+1} - f_n \leq \frac{\mu_2 - r}{\lambda/p + r - \mu_2}\epsilon^n \tag{19}$$

for all $n \geq 0$*, where* $\epsilon = \frac{\lambda(1-p)}{\lambda + rp - \mu_2 p}$*. Consequently,*

$$f_n \leq f_\infty \leq f_n + \frac{\mu_2 - r}{\lambda + r - \mu_2}\epsilon^n \tag{20}$$

holds for all $n \geq 0$*.*

Proof. To prove (19) we use an induction argument. First note that

$$\mathcal{A}(D+y) + \frac{\lambda(1-p)}{p} f_0(S(y)) = (\mu_1 - r)y + D(\mu_2 - r - \lambda/p) + \lambda/p \le 0$$

holds for $D = \frac{\lambda}{\lambda + rp - \mu_2 p}$. From similar arguments as in the verification results above it then follows that $f_1(y) = Jf_0(y) \le \frac{\lambda}{\lambda+rp-\mu_2 p} + y$ and hence $f_1(y) - f_0(y) \le \frac{\mu_2 - r}{\lambda/p + r - \mu_2}$. Consequently (19) holds for $n = 0$. Now assume (19) holds for some $n \ge 0$. Then

$$\begin{aligned} & f_{n+2}(y) - f_{n+1}(y) \\ \le \;& \sup_\tau \mathbb{E}_y \left[\frac{\lambda(1-p)}{p} \int_0^\tau e^{(\mu_2 - r - \lambda/p)t} \left(f_{n+1} - f_n\right)(S(Y_t))\, dt \right] \\ \le \;& \frac{\mu_2 - r}{\lambda/p + r - \mu_2} \epsilon^n \frac{\lambda(1-p)}{p} \int_0^\infty e^{(\mu_2 - r - \lambda/p)t}\, dt \\ = \;& \frac{\mu_2 - r}{\lambda/p + r - \mu_2} \epsilon^{n+1}, \end{aligned}$$

so (19) holds for all $n \ge 0$ by induction.

Finally, note that

$$f_\infty - f_n = \sum_{k=n}^\infty f_{k+1} - f_k \le \sum_{k=n}^\infty \frac{\mu_2 - r}{\lambda/p + r - \mu_2} \epsilon^k = \frac{\mu_2 - r}{\lambda + r - \mu_2} \epsilon^n,$$

so (20) is a consequence of (19). □

Remark. Note that $\epsilon = \frac{\lambda(1-p)}{\lambda + rp - \mu_2 p} < 1$. Consequently, the sequence f_n converges uniformly to f_∞, and the rate of convergence is exponential.

We now prove a corresponding result for the optimal boundaries b_n.

Theorem 7.2. *(Rate of convergence of b_n to b_∞.) The inequality*

$$0 \le b_{n+1} - b_n \le K\epsilon^n \tag{21}$$

holds for all $n \ge 1$, where $K = \frac{(1-\gamma_1)(\mu_2 - r)(r + \lambda/p - \mu_2)}{-\gamma_1 (r - \mu_1)(r + \lambda - \mu_2)}$. Consequently,

$$b_n \le b_\infty \le b_n + \frac{K\epsilon^n}{1-\epsilon}. \tag{22}$$

Proof. For $n \ge 1$, define (compare the proof of Lemma 5.3)

$$l_n(z) = (\mu_1 - r - \lambda/p)z + \mu_2 - r - \lambda/p + \frac{\lambda(1-p)}{p} f_{n-1}(S(z)),$$

and

$$h_n(y) = \int_0^y \frac{l_n(z)}{z\varphi(z)}\, dz.$$

With this notation we have $h_n(b_n) = h_{n+1}(b_{n+1}) = 0$, and by the properties of h_n we have $b_{n+1} - b_n \ge 0$ so the first inequality in (21) follows immediately.

