Swedish Summer PDEs

August 26-28, Room F11, KTH, Stockholm

Abstracts

The fractional unstable obstacle problem

Mark Allen
Brigham Young University, Provo

We study a model for combustion on a boundary, specifically, $(-\Delta)^s u = \chi_{\{u>c\}}$ for $0 < s < 1$ and some constant c. Our main object of study is the free boundary $\partial\{u > c\}$. We study the behavior of the free boundary as well as the stability of certain solutions. We show that when $s \leq 1/2$ these solutions are stable; however, when $s > 1/2$ these solutions are not stable. This is joint work with Mariana Smit Vega Garcia.

The structure of the regular and the singular set of the free boundary in the obstacle problem for fractional heat equation

Agnid Banerjee
TIFR Centre, Bangalore

In this talk, I will discuss the structure of the free boundary in the obstacle problem for fractional powers of the heat operator. Our results are derived from the study of a lower dimensional obstacle problem for a class of local, but degenerate, parabolic equations. The analysis will be based on new Almgren, Weiss and Monneau type monotonicity formulas and the associated blow-up analysis. This is a joint work with D. Danielli, N. Garofalo and A. Petrosyan.

The Dominative p-Laplacian

Karl Brustad
Alto University, Helsinki

In 2003 Crandall and Zhang discovered an unexpected superposition principle in the p-Laplace equation. Namely that any linear combination of translates of the fundamental solution (with positive coefficients) is again a p-superharmonic function in $\mathbb{R}^n$. This was surprising because the p-Laplacian is nonlinear and properties of solutions are therefore rarely preserved under addition. The result was improved and the proof was somewhat simplified by Lindqvist and Manfredi in 2008, but the underlying reasons behind the principle and how far it could be extended, was still not clear.

It turns out that the superposition principle is governed by the supersolutions of the dominative p-Laplace equation. This is a sublinear elliptic operator that dominates the normalized p-Laplacian but share its fundamental solution. The superposition principle then follows immediately.

With this auxiliary operator at hand, further insight is possible. We shall, for example, classify the *superposition-able* p-harmonic functions, and also, with a generalization of the dominative p-Laplacian, give a new proof of a superposition principle in the doubly nonlinear diffusion equation.

Boundary Regularity for the Free Boundary in the One-phase Problem.

Héctor Chang
CIMAT, Guanajuato

We consider the Bernoulli one-phase free boundary problem in a domain Ω and show that the free boundary F is $C^{1,1/2}$ regular in a neighborhood of the fixed boundary $\partial\Omega$. We achieve this by relating the behavior of F near $\partial\Omega$ to a Signorini-type obstacle problem.

An Epiperimetric Approach to Isolated Singularities

Max Engelstein
MIT

In this talk, we will present the first uniqueness of blowups result for minimizers of the Alt-Caffarelli functional. In particular, we prove that the tangent object is unique at isolated singular points in the free boundary. Our main tool is a new approach to proving (log-)epiperimetric inequalities at isolated singularities. This epiperimetric inequality differs from previous ones in that it holds without any additional assumptions on the symmetries of the tangent object.

If we have time, we will also discuss how this method allows us to recover some uniqueness of blow-ups results in the minimal surfaces setting, particularly those of Allard-Almgren ('81) and Leon Simon ('83). This is joint work with Luca Spolaor (UCSD) and Bozhidar Velichkov (U. Naples).

Lipschitz bounds and non-uniformly elliptic functionals

Cristiana De Filippis
University of Oxford

We consider integral functionals of the type

$$w \mapsto \int_\Omega [F(x, Dw) - f \cdot w] \, dx$$

and investigate two main issues:

- the Euler-Lagrange equation of the functional is non-uniformly elliptic, in the sense that the ellipticity ratio evaluated on balls B

$$\mathcal{R}(z, B) := \frac{\sup_{x\in B} \text{ of the highest eigenvalue of } \partial_z^2 F(x, z)}{\inf_{x\in B} \text{ of the lowest eigenvalue of } \partial_z^2 F(x, z)}$$

  may blow up when $|z| \to \infty$;

- the minimal regularity to assume on the datum f and on the map $x \mapsto F(x, \cdot)$ in order to guarantee Lipschitz continuity of minima .

