Stopping problems with an unknown state

Erik Ekström
Department of Mathematics, Uppsala University

Yuqiong Wang
Department of Mathematics, Uppsala University

June 12, 2023

Abstract

We extend the classical setting of an optimal stopping problem under full information to include for problems with an unknown state. The framework allows the unknown state to influence (i) the drift of the underlying process, (ii) the payoff functions, and (iii) the distribution of the time horizon. Since the stopper is assumed to observe the underlying process and the random horizon, this is a two-source learning problem. Assigning a prior distribution for the unknown state, standard filtering theory can be employed to embed the problem in a Markovian framework with one additional state-variable representing the posterior of the unknown state. We provide a convenient formulation of this Markovian problem, based on a measure change technique that decouples the underlying process from the new state variable. Moreover, we show by means of several novel examples that this reduced formulation can be used to solve problems explicitly.

1 Introduction

In most literature on optimal stopping theory, the stopper acts under full information about the underlying system. In some applications, however, information is a scarce resource, and the stopper then needs to base her decision only on the information available upon stopping. We study stopping problems of the type

$$\sup_{\tau} \mathbb{E}\left[g(\tau, X_\tau, \theta)1_{\{\tau<\gamma\}} + h(\gamma, X_\gamma, \theta)1_{\{\gamma\leq\tau\}}\right], \tag{1}$$

where X is a diffusion process; here g and h are given functions representing the payoff if stopping occurs before or after the random time horizon γ, respectively, and θ is a Bernoulli random variable representing the unknown state. This unknown state may influence the drift of the diffusion process X, the distribution of the random horizon γ and the payoff functions g and h.

Cases with $g(t, x, \theta) = g(t, \theta)$ and $h(t, x, \theta) = h(t, \theta)$ are closely related to statistical problems, where the process X merely serves as an observation process but does not affect the payoff upon stopping. A classical example is the sequential testing problem for a Wiener process, see [21] for a perpetual version and [19] for a version with a random horizon. Cases with $g(t, x, \theta) = g(t, x)$ and $h(t, x, \theta) = h(t, x)$, on the other hand, where the unknown state does not affect the payoff directly but only implicitly via the dynamics of X, have been studied mainly in the financial literature. For example, American options with incomplete information about the drift of the underlying process have been studied in [4] and [7], and a liquidation problem has been studied in [5]. Related literature include optimal stopping for regime-switching models (see [12] and [23]), studies of models containing change-points ([9] and [13]), a study allowing for an arbitrary distribution of the unknown state ([6]), problems of stochastic control ([16]) and singular control ([2]), and stochastic games ([3]) under incomplete information. Stopping problems with a random time horizon are studied in, for example, [1] and [17], where the authors consider models with a random finite time horizon, but with state-independent distributions; for a study with a state-dependent random horizon, see [8].

In the current article, we study the optimal stopping problem using the general formulation in (1), which is flexible enough to accommodate several new examples. In particular, the notion of a state-dependent random horizon appears to be largely unstudied, even though it is a natural ingredient in many applications. Indeed, consider a situation where the unknown state is either "good" ($\theta = 1$) or "bad" ($\theta = 0$) for an agent who is thinking of investing in a certain business opportunity. Since agents are typically subject to competition, the business opportunity would eventually disappear, and the rate with which it does so would typically be larger in the "good" state than in the "bad" state. The disappearance of a business opportunity is incorporated in our set-up by choosing the compensation $h \equiv 0$.

In some applications, it is more natural to have a random state-dependent horizon at which the stopper is forced to stop (as opposed to missing out on the opportunity). For example, in modeling of financial contracts with recall risk (see, e.g., [11]), the party who makes the recall would decide on a time point at which the positions at hand have to be terminated. Consequently, problems with $h = g$ can be viewed as problems of forced stopping. More generally, the random horizon can be useful in models with competition, where $h \leq g$ corresponds to situations with first-mover advantage, and $h \geq g$ to situations with second-mover advantage.

We first apply classical filtering methods (see, e.g., [18]) to the stopping problem (1), which allows us to re-formulate the stopping problem in terms of a two-dimensional state process (X, Π), where Π is the probability of one of the states conditional on observations. Then a measure-change technique is employed, where the dynamics of the diffusion process X under the new measure are unaffected by the unknown state, whereas the Radon-Nikodym derivative can be fully expressed in terms of Π. Finally, it is shown how the general set-up, with two spatial dimensions, can be reduced further in specific examples. In fact, we provide three different examples (a hiring problem, a problem of optimal closing of a short position, and a sequential testing problem with random horizon) where it turns out that the spatial dimension is one-dimensional so that the problems are amenable to further analysis. The examples are mainly of motivational character, and in order not to burden the presentation with too many details, we content ourselves by providing the reduction to one spatial dimension – a detailed study of the corresponding one-dimensional problem can then be performed using standard methods of optimal stopping theory.

2 Problem specification

We consider a Bayesian set-up where one observes a diffusion process X in continuous time, the drift of which depends on an unknown state θ that takes values 0 and 1 with probabilities $1 - \pi$ and π, respectively, with $\pi \in [0, 1]$. Given payoff functions g and h, the problem is to stop the process so as to maximize the expected reward in (1). Here the random horizon γ has a state-dependent distribution, but is independent of the noise of X.