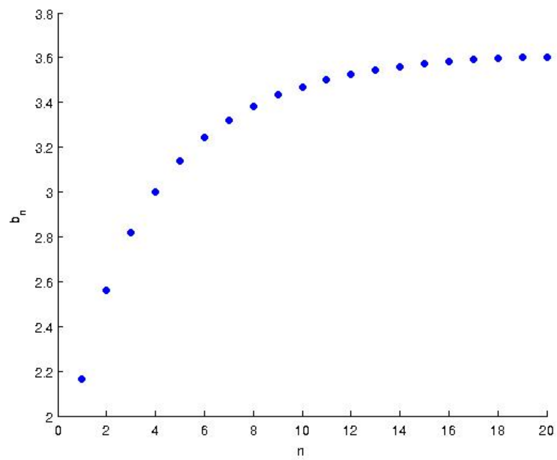

FIGURE 4. The convergence of the optimal boundaries b_1 to b_{20}, for the same parameter values as in Figure 2.

Take $y \in [b_n, b_{n+1}]$ and observe that for such a y we have $f_n(S(y)) = 1 + S(y) = (1+y)/(1-p)$. Therefore

$$l_{n+1}(y) = (\mu_1 - r)y + \mu_2 - r,$$

so

$$l_{n+1}(y) \leq l_{n+1}(b_n) = (\mu_1 - r)b_n + \mu_2 - r \leq 0$$

since $b_n \geq \hat{B}$. Consequently,

$$h'_{n+1}(y) = \frac{l_{n+1}(y)}{y\varphi(y)} \leq \frac{-(r + (r-\mu_1)b_n - \mu_2)}{b_{n+1}\varphi(b_n)}. \tag{23}$$

Furthermore,

$$\begin{aligned} 0 &\leq h_{n+1}(b_n) = h_{n+1}(b_n) - h_n(b_n) \\ &= \frac{\lambda(1-p)}{p} \int_0^{b_n} \frac{f_n(S(y)) - f_{n-1}(S(y))}{y\varphi(y)}\, dy \\ &\leq \frac{\lambda(1-p)}{p} \frac{\mu_2 - r}{\lambda/p + r - \mu_2} \epsilon^{n-1} \int_0^{b_n} y^{-1-\gamma_1} dy \\ &= \frac{(\mu_2 - r)\epsilon^n}{-\gamma_1} \frac{1}{\varphi(b_n)}, \end{aligned}$$

where the second inequality follows from (19). This, combined with (23) and the bounds for b_n and b_{n+1} determined in the remark after Lemma 5.3, yields

$$0 \leq b_{n+1} - b_n \leq \frac{(\mu_2 - r)\epsilon^n}{-\gamma_1} \frac{b_{n+1}}{-(r + (r-\mu_1)b_n - \mu_2)} \leq K\epsilon^n.$$

This finishes the proof of (21).

Finally,

$$b_\infty - b_n = \sum_{k=n}^{\infty} b_{k+1} - b_k \le K \sum_{k=n}^{\infty} \epsilon^k = K \frac{\epsilon^n}{1-\epsilon},$$

which proves (22). □

8. Parameter dependencies

It is intuitively clear that the value V is decreasing in the initial probability π, in the jump rate λ and increasing in the parameter p. Moreover, a large volatility σ gives large (trendless) fluctuations of X, thereby making inference about the drift more noisy, so V should be decreasing in σ. Similarly, increasing μ_2 gives a larger drift as well as widens the gap between the drifts, so inference is faster and the value should then be increasing in μ_2. Finally, increasing μ_1 gives on one hand a larger drift, but on the other hand a slower inference, so the dependence on μ_1 is not a priori clear.

In the current section we confirm the above intuitive statements about π, λ, p, σ and μ_2. As indicated above, we do not know whether there is a monotone dependence on μ_1.

Theorem 8.1. *The value V is non-increasing in π, in the volatility σ and in the jump rate λ, and it is non-decreasing in the initial drift μ_2 and in the parameter p. Consequently, the optimal liquidation level B is non-increasing in σ and λ and non-decreasing in μ_2 and p.*

Proof. First note that $U \ge 1+\phi$ and $U' \le 1$ imply $(1+\phi)U'(\phi) - U(\phi) \le 0$. Therefore it follows from the relation $V = \frac{x}{1+\phi} U$ that

$$\frac{\partial V}{\partial \phi} = x \frac{\partial}{\partial \phi} \frac{U(\phi)}{1+\phi} = x \frac{(1+\phi)U'(\phi) - U(\phi)}{(1+\phi)^2} \le 0.$$

Since $\pi = \frac{\phi}{1+\phi}$ is increasing in ϕ, V is non-increasing also as a function of π.

