Obstacle problem for the fractional heat equation

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(joint work with Donatella Danielli, Nicola Garofalo and Arshak Petrosyan)

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This problem arises in elasticity, probability and math finance etc.

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For a given $u \in \mathrm{Dom}(-\Delta)^s$ one considers the function $U(x, y)$ that solves the so-called extension problem with $a = 1 - 2s$

$$\begin{cases} \mathrm{div}_X(y^a \nabla_X U) = 0 & x \in \mathbb{R}^n, y > 0, \\ U(x,0) = u(x). \end{cases} \tag{0.1}$$

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Then, one has the following weighted Dirichlet-to-Neumann relation:

$$-\frac{2^{-a}\Gamma\left(\frac{a+1}{2}\right)}{\Gamma\left(\frac{1-a}{2}\right)} \lim_{y \to 0^+} y^a \frac{\partial U}{\partial y}(x, y) = (-\Delta)^s u(x). \tag{0.2}$$

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This is a thin obstacle problem since now the obstacle, the coincidence set and the free boundary are confined to the thin manifold $M = \mathbb{R}^n \times \{0\}$ which bounds the thick space $\mathbb{R}^{n+1}_+$.

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In this lecture my focus will be on a thin obstacle problem for a class of degenerate parabolic equations which enter in the analysis of the obstacle problem for fractional powers of the heat operator $(\partial_t - \Delta)^s$, $0 < s < 1$. Certain qualitative properties were studied using this connection by Athanasopoulos-Caffarelli-Milakis (ACM).

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Analogous to the elliptic case, this is possible thanks to the extension problem for $(\partial_t - \Delta)^s$ recently established by