The above set-up can be realised by considering a probability space $(\Omega, \mathcal{F}, \mathbb{P}_\pi)$ that hosts a standard Brownian motion W and an independent Bernoulli-distributed random variable θ with $\mathbb{P}_\pi(\theta = 1) = \pi = 1 - \mathbb{P}_\pi(\theta = 0)$. Additionally, we let γ be a random time (possible infinite) independent of W and with state-dependent survival distribution

$$\mathbb{P}_\pi(\gamma > t | \theta = i) = F_i(t),$$

where F_i is continuous and non-increasing with $F_i(0) = 1$ and $F_i(t) > 0$ for all $t \geq 0$, $i = 0, 1$. We remark that we include the possibility that $F_i \equiv 1$ for some $i \in \{0, 1\}$ (or for both), corresponding to an infinite horizon. We then have

$$\mathbb{P}_\pi = (1 - \pi)\mathbb{P}_0 + \pi \mathbb{P}_1, \tag{2}$$

where $\mathbb{P}_0(\cdot) = \mathbb{P}_\pi(\cdot|\theta = 0)$ and $\mathbb{P}_1(\cdot) = \mathbb{P}_\pi(\cdot|\theta = 1)$.

Now consider the equation

$$dX_t = \mu(X_t, \theta)\, dt + \sigma(X_t)\, dW_t. \tag{3}$$

Here $\mu(\cdot,\cdot) : \mathbb{R} \times \{0, 1\} \to \mathbb{R}$ is a given function of the unknown state θ and the current value of the underlying process; we denote by $\mu_0(x) = \mu(x, 0)$ and $\mu_1(x) = \mu(x, 1)$. The diffusion coefficient $\sigma(\cdot) : \mathbb{R} \to (0, \infty)$ is a given function of x, independent of the unknown state θ. We assume that the functions μ_0, μ_1 and σ satisfy standard Lipschitz conditions so that the existence and uniqueness of a strong solution X is guaranteed. We are also given two functions

$$g(\cdot,\cdot,\cdot) : [0, \infty) \times \mathbb{R} \times \{0, 1\} \to \mathbb{R}$$

and

$$h(\cdot,\cdot,\cdot) : [0, \infty) \times \mathbb{R} \times \{0, 1\} \to \mathbb{R},$$

which we refer to as the payoff functions. We assume that g_i and h_i are continuous for $i = 0, 1$, where we use the notation $g_i(\cdot,\cdot) := g(\cdot,\cdot,i)$ and $h_i(\cdot,\cdot) := h(\cdot,\cdot,i)$ to denote the payoff functions on the event $\{\theta = i\}$, $i = 0, 1$.

Denote by $\mathcal{F}^X$ the smallest right-continuous filtration that makes X adapted, and let $\mathcal{T}^X$ be the set of $\mathcal{F}^X$-stopping times. Similarly, denote by $\mathcal{F}^{X,\gamma}$ the smallest right-continuous filtration to which both X and the process $1_{\{\cdot\geq\gamma\}}$ are adapted, and let $\mathcal{T}^{X,\gamma}$ be the set of $\mathcal{F}^{X,\gamma}$-stopping times.

We now consider the optimal stopping problem

$$V = \sup_{\tau\in\mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau<\gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma\leq\tau\}} \right]. \tag{4}$$

In (4), and in similar expressions throughout the paper, we use the convention that $h(\tau, X_\tau, \theta) := 0$ on the event $\{\tau = \gamma = \infty\}$. We further assume that the integrability condition

$$\mathbb{E}_\pi \left[\sup_{t\geq 0} \{ |g(t, X_t, \theta)| + |h(t, X_t, \theta)| \} \right] < \infty$$

holds.

Remark 1. *The unknown state θ in the stopping problem (4) influences*

(i) the drift of the process X,

(ii) the payoffs g and h, and

(iii) the survival distribution of the random horizon γ.

More precisely, on the event $\{\theta = 0\}$ the drift of X is $\mu_0(\cdot)$, the payoff functions are $g_0(\cdot,\cdot)$ and $h_0(\cdot,\cdot)$, and the random horizon has survival distribution function $F_0(\cdot)$; on the event $\{\theta = 1\}$, the drift is μ_1, the payoff functions are $g_1(\cdot,\cdot)$ and $h_1(\cdot,\cdot)$, and the random horizon has survival distribution function $F_1(\cdot)$.

Remark 2. *In (4), the payoff on the event $\{\tau = \gamma\}$ is specified in terms of h. In some applications, however, one may want to use the alternative formulation*

$$U := \sup_{\tau\in\mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau\leq\gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma<\tau\}} \right]. \tag{5}$$

If $g \geq h$, then we have

$$U = \sup_{\tau\in\mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau<\gamma\}} + g(\gamma, X_\gamma, \theta) 1_{\{\gamma\leq\tau\}} \right]$$

since $\tau \wedge \gamma \in \mathcal{T}^{X,\gamma}$ for every $\tau \in \mathcal{T}^{X,\gamma}$, and thus the formulation (5) is contained in the formulation (4).

Similarly, if $g \leq h$, then

$$U = \sup_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma \leq \tau\}} \right] = V$$

where the first equality uses that for a given stopping time τ, the time

$$\tau' = \begin{cases} \tau & \text{on } \{\tau < \gamma\} \\ \infty & \text{on } \{\tau \geq \gamma\} \end{cases}$$

is also a stopping time. In the general case where no ordering between g and h is given, however, the formulation (5) is more involved; such cases are not covered by the results of the current article.

3 A useful reformulation of the problem

In this section we rewrite the optimal stopping problem (4) with incomplete information as an optimal stopping problem with respect to stopping times in $\mathcal{T}^X$ and with complete information.

