The fractional unstable obstacle problem

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Similar Problems

Combustion Model

A model for interior solid combustion is given by

$$\partial_t u - \Delta u = \chi_{\{u>0\}}.$$

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Monneau and Weiss studied the elliptic problem

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Comparison to the Obstacle Problem

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However, the sign change gives many differences.

- $u \notin C^{1,1}$ (an example constructed by Andersson and Weiss)
- Use the implicit function theorem on the free boundary $\{u = 0\}$ where the gradient does not vanish.
- Points of interest are where the gradient does vanish.

Singular Points

Solutions to

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may be found by minimizing the functional

$$\int |\nabla v|^2 - 2\max(v, 0).$$

- The gradient never vanishes on the free boundary for minimizers of the functional (no singular points).
- The free boundary is real analytic and $u \in C^{1,1}$.

Two-phase Fractional Obstacle Problem

With Lindgren and Petrosyan studied minimizers of the functional

$$\int_{\Omega^+} |\nabla v|^2 x_n^a + 2\int_{\Omega'} \lambda_+ v^+ + \lambda_- v^-.$$

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For the unstable fractional obstacle problem, we study minimizers of the functional

$$J_a(v, \lambda_+, \lambda_-) := \int_{\Omega^+} |\nabla v|^2 x_n^a - 2\int_{\Omega'} \left(\lambda_+ v^+ + \lambda_- v^-\right)\, d\mathcal{H}^{n-1}.$$

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When considering the extension, minimizers are solutions to

$$(-\Delta)^s u = \chi_{\{u>0\}}.$$

Extension Operator

We consider the domain $U \times \mathbb{R}^+$, and write $(x', x_n) \in \mathbb{R}^n$ with $x' \in \mathbb{R}^{n-1}$ and $x_n \in \mathbb{R}$. Let F solve

$$\text{div}(x_n^a \nabla F(x', x_n)) = 0 \text{ in } U \times \mathbb{R}$$
$$F(x', 0) = f(x')$$
$$\lim_{x_n \to \infty} F(x', x_n) = 0.$$

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Then

$$(-\Delta)^s f(x) = c_{N,a} \lim_{x_n \to 0} x_n^a \partial_{x_n} F(x', x_n)$$

where $c_{N,a}$ is a negative constant depending on dimension $N = n - 1$ and a, were s and a are related by $2s = 1 - a$.

Contrasting properties

- For the fractional unstable obstacle problem $\partial\{u(\,\cdot\,,0) > 0\} = \partial\{u(\,\cdot\,,0) < 0\}$.
- For the two-phase fractional obstacle problem $\partial\{u(\,\cdot\,,0) > 0\} \cap \partial\{u(\,\cdot\,,0) < 0\} = \emptyset$ when $a \geq 0$ $(s \leq 1/2)$.

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- For the two-phase fractional obstacle problem minimizers always achieve the optimal regularity $C^{0,1-a}$ or $C^{1,-a}$.
- For the fractional unstable obstacle problem minimizers may not achieve the optimal Lipschitz regularity.

Properties of Minimizers

Always a "two-phase" problem

u is a minimizer of $J_a(v, \lambda_+, \lambda_-)$ if and only if $u + cx_n^{1-a}$ is a minimizer of $J_a(w, \lambda_+ - c(1-a), \lambda_- + c(1-a))$ for any constant c such that $-\lambda_- \leq c(1-a) \leq \lambda_+$.

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Consequently, one may study minimizers of the energy functional

$$J_a(u) := \int_{\Omega^+} |\nabla u|^2 x_n^a - 2\int_{\Omega'} u^-,$$

Nondegeneracy Properties

Let u be a minimizer. Then

$$\sup_{B'_r(x_0,0)} u \geq C r^{1-a} \text{ for every } r < R$$

where C is a constant depending only on dimension n and s.

Regularity

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- $u \in C^{1-a}$ for $a < 0$ $(s > 1/2)$.
- $u \in C^{0,\alpha}$ for all $\alpha < 1$ if $a = 0$ $(s = 1/2)$.
- There is a stable solution which is not Lipschitz.
- We have not shown whether the solution is a minimizer of the functional or not.

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- Research question: show higher regularity of the free boundary when the gradient does not vanish.

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- When $a < 0$ $(s > 1/2)$, use the implicit function theorem whenever the gradient does not vanish.
- Research question: show higher regularity of the free boundary when the gradient does not vanish.
- Research question: study singular points of the free boundary when the gradient does vanish.

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- With the extension the free boundary has Hausdorff dimension $n - 2$.

2nd variational formula

If u is a minimizer and $w \in H_0^1(a, B_r(x_0))$,

$$0 \leq \int_{B_r^+(x_0)} |\nabla w|^2 x_n^a - 2 \int_{\{u=0\} \cap B_r'(x_0)} \frac{w^2}{|\nabla u|} d\mathcal{H}^{n-2}.$$

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This formula may seem strange because w is only evaluated on a set of co-dimension 2. However, when $-1 < a < 0$ sets of Hausdorff dimension $n - 2$ may have positive capacity, and consequently, w will have a trace on such sets.

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- For the unstable obstacle problem, a scaling argument shows the second term goes to $-\infty$.
- When $1/2 < s < 1$, both sides scale the same.
- A scaling argument won't give a contradiction.

Homogeneous solutions

At a singular point perform a blow-up

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- We have nondegeneracy even though we don't assume it is a minimizer
- Nondegeneracy and optimal regularity come from $1/2 < s < 1$.
- In the limit obtain a homogeneous solution of degree $1 - a$.

Sobolev/Trace Inequality

There will exist a constant C such that

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- Using u somehow as a test function w (such as a derivative of u) seems promising, but doesn't help.
- One must know the structure of the free boundary.
- One must also know the gradient along the free boundary.

Local vs global minimizers

We constructed solutions which are locally minimizers, but not a minimizer on the entire domain.

Symmetric solutions with singular points

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- We use odd reflection to construct solutions with singular points.
- Are these solutions local minimizers?

Symmetric solutions are not local minimizers

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Symmetric solutions are not local minimizers

- We use the explicit structure of the free boundary to determine the gradient on the free boundary.
- We construct a test function w which is a-harmonic everywhere except one line segment of the free boundary.
- We show that the second variational-formula is violated.
- We use a comparison principle to show that any symmetric solution is not a local minimizer.

Symmetric solutions are stable

We use the explicit structure of the free boundary (line segments) as well as the proof of the second variation to conclude that

$$A_0^t = \int_{B_r^+} |\nabla w|^2 x_n^a - 4 \int_0^1 \int_0^\omega \int_{B_r' \cap \{u + \tau t w = 0\}} \frac{w^2}{|\nabla u + \tau t w|} d\mathcal{H}^{n-2} d\tau d\omega.$$

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For fixed w, as $t \to 0$, $|\nabla u + \tau t w| \to \infty$, so that for fixed w,

$$A_0^t > 0$$

for any $t \leq t_0$ with t_0 depending on w.

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Future Directions

- Higher regularity of the free boundary when $1/2 < s < 1$.
- Regularity of the free boundary away from singular points for $0 < s \leq 1/2$.
- Study of singular points in higher dimensions when $1/2 < s < 1$.
- Determining if singular points can occur for minimizers when $0 < s \leq 1/2$.

Thank You.