Now, let $\sigma_2 \ge \sigma_1 > 0$, $\lambda_1 \le \lambda_2$, $\mu_{2,1} \ge \mu_{2,2}$ and $p_1 \ge p_2$, and let $U_i, B_i, \mathcal{L}_i, i = 1,2$ be the corresponding value functions, optimal exercise boundaries and linear operators, respectively. Also let $\omega_i = (\mu_{2,i} - \mu_1)/\sigma_i$, and note that $\omega_1 \ge \omega_2$. For $\phi < B_2$ we have

$$\begin{aligned}
\mathcal{L}_1 U_2(\phi) &= \mathcal{L}_2 U_2(\phi) + \mathcal{L}_1 U_2(\phi) - \mathcal{L}_2 U_2(\phi) \\
&= \frac{(\omega_1^2 - \omega_2^2)}{2} \phi^2 U_2''(\phi) + (\mu_{2,1} - \mu_{2,2})(U_2(\phi) - \phi U_2'(\phi)) \\
&\quad + (\frac{\lambda_2}{p_2} - \frac{\lambda_1}{p_1}) U_2(\phi) + \lambda_1 \frac{1-p_1}{p_1} U_2(\frac{p_1+\phi}{1-p_1}) - \lambda_2 \frac{1-p_2}{p_2} U_2(\frac{p_2+\phi}{1-p_2}) \\
&\ge 0,
\end{aligned}$$

where we used $\mathcal{L}_2 U_2(\phi) = 0$, the convexity of U_2, $U_2' \le 1$ and $U_2 \ge 1+\phi$.

Now, let Φ be the solution of (12) with $\sigma = \sigma_1$, $\lambda = \lambda_1$, $\mu_2 = \mu_{2,1}$ and $p = p_1$, and let $\tau_{B_2} = \inf\{t \ge 0 : \Phi_t \ge B_2\}$ be the first passage time over

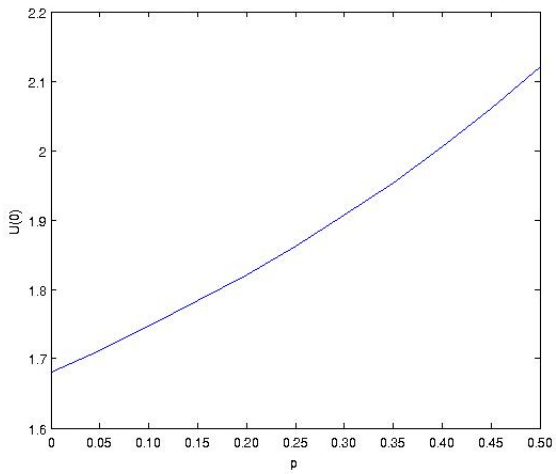

FIGURE 5. $U(0)$ plotted as a function of p. The other parameter values are as in Figure 2. In the limit as $p \to 0$, the case of incomplete information, as studied in [7], is obtained.

B_2. By Ito's formula,

$$e^{(\mu_{2,1}-\lambda_1-r)(t\wedge\tau_{B_2})}U_2(\Phi_{t\wedge\tau_{B_2}}) = U_2(\phi)+\int_0^{t\wedge\tau_{B_2}} e^{(\mu_{2,1}-\lambda_1-r)s}\mathcal{L}_1 U_2(\Phi_s)\,ds+M_t,$$

where M_t is a martingale (compare the proof of Theorem 4.1). It follows that

$$\begin{aligned} U_2(\phi) &\leq \mathbb{E}_\phi^{\mathbb{Q}}\left[e^{(\mu_{2,1}-\lambda_1-r)\tau_{B_2}}U_2(\Phi_{\tau_{B_2}})\right] \\ &= \mathbb{E}_\phi^{\mathbb{Q}}\left[e^{(\mu_{2,1}-\lambda_1-r)\tau_{B_2}}(1+\Phi_{\tau_{B_2}})\right] \leq U_1(\phi). \end{aligned}$$

Consequently, U (hence also V) exhibits the claimed monotone dependencies.

The indicated dependencies for the optimal liquidation level B now follow immediately since $B = \inf\{\phi > 0 : U(\phi) = 1+\phi\}$. □

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