First consider the stopping problem

$$\hat{V} = \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma \leq \tau\}} \right], \tag{6}$$

where the supremum is taken over $\mathcal{F}^X$-stopping times. Since $\mathcal{T}^X \subseteq \mathcal{T}^{X,\gamma}$, we have $\hat{V} \leq V$. On the other hand, by a standard argument, cf. [1] or [17], we also have the reverse inequality, so $\hat{V} = V$. Indeed, first recall that for any $\tau \in \mathcal{T}^{X,\gamma}$ there exists $\tau' \in \mathcal{T}^X$ such that $\tau \wedge \gamma = \tau' \wedge \gamma$, see [20, page 378]. Consequently, $\tau = \tau'$ on $\{\tau < \gamma\} = \{\tau' < \gamma\}$ and $\tau \wedge \gamma = \tau' \wedge \gamma = \gamma$ on $\{\tau \geq \gamma\} = \{\tau' \geq \gamma\}$, so

$$\mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau < \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma \leq \tau\}} \right] = \mathbb{E}_\pi \left[g(\tau', X_{\tau'}, \theta) 1_{\{\tau' < \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma \leq \tau'\}} \right],$$

from which $\hat{V} = V$ follows. Moreover, if $\tau' \in \mathcal{T}^X$ is optimal in (6), then it is also optimal in (4).

Remark 3. *Since we assume that the survival distributions are continuous, we have*

$$\mathbb{P}_\pi(\tau = \gamma < \infty) = 0$$

for any $\tau \in \mathcal{T}^X$. Consequently, we can alternatively write

$$\hat{V} = \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi \left[g(\tau, X_\tau, \theta) 1_{\{\tau \leq \gamma\}} + h(\gamma, X_\gamma, \theta) 1_{\{\gamma < \tau\}} \right].$$

To study the stopping problem (4), or equivalently, the optimal stopping problem (6), we introduce the conditional probability process

$$\Pi_t := \mathbb{P}_\pi(\theta = 1 | \mathcal{F}_t^X)$$

and the corresponding probability ratio process

$$\Phi_t := \frac{\Pi_t}{1 - \Pi_t}. \tag{7}$$

Note that $\Pi_0 = \pi$ and $\Phi_0 = \varphi := \pi/(1-\pi)$, $\mathbb{P}_\pi$−a.s.

Proposition 4. *We have*

$$V = \sup_{\tau\in\mathcal{T}^X} \mathbb{E}_\pi\Big[g_0(\tau,X_\tau)(1-\Pi_\tau)F_0(\tau) + g_1(\tau,X_\tau)\Pi_\tau F_1(\tau) - \int_0^\tau h_0(t,X_t)(1-\Pi_t)dF_0(t) - \int_0^\tau h_1(t,X_t)\Pi_t dF_1(t)\Big]. \quad (8)$$

Moreover, if $\tau\in\mathcal{T}^X$ is optimal in (8), then it is also optimal in (4).

Proof. Denote by τ a stopping time in $\mathcal{T}^X$. Using the tower property we find that

$$\begin{aligned}
\mathbb{E}_\pi\left[g(\tau,X_\tau,\theta)1_{\{\tau<\gamma\}}\right] &= \mathbb{E}_\pi\left[\mathbb{E}_\pi\left[g_i(\tau,X_\tau)1_{\{\tau<\gamma\}}|\mathcal{F}^X_\tau\right]\right]\\
&= \mathbb{E}_\pi\left[\mathbb{E}_\pi\left[g(\tau,X_\tau,\theta)1_{\{\tau<\gamma\}}|\mathcal{F}^X_\tau,\theta=0\right](1-\Pi_\tau)\right.\\
&\qquad \left.+\mathbb{E}_\pi\left[g(\tau,X_\tau,\theta)1_{\{\tau<\gamma\}}|\mathcal{F}^X_\tau,\theta=1\right]\Pi_\tau\right]\\
&= \mathbb{E}_\pi\left[g_0(\tau,X_\tau)\mathbb{P}_\pi\left(\tau<\gamma|\mathcal{F}^X_\tau,\theta=0\right)(1-\Pi_\tau)\right.\\
&\qquad \left.+\, g_1(\tau,X_\tau)\mathbb{P}_\pi\left(\tau<\gamma|\mathcal{F}^X_\tau,\theta=1\right)\Pi_\tau\right]\\
&= \mathbb{E}_\pi\left[g_0(\tau,X_\tau)(1-\Pi_\tau)F_0(\tau)+g_1(\tau,X_\tau)\Pi_\tau F_1(\tau)\right],
\end{aligned}$$

where we for the last equality recall that γ and τ are independent on the event $\{\theta=i\}$, $i=0,1$.

Similarly, for the second term we use

$$\mathbb{P}_\pi(\theta=0,\gamma\ge t|\mathcal{F}^X_\tau) = (1-\Pi_\tau)F_0(t) \quad \& \quad \mathbb{P}_\pi(\theta=1,\gamma\ge t|\mathcal{F}^X_\tau) = \Pi_\tau F_1(t)$$

so that

$$\begin{aligned}
\mathbb{E}_\pi\left[h(\gamma,X_\gamma,\theta)1_{\{\gamma\le\tau\}}\right] &= \mathbb{E}_\pi\left[\mathbb{E}_\pi\left[h(\gamma,X_\gamma,\theta)1_{\{\gamma\le\tau\}}|\mathcal{F}^X_\tau\right]\right]\\
&= -\mathbb{E}_\pi\left[\int_0^\infty \mathbb{E}_\pi\left[h(\gamma,X_\gamma,\theta)1_{\{\gamma\le\tau\}}|\mathcal{F}^X_\tau,\theta=0,\gamma=t\right](1-\Pi_\tau)dF_0(t)\right.\\
&\qquad \left.+\int_0^\infty \mathbb{E}_\pi\left[h(\gamma,X_\gamma,\theta)1_{\{\gamma\le\tau\}}|\mathcal{F}^X_\tau,\theta=1,\gamma=t\right]\Pi_\tau dF_1(t)\right]\\
&= -\mathbb{E}_\pi\left[\int_0^\infty h_0(t,X_t)1_{\{t\le\tau\}}(1-\Pi_\tau)dF_0(t)\right.\\
&\qquad \left.+\int_0^\infty h_1(t,X_t)1_{\{t\le\tau\}}\Pi_\tau dF_1(t)\right]\\
&= -\mathbb{E}_\pi\left[\int_0^\tau h_0(t,X_t)(1-\Pi_t)dF_0(t)+\int_0^\tau h_1(t,X_t)\Pi_t dF_1(t)\right],
\end{aligned}$$

where the last equality holds due to martingality of Π. The optimal stopping problem (4) therefore coincides with the stopping problem

$$\sup_{\tau\in\mathcal{T}^X} \mathbb{E}_\pi\Big[g_0(\tau,X_\tau)(1-\Pi_\tau)F_0(\tau) + g_1(\tau,X_\tau)\Pi_\tau F_1(\tau) - \int_0^\tau h_0(t,X_t)(1-\Pi_t)dF_0(t) - \int_0^\tau h_1(t,X_t)\Pi_t dF_1(t)\Big].$$

□

It is well-known (see, e.g., [22, p. 180-181] or [10, p. 522]) that the posterior probability is given by

$$\Pi_t = \frac{\frac{\pi}{1-\pi}L_t}{1+\frac{\pi}{1-\pi}L_t},$$

where

$$L_t := \exp\left\{\int_0^t \frac{\mu_1(X_s)-\mu_0(X_s)}{\sigma^2(X_s)}\,dX_s - \frac{1}{2}\int_0^t \frac{\mu_1^2(X_s)-\mu_0^2(X_s)}{\sigma^2(X_s)}\,ds\right\}.$$

Therefore, by an application of Ito's formula, the pair (X, Π) satisfies

$$\begin{cases} dX_t = (\mu_0(X_t) + (\mu_1(X_t) - \mu_0(X_t))\Pi_t)\, dt + \sigma(X_t) d\hat{W}_t \\ d\Pi_t = \omega(X_t)\Pi_t(1-\Pi_t)\, d\hat{W}_t, \end{cases} \tag{9}$$

where $\omega(x) := (\mu_1(x) - \mu_0(x))/\sigma(x)$ is the *signal-to-noise ratio* and

$$\hat{W}_t := \int_0^t \frac{dX_t}{\sigma(X_s)} - \int_0^t \frac{1}{\sigma(X_t)} (\mu_0(X_s) + (\mu_1(X_s) - \mu_0(X_s))\, \Pi_s)\; ds$$

is the so-called innovations process; by P. Lévy's theorem, $\hat{W}$ is a $\mathbb{P}_\pi$-Brownian motion. By our non-degeneracy and Lipschitz assumptions on the coefficients, there exists a unique strong solution to the system (9), and the pair (X, Π) is a strong Markov process. Moreover, using Ito's formula, it is straightforward to check that the likelihood ratio process Φ in (7) satisfies

$$d\Phi_t = \frac{\mu_1(X_t) - \mu_0(X_t)}{\sigma^2(X_t)} \Phi_t (dX_t - \mu_0(X_t)\, dt). \tag{10}$$

4 A measure change

In the current section, we provide a measure change that decouples X from Π. This specific measure change technique was first used in [15], and has afterwards been applied by several authors (see [2], [5], [7], [14]).

Lemma 5. *For $t \in [0, \infty)$, denote by $\mathbb{P}_{\pi,t}$ the measure $\mathbb{P}_\pi$ restricted to $\mathcal{F}_t^X$. We then have*

$$\frac{d\mathbb{P}_{0,t}}{d\mathbb{P}_{\pi,t}} = \frac{1+\varphi}{1+\Phi_t}.$$

Proof. For any $A \in \mathcal{F}_t^X$ we have

$$\mathbb{E}_\pi\left[(1-\Pi_t)1_A\right] = \mathbb{E}_\pi\left[1_{\{\theta=0\}}1_A\right] = (1-\pi)\mathbb{E}_0\left[1_A\right]$$

by (2). Consequently,

$$\frac{d\mathbb{P}_{0,t}}{d\mathbb{P}_{\pi,t}} = \frac{1-\Pi_t}{1-\pi} = \frac{1+\varphi}{1+\Phi_t}.$$

□

Since $1 - \Pi_\tau = 1/(1+\Phi_t)$ and $\Pi_t = \frac{\Phi_t}{1+\Phi_t}$, it is now clear that the stopping problem (4) (or, equivalently, problem (8)) can be written

$$\begin{aligned} V \;&=\; \frac{1}{1+\varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_0 \Big[g_0(\tau, X_\tau) F_0(\tau) + g_1(\tau, X_\tau)\Phi_\tau F_1(\tau) \\ &\qquad - \int_0^\tau h_0(t, X_t) dF_0(t) - \int_0^\tau h_1(t, X_t)\Phi_t dF_1(t) \Big], \end{aligned}$$

where the expected value is with respect to $\mathbb{P}_0$, under which the process (X, Φ) is strong Markov and satisfies

$$\begin{cases} dX_t = \mu_0(X_t)\, dt + \sigma(X_t)\, dW_t \\ d\Phi_t = \omega(X_t)\Phi_t\, dW_t \end{cases} \tag{11}$$

(cf. (3) and (10)).

Next we introduce the process

$$\Phi_t^\circ := \frac{F_1(t)}{F_0(t)} \Phi_t, \tag{12}$$

so that

$$V = \frac{1}{1+\varphi}\sup_{\tau\in\mathcal{T}^X}\mathbb{E}_0\Big[F_0(\tau)\left(g_0(\tau,X_\tau)+g_1(\tau,X_\tau)\Phi^\circ_\tau\right) - \int_0^\tau h_0(t,X_t)dF_0(t) - \int_0^\tau \frac{F_0(t)}{F_1(t)}h_1(t,X_t)\Phi^\circ_t dF_1(t)\Big]. \tag{13}$$

Note that the process Φ° satisfies

$$d\Phi^\circ_t = \frac{1}{f(t)}\Phi^\circ_t\, df(t) + \omega(X_t)\Phi^\circ_t dW_t$$

under $\mathbb{P}_0$, where $f(t) = F_1(t)/F_0(t)$.

Remark 6. *The process Φ° is the likelihood ratio given observations of the processes X and $1_{\{\cdot\geq\gamma\}}$ on the event $\{\gamma > t\}$. Indeed, for $t \leq T$, defining*

$$\Pi^\circ_t := \mathbb{P}_\pi(\theta = 1|\mathcal{F}^X_t, \gamma > t) = \frac{\mathbb{P}_\pi(\theta=1,\gamma>t|\mathcal{F}^X_t)}{\mathbb{P}_\pi(\gamma>t|\mathcal{F}^X_t)} = \frac{\Pi_t F_1(t)}{\Pi_t F_1(t) + (1-\Pi_t)F_0(t)},$$

we have

$$\Pi^\circ_t = \frac{\Phi^\circ_t}{\Phi^\circ_t + 1}.$$

We summarise our theoretical findings in the following theorem.

Theorem 7. *Denote by*

$$v = \sup_{\tau\in\mathcal{T}^X}\mathbb{E}_0\Big[F_0(\tau)\left(g_0(\tau,X_\tau)+g_1(\tau,X_\tau)\Phi^\circ_\tau\right) - \int_0^\tau h_0(t,X_t)dF_0(t) - \int_0^\tau \frac{F_0(t)}{F_1(t)}h_1(t,X_t)\Phi^\circ_t dF_1(t)\Big], \tag{14}$$

where (X,Φ°) is given by (11) and (12). Then $V = v/(1+\varphi)$, where $\varphi = \pi/(1-\pi)$. Moreover, if $\tau\in\mathcal{T}^X$ is an optimal stopping in (14), then it is also optimal in the original problem (4).

Remark 8. *Under $\mathbb{P}_0$, the three-dimensional process (t, X, Φ°) is strong Markov, and the stopping problem (14) can be naturally embedded in a setting allowing for an arbitrary starting point (t,x,φ). In the sections below we consider examples that can be reduced to problems that only depend on Φ°, where Φ° is a one-dimensional Markov process, which simplifies the embedding.*

5 An example: a hiring problem

In this section we consider a (simplistic) version of a hiring problem. To describe this, consider a situation where a company tries to decide whether or not to employ a certain candidate, where there is considerable uncertainty about the candidate's ability. The candidate is either of a 'stong type' or of a 'weak type', and during the employment procedure, tests are performed to find out which is the true state. At the same time, the candidate is potentially lost for the company as he/she may receive other offers. Moreover, the rate at which such offers are presented, may depend on the ability of the candidate; for example, a candidate of the strong type could be more likely to be recruited to other companies than a candidate of the weak type.

To model the above hiring problem, we let $h_i \equiv 0$, $i = 0, 1$, and

$$g(t,x,\theta) = \begin{cases} -e^{-rt}c & \text{if } \theta = 0 \\ e^{-rt}d & \text{if } \theta = 1 \end{cases}$$

where c and d are positive constants representing the overall cost and benefit of hiring the candidate, respectively, and $r > 0$ is a constant discount rate. To learn about the unknown state θ, tests are performed and represented as a Brownian motion

$$X_t = \mu(\theta)t + \sigma W_t$$

with state-dependent drift

$$\mu(\theta) = \begin{cases} \mu_0 & \text{if } \theta = 0 \\ \mu_1 & \text{if } \theta = 1, \end{cases}$$

where $\mu_0 < \mu_1$. We further assume that the survival probabilities F_0 and F_1 decay exponentially in time, i.e.

$$F_0(t) = e^{-\lambda_0 t} \qquad \& \qquad F_1(t) = e^{-\lambda_1 t},$$

where $\lambda_0, \lambda_1 \geq 0$ are known constants. The stopping problem (4) under consideration is thus

$$V = \sup_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[e^{-r\tau} \left(d1_{\{\theta=1\}} - c1_{\{\theta=0\}} \right) 1_{\{\tau<\gamma\}} \right],$$

where $\pi = \mathbb{P}_\pi(\theta = 1)$.

By Theorem 7, we have

$$V = \frac{1}{1+\varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_0 \left[e^{-(r+\lambda_0)\tau} \left(\Phi^\circ_\tau d - c \right) \right],$$

where the underlying process Φ° is a geometric Brownian motion satisfying

$$d\Phi^\circ_t = -(\lambda_1 - \lambda_0)\Phi^\circ_t \, dt + \omega \Phi^\circ_t \, dW.$$

Clearly, the value of the stopping problem is

$$V = \frac{d}{1+\varphi} \sup_{\tau \in \mathcal{T}^X} \mathbb{E}_0 \left[e^{-(r+\lambda_0)\tau} \left(\Phi^\circ_\tau - \frac{c}{d} \right) \right] = \frac{d}{1+\varphi} V^{Am}(\varphi),$$

where V^{Am} is the value of the American call option with underlying Φ° starting at $\Phi^\circ_0 = \varphi$ and with strike $\frac{c}{d}$. Standard stopping theory gives that the corresponding value is

$$V = \begin{cases} \frac{db^{1-\eta}}{\eta(1+\varphi)} \varphi^\eta, & \varphi < b, \\ \frac{d}{1+\varphi}(\varphi - \frac{c}{d}), & \varphi \geq b, \end{cases}$$

where $\eta > 1$ is the positive solution of the quadratic equation

$$\frac{\omega^2}{2} \eta(\eta - 1) + (\lambda_0 - \lambda_1)\eta - (r + \lambda_0) = 0,$$

and $b = \frac{c\eta}{d(\eta-1)}$. Furthermore,

$$\tau := \inf\{t \geq 0 : \Phi^\circ_t \geq b\}$$

is an optimal stopping time. More explicitly, in terms of the process X we have

$$\tau = \inf \left\{ t \geq 0 : X_t \geq x + \frac{\sigma}{\omega} \left(\ln \left(\frac{b}{\varphi} \right) + (\lambda_1 - \lambda_0)t \right) + \frac{\mu_0 + \mu_1}{2} t \right\},$$

where $\omega := (\mu_1 - \mu_0)/\sigma$.

6 An example: closing a short position

In this section we study an example of optimal closing of a short position under recall risk, cf. [11]. We consider a short position in an underlying stock with unknown drift, where the random horizon corresponds to a time point at which the counterparty recalls the position. Naturally, the counterparty favours a large drift, and we thus assume that the risk of recall is greater in the state with a small drift. A similar model (but with no recall risk) was studied in [5].

Let the stock price be modeled by geometric Brownian motion with dynamics

$$dX_t = \mu(\theta)X_t\,dt + \sigma X_t\,dW_t,$$

where the drift is state-dependent with $\mu(0) = \mu_0 < \mu_1 = \mu(1)$, and σ is a known constant. We let $g(t,x,\theta) = h(t,x,\theta) = xe^{-rt}$ and consider the stopping problem

$$V = \inf_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi[e^{-r\tau\wedge\gamma}X_{\tau\wedge\gamma}],$$

where

$$F_i(t) := \mathbb{P}_\pi(\gamma > t|\theta = i) = e^{-\lambda_i t},$$

with $\lambda_0 > 0 = \lambda_1$ and

$$\mathbb{P}_\pi(\theta = 1) = \pi = 1 - \mathbb{P}_\pi(\theta = 0).$$

Here r is a constant discount rate; to avoid degenerate cases, we assume that $r \in (\mu_0, \mu_1)$.

Then the value function can be written as $V = v/(1+\varphi)$, where

$$v = \inf_{\tau\in\mathcal{T}^X} \mathbb{E}_0\Big[e^{-r\tau}F_0(\tau)X_\tau\,(1+\Phi^\circ_\tau) + \lambda_0\int_0^\tau e^{-rt}F_0(t)X_t\,(1+\Phi^\circ_t)\;dt\Big], \tag{15}$$

with $d\Phi^\circ_t = \lambda_0\Phi^\circ_t dt + \omega\Phi^\circ_t\,dW_t$ and $\Phi^\circ_0 = \varphi$. Here $\omega = \frac{\mu_1-\mu_0}{\sigma}$.

Another change of measure will remove the occurrencies of X in (15). In fact, let $\tilde{\mathbb{P}}$ be a measure with

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}_0}\bigg|_{\mathcal{F}_t} = e^{-\frac{\sigma^2}{2}t+\sigma W_t},$$

so that $\tilde{W}_t = -\sigma t + W_t$ is a $\tilde{\mathbb{P}}-$Brownian motion. Then

$$v = x\inf_{\tau\in\mathcal{T}^X} \tilde{\mathbb{E}}\Big[e^{-(r+\lambda_0-\mu_0)\tau}\,(1+\Phi^\circ_\tau) + \lambda_0\int_0^\tau e^{-(r+\lambda_0-\mu_0)t}\,(1+\Phi^\circ_t)\;dt\Big], \tag{16}$$

with

$$d\Phi^0_t = (\lambda_0+\sigma\omega)\Phi^0_t\,dt + \omega\Phi^0_t\,d\tilde{W}.$$

The optimal stopping problem (16) is a one-dimensional time-homogeneous problem, and is thus straightforward to analyze using standard stopping theory. Indeed, setting

$$\bar{v} := \frac{v}{x} = \inf_{\tau\in\mathcal{T}^X} \tilde{\mathbb{E}}\Big[e^{-(r+\lambda_0-\mu_0)\tau}\,(1+\Phi^\circ_\tau) + \lambda_0\int_0^\tau e^{-(r+\lambda_0-\mu_0)t}\,(1+\Phi^\circ_t)\;dt\Big],$$

the associated free-boundary problem is to find $(\bar{v}, B)$ such that

$$\begin{cases} \frac{\omega^2\varphi^2}{2}\bar{v}_{\varphi\varphi} + (\lambda_0+\mu_1-\mu_0)\varphi\bar{v}_\varphi - (r+\lambda_0-\mu_0)\bar{v} + \lambda_0(1+\varphi) = 0 & \varphi < B \\ \bar{v}(\varphi) = 1+\varphi & \varphi \geq B \\ \bar{v}_\varphi(B) = 1, & \end{cases} \tag{17}$$

and such that $\bar{v} \leq 1+\varphi$. Solving the free-boundary problem (17) gives

$$B = \frac{\eta(r-\mu_0)(\mu_1-r)}{(1-\eta)(r+\lambda_0-\mu_0)(\lambda_0+\mu_1-r)}$$

and

$$\bar{v}(\varphi) = \begin{cases} \frac{r-\mu_0}{(1-\eta)(r+\lambda_0-\mu_0)} \left(\frac{\varphi}{B}\right)^\eta - \frac{\lambda_0}{\mu_1 - r}\varphi + \frac{\lambda_0}{r+\lambda_0-\mu_0} & \varphi < B \\ 1+\varphi & \varphi \geq B, \end{cases}$$

where $\eta < 1$ is the positive solution of the quadratic equation

$$\frac{\omega^2}{2}\eta(\eta-1) + (\lambda_0 + \mu_1 - \mu_0)\eta - (r + \lambda_0 - \mu_0) = 0.$$

Note that $\bar{v}$ is concave since it satisfies the smooth-fit condition at B and since $\eta < 1$, so $\bar{v}(\varphi) \leq 1 + \varphi$. A standard verification argument then gives that $V = \frac{x}{1+\varphi}\bar{v}(\varphi)$, and

$$\tau_B := \inf\{t \geq 0 : \Phi_t^\circ \geq B\} = \inf\{t \geq 0 : \Phi_t \geq Be^{-\lambda_0 t}\}$$

is optimal in (15).

7 An example: a sequential testing problem with a random horizon

Consider the sequential testing problem for a Wiener process, i.e. the problem of determining as quickly, and accurately, an unknown drift θ from observations of the process

$$X_t = \theta t + \sigma W_t.$$

Similar to the classical version (see [21]), we assume that θ is Bernoulli distributed with $\mathbb{P}_\pi(\theta = 1) = \pi = 1 - \mathbb{P}_\pi(\theta = 0)$, where $\pi \in (0,1)$. In [19], the sequential testing problem has been studied under a random horizon. Here we consider an instance of a testing problem which further extends the set-up by allowing the distribution of the random horizon to depend on the unknown state.

More specifically, we assume that when $\theta = 1$, then the horizon γ is infinite, i.e. $F_1(t) = 1$ for all t; and when $\theta = 0$, the time horizon is exponentially distributed with rate λ, i.e. $F_0(t) = e^{-\lambda t}$. Mimicking the classical formulation of the problem, we study the problem of minimizing

$$\mathbb{P}_\pi(\theta \neq d) + c\mathbb{E}_\pi[\tau]$$

over all stopping times $\tau \in \mathcal{T}^{X,\gamma}$ and $\mathcal{F}_\tau^{X,\gamma}$-measurable decision rules d with values in $\{0,1\}$. By standard methods, the above optimization problem reduces to a stopping problem

$$V = \inf_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[\hat{\Pi}_\tau \wedge (1 - \hat{\Pi}_\tau) + c\tau\right],$$

where

$$\hat{\Pi}_t := \mathbb{P}_\pi(\theta = 1 | \mathcal{F}_t^{X,\gamma}).$$

Moreover, the process $\hat{\Pi}$ satisfies

$$\hat{\Pi}_t = \begin{cases} \Pi_t^\circ & t < \gamma \\ 0 & t \geq \gamma, \end{cases}$$

where

$$\Pi_t^\circ = \frac{\Pi_t}{\Pi_t + (1-\Pi_t)e^{-\lambda t}} = \frac{\mathbb{P}_\pi(\theta = 1|\mathcal{F}_t^X)}{\mathbb{P}_\pi(\theta = 1|\mathcal{F}_t^X) + (1 - \mathbb{P}_\pi(\theta = 1|\mathcal{F}_t^X))e^{-\lambda t}},$$

and it follows that

$$\begin{aligned} V &= \inf_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[\hat{\Pi}_\tau \wedge (1 - \hat{\Pi}_\tau) + c\tau\right] \\ &= \inf_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[(\Pi_\tau^\circ \wedge (1 - \Pi_\tau^\circ))\, 1_{\{\tau < \gamma\}} + c(\tau \wedge \gamma)\right] \\ &= \inf_{\tau \in \mathcal{T}^{X,\gamma}} \mathbb{E}_\pi \left[(\Pi_\tau^\circ \wedge (1 - \Pi_\tau^\circ) + c\tau)\, 1_{\{\tau < \gamma\}} + c\gamma 1_{\{\gamma \leq \tau\}}\right]. \end{aligned}$$

In other words, $g_i(t,\pi) = \pi \wedge (1-\pi) + ct$ and $h_i(t,\pi) = ct$ for $i \in \{0,1\}$. Following the general methodology leading up to Theorem 7, we find that

$$
\begin{aligned}
V &= \inf_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi \left[(\Pi^\circ_\tau \wedge (1-\Pi^\circ_\tau) + c\tau)\left((1-\Pi_\tau)F_0(\tau) + \Pi_\tau\right) - c\int_0^\tau t(1-\Pi_t)\, dF_0(t) \right] \\
&= \inf_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi \Bigg[(\Pi^\circ_\tau \wedge (1-\Pi^\circ_\tau) + c\tau)\left((1-\Pi_\tau)F_0(\tau) + \Pi_\tau\right) - c\tau(1-\Pi_\tau)F_0(\tau) \\
&\qquad\qquad -c\int_0^\tau F_0(t)\, d\left(t(1-\Pi_t)\right) \Bigg] \\
&= \inf_{\tau \in \mathcal{T}^X} \mathbb{E}_\pi \left[(\Pi^\circ_\tau \wedge (1-\Pi^\circ_\tau))\left((1-\Pi_\tau)F_0(\tau) + \Pi_\tau\right) + c\int_0^\tau \left((1-\Pi_t)F_0(t) + \Pi_t\right) dt \right] \\
&= \frac{1}{1+\varphi} \inf_{\tau \in \mathcal{T}^X} \mathbb{E}_0 \left[F_0(\tau)\left(\Phi^\circ_\tau \wedge 1\right) + c\int_0^\tau F_0(t)(1+\Phi^\circ_t)\, dt \right].
\end{aligned}
$$

Here $\Phi^\circ := \Pi^\circ/(1-\Pi^\circ)$ satisfies

$$
d\Phi^\circ_t = \lambda \Phi^\circ_t\, dt + \omega \Phi^\circ_t dW_t,
$$

where $\omega = \frac{1}{\sigma}$.