We discuss several results and conditions involving potential estimates, Lavrentiev phenomenon, limiting Lorentz spaces and continuity conditions. The talk is based on joint works with G. Mingione (Parma).

On the (in)stability of the identity map in optimal transportation

Yash Jhaveri
IAS, Princeton

In the optimal transport problem, it is well-known that the geometry of the target domain plays a crucial role in the regularity of the optimal transport. In the quadratic cost case, for instance, Caffarelli showed that having a convex target domain is essential in guaranteeing the optimal transport's continuity. In this talk, we shall explore how, quantitatively, important convexity is in producing continuous optimal transports.

Higher integrability for doubly nonlinear parabolic equations

Juha Kinnunen
Alto University, Helsinki

We discuss a local higher integrability result for the spatial gradient of weak solutions to doubly nonlinear parabolic equations of the type

$$\left(|u|^{p-2}u\right)_t - \operatorname{div}\left(|Du|^{p-2}Du\right) = 0$$

in the range

$$\max\left\{\frac{2n}{n+2}, 1\right\} < p < \frac{2n}{(n-2)_+},$$

where $n \in \mathbb{N}$ is the spatial dimension. In [bdks20019] we show that a gradient of a nonnegative weak solution to a doubly nonlinear equation belongs locally to a slightly higher Sobolev space than assumed a priori with a reverse Hölder type estimate. The range may seem unexpected, but the lower bound also appears in the higher integrability for the parabolic p-Laplace equation [kl2000], while the upper bound is the same as for the porous medium equation in the fast diffusion range [gs2018] and [bds2019].

The equation is homogeneous in the sense that solutions are invariant under multiplication by constants, but constants cannot be added without destroying the property of being a solution. The key ingredient in the proof of our main result is an appropriate intrinsic geometry that depends on the the solution as well as its spatial gradient and thus allows us to rebalance the mismatch between the function and its gradient in the equation. Variants of this techinque have been successfully used in proving the correspoding results for the parabolic p-Laplace equation [kl2000] and for the porous medium equation [gs2016], [gs2018], [bdks2018] and [bds2019]. Related results and open questions are also discussed.

bdks20019 V. Bögelein, F. Duzaar, J. Kinnunen and C. Scheven, Higher integrability for doubly nonlinear parabolic systems (submitted).

bdks2018 V. Bögelein, F. Duzaar, R. Korte and C. Scheven, The higher integrability of weak solutions of porous medium systems, *Adv. Nonlinear Anal.* **8**(1): 1004–1034, 2019.

bds2019 V. Bögelein, F. Duzaar and C. Scheven, Higher integrability for the singular porous medium system (submitted).

gs2016 U. Gianazza and S. Schwarzacher, Self-improving property of degenerate parabolic equations of porous medium-type, *Amer. J. Math.* (to appear).

gs2018 U. Gianazza and S. Schwarzacher, Self-improving property of the fast diffusion equation (submitted)

kl2000 J. Kinnunen and J.L. Lewis, Higher integrability for parabolic systems of p-Laplacian type, *Duke Math. J.*, **102**(2): 253–271, 2000.

A spiral interface with positive Alt-Caffarelli-Friedman limit at the origin

Dennis Kriventsov
Rutgers University, New Jersey

I will discuss an example of a pair of continuous nonnegative subharmonic functions, each vanishing where the other is positive, which have a strictly positive limit for the Alt-Caffarelli-Friedman monotonicity formula at the origin, but for which the origin is not a point of differentiability for the boundary of their supports. Time permitting, I will also discuss some further progress on related problems. This is based on joint work with Mark Allen.