Standard stopping theory can now be applied to solve the sequential testing problem with a random horizon. Setting

$$
v(\varphi) := \inf_{\tau \in \mathcal{T}^X} \mathbb{E}_0 \left[F_0(\tau)\left(\Phi^\circ_\tau \wedge 1\right) + c\int_0^\tau F_0(t)(1+\Phi^\circ_t)\, dt \right],
$$

where $\Phi^\circ_0 = \varphi$, one expects a two-sided stopping region $(0,A] \cup [B,\infty)$, and v to satisfy

$$
\begin{cases}
\frac{1}{2}\omega^2\varphi^2 v_{\varphi\varphi} + \lambda\varphi v_\varphi - \lambda v + c(1+\varphi) = 0, & \varphi \in (A,B) \\
v(A) = A \\
v_\varphi(A) = 1 \\
v(B) = 1 \\
v_\varphi(B) = 0
\end{cases}
$$

for some constants A, B with $0 < A < 1 < B$. The general solution of the ODE is easily seen to be

$$
v(\varphi) = C_1 \varphi^{-\frac{2\lambda}{\omega^2}} + C_2\varphi + \frac{c}{\lambda} - \frac{c}{\lambda + \frac{1}{2}\omega^2}\varphi \ln(\varphi)
$$

where C_1, C_2 are arbitrary constants. Since the stopping region is two-sided, explicit solutions are not expected. Instead, using the four boundary conditions, equations for the unknowns C_1, C_2, A and B can be derived using standard methods; we omit the details.

Acknowledgement 9. *We thank two anonymous referees for their thorough reading and constructive criticism. The feedback helped us to improve considerably the presentation and the arguments.*

References

[1] A. Chakrabarty and X. Guo. Optimal stopping times with different information levels and with time uncertainty. In *Stochastic Analysis and Applications to Finance: Essays in Honour of Jia-an Yan*, pages 19–38. World Scientific, 2012.

[2] T. De Angelis. Optimal dividends with partial information and stopping of a degenerate reflecting diffusion. *Finance Stoch.*, 24(1):71–123, 2020.

[3] T. De Angelis, F. Gensbittel, and S. Villeneuve. A Dynkin game on assets with incomplete information on the return. *Math. Oper. Res.*, 46(1):28–60, 2021.

[4] J.-P. Décamps, T. Mariotti, and S. Villeneuve. Investment timing under incomplete information. *Math. Oper. Res.*, 30(2):472–500, 2005.

[5] E. Ekström and B. Lu. Optimal selling of an asset under incomplete information. *Int. J. Stoch. Anal.*, pages Art. ID 543590, 17, 2011.

[6] E. Ekström and J. Vaicenavicius. Optimal liquidation of an asset under drift uncertainty. *SIAM J. Financial Math.*, 7(1):357–381, 2016.

[7] E. Ekström and M. Vannestål. American options and incomplete information. *Int. J. Theor. Appl. Finance*, 22(6):1950035, 14, 2019.

[8] S. Engehagen, M. Hornslien, M. Lavrutich, and S. Tønnessen. Optimal harvesting of farmed salmon during harmful algal blooms. *Marine Policy*, 129:104528, 2021.

[9] P. V. Gapeev. Pricing of perpetual American options in a model with partial information. *Int. J. Theor. Appl. Finance*, 15(1):1250010, 21, 2012.

[10] P. V. Gapeev and A. N. Shiryaev. On the sequential testing problem for some diffusion processes. *Stochastics*, 83(4-6):519–535, 2011.

[11] K. Glover and H. Hulley. Short selling with margin risk and recall risk. *Int. J. Theor. Appl. Finance*, 25(2):Paper No. 2250007, 33, 2022.

[12] X. Guo and Q. Zhang. Closed-form solutions for perpetual American put options with regime switching. *SIAM J. Appl. Math.*, 64(6):2034–2049, 2004.

[13] V. Henderson, K. Kladívko, M. Monoyios, and C. Reisinger. Executive stock option exercise with full and partial information on a drift change point. *SIAM J. Financial Math.*, 11(4):1007–1062, 2020.

[14] P. Johnson and G. Peskir. Sequential testing problems for Bessel processes. *Trans. Amer. Math. Soc.*, 370(3):2085–2113, 2018.

[15] M. Klein. Comment on "Investment timing under incomplete information". *Math. Oper. Res.*, 34(1):249–254, 2009.

[16] P. Lakner. Utility maximization with partial information. *Stochastic Process. Appl.*, 56(2):247–273, 1995.

[17] J. Lempa and P. Matomäki. A Dynkin game with asymmetric information. *Stochastics*, 85(5):763–788, 2013.

[18] R. S. Liptser and A. N. Shiryaev. *Statistics of random processes. II*, volume 6 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, expanded edition, 2001. Applications, Translated from the 1974 Russian original by A. B. Aries, Stochastic Modelling and Applied Probability.

[19] A. Novikov and J. L. Palacios-Soto. Sequential hypothesis tests under random horizon. *Sequential Anal.*, 39(2):133–166, 2020.

[20] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2005. Second edition. Version 2.1, Corrected third printing.

[21] A. N. Shiryaev. Two problems of sequential analysis. *Cybernetics*, 3(2):63–69 (1969), 1967.

[22] A. N. Shiryaev. *Optimal stopping rules*, volume 8 of *Stochastic Modelling and Applied Probability*. Springer-Verlag, Berlin, 2008. Translated from the 1976 Russian second edition by A. B. Aries, Reprint of the 1978 translation.

[23] J. Vaicenavicius. Asset liquidation under drift uncertainty and regime-switching volatility. *Appl. Math. Optim.*, 81(3):757–784, 2020.