Relaxed Euler systems and convergence to Navier-Stokes equations

Yue-Jun Peng
Laboratoire de Mathématiques Blaise Pascal, Université Clermont Auvergne

We consider the approximation of Navier-Stokes equations for a Newtonian fluid by Euler type systems with relaxation both in compressible and incompressible cases. This requires to decompose the second-order derivative terms of the velocity into first-order ones. Usual decompositions lead to approximate systems with tensor variables. We construct approximate systems with vector variables by using Hurwitz-Radon matrices. These systems are written in the form of balance laws and admit strictly convex entropies, so that they are symmetrizable hyperbolic. For smooth solutions, we prove the convergence of the approximate systems to the Navier-Stokes equations in uniform time intervals. Global-in-time convergence is also shown for the initial data near constant equilibrium states of the systems. These convergence results are established not only for the approximate systems with vector variables but also for those with tensor variables.

Improved regularity along singular sets

Edgard Pimentel
PUC-Rio

In this talk we discuss gains of regularity for the solutions to degenerate diffusions, as they reach a non-physical free boundary. Our findings demonstrate that along appropriate singular sets, solutions present improved regularity. We have recourse to geometric methods, importing information from auxiliary equations for which a richer regularity theory is available. The models we examine include the porous medium equation and diffusions in the double-divergence form. We conclude the talk by presenting further directions of research, as well as a number of interesting open problems.

Stability of extremals of the Riesz potential

Aldo Pratelli
University of Pisa

The Riesz potential has attracted a lot of interest in the last years, also in view of its role in physics, as it is a good approximation of the repulsive force between particles. As one can easily imagine, the ball is the unique maximizer of this potential among sets of given volume. Nevertheless, for small volumes the ball is also the unique minimizer of the total energy given by the sum of the Riesz potential and the perimeter. We will present a general overview of the question, together with some very recent results. Joint work with Nicola Fusco (Naples).

On the derivative of fractional maximal function on domains

Olli Saari
University of Bonn

Fractional maximal function shares the L^p mapping properties of a Riesz potential. Up to one derivative, this phenomenon carries over to the Sobolev scale. When the analysis is restricted to an open proper subset of the Euclidean n-space, much less is known. I will discuss partial results towards understanding the role of the domain in these questions. This is based on joint work with J.P. Ramos and J. Weigt.

Analyzing a special case of the Hele-Shaw flow using integro-differential operators.

Russell Schwab
Michigan State University, East Lansing

We will demonstrate that the solutions of an evolutionary free boundary problem, called Hele-Shaw, can be analyzed by an equivalent fractional parabolic equation in one fewer dimensions (a nonlinear version of the fractional (1/2)-heat equation). In order to obtain useful results from the parabolic interpretation of the problem, one must have precise information about the drift vectors and Levy measures that characterize the fractional operator that drives the equation. We will explain what happens in this context in the case of Hele-Shaw with an initial boundary that is the graph of a function whose derivative enjoys a Dini modulus of continuity. This is ongoing and joint work with Farhan Abedin.

Understanding the singular set in the obstacle problem

Joaquim Serra
ETH, Zürich

I will explain recent results with A. Figalli and X. Ros-Oton on the structure of the singular set in the obstacle problem. As an application of the new results we prove a strong version of Shaeffer's conjecture in three dimensions.

A Qualitative Description of Extremals for Morrey's Inequality

Frank Seuffert
University of Pennsylvania

The seminorm form of Morrey's inequality is summarized as follows: Let $u \in L^1_{loc}(\mathbb{R}^n)$ be such that $Du \in L^p(\mathbb{R}^n)$ and $p > n$. Then there is some $C > 0$ depending only on n and p such that

$$C\|Du\|_p \geq [u]_{C^{0,1-n/p}} \tag{1}$$

where $[\cdot]_{C^{0,1-n/p}}$ is the $(1 - n/p)$-Hölder seminorm given by $[u]_{C^{0,1-n/p}} := \sup_{x \neq y} \left\{ \frac{|u(x)-u(y)|}{|x-y|^{1-n/p}} \right\}$. This inequality was (essentially) proven 80 years ago by C. B. Morrey Jr. However, until recently, nothing was known about extremals or the sharp constant of Morrey's inequality. In a recent project, R. Hynd and I proved the existence of extremals and some of their qualitative characteristics. The key to our results is to show that a function, v, is an extremal of Morrey's Inequality if and only if it satisfies a PDE:

$$-\Delta_p v = c(\delta_x - \delta_y) \tag{2}$$

where δ_x and δ_y are dirac masses at some $x, y \in \mathbb{R}^n$ and c is any nonzero constant. The points x and y in (2) are essential in the structure of v. In a recent project, we show that x and y are the unique pair of points where v achieves its $(1-n/p)$-Hölder seminorm, they are the points where v achieves its absolute maximum and minimum, and v is analytic except at x and y. Moreover, using the PDE, (2), we are able to show that extremals of Morrey's inequality are cylindrically symmetric (if $n \geq 3$) or evenly symmetric (if $n = 2$) about the line containing x and y, reflectionally antisymmetric (up to addition by a constant), and unique up to operations that are invariant on the ratio of the seminorms in (1). We also give explicit solutions for extremals when $n = 1$ and some numeraical approximations of extremals for $n = 2$ and $p = 4$. This work is a collaboration with R. Hynd.

Regularity of almost minimizers with free boundary

Mariana Smit Vega Garcia
Western Washington University, Seattle

We study almost minimizer for functionals which yield a free boundary, as in the work of Alt-Caffarelli and Alt-Caffarelli-Friedman. The almost minimizing property can be understood as the defining characteristic of a minimizer in a problem which explicitly takes noise into account. In this talk, we discuss the regularity of almost minimizers to energy functionals with variable coefficients. This is joint work with G. David, M. Engelstein & T. Toro.

FuMaPr N. Fusco, F. Maggi, A. Pratelli *The sharp quantitative isoperimetric inequality*, Ann. of Math. (2) **168 (3)** (2008), 941–980.

FiMaPr A. Figalli, F. Maggi, A. Pratelli, *A mass transportation approach to quantitative isoperimetric inequalities*, Invent. Math. **182** (2010), 167–211

CiLe M. Cicalese, G.P.. Leonardi *A selection principle for the sharp quantitative isoperimetric inequality*, Arch. Ration. Mech. Anal. **206** (2012), 617–643

On the spatial decay of the Boltzmann equation with hard potentials

Haitao Wang
Jiao Tong University, Shanghai

We study the spatial decay of the solution to the Boltzmann equation with hard potentials for both linear and nonlinear problems. For the nonlinear study, we get the spatial behavior by using the nonlinear weighted energy estimate. This result can be viewed as a supplement to the existing well-posedness and time decay results. For the linear problem, we get the space-time pointwise behavior under some slow velocity decay assumption, which extends the classical results from hard sphere to hard potential. Both results reveal that hard sphere and hard potential models differ in their spatial behaviors. This is a joint work with Yu-Chu Lin and Kung-Chien Wu.

Analysis on subsonic steady solutions with both fixed and free boundaries

Chunjing Xie
Jiao Tong University, Shanghai

In this talk, we will first discuss the recent progress on subsonic steady flow in nozzles or past a body. Then we discuss the subsonic steady flows involving free boundaries.

Regularity of the singular set in the fully nonlinear obstacle problem

Hui Yu
Columbia University, New York

In this talk we describe a method to study the singular set in the obstacle problem. This method does not depend on monotonicity formulae and works for fully nonlinear elliptic operators.

This is based on joint work with Ovidiu Savin from Columbia University.

A fundamental inequality for the p-Laplacian and the ∞-Laplacian

Yi Zhang
ETH, Zürich

In a recent paper with Dong, Fa and Yuan, we discovered a fundamental inequality which is very useful to study the second order Sobolev regularity of equations involving p-Laplacian. It generalized the key identity for the higher order Sobolev regularity of planar infinity harmonic functions by Koch, Zhou and the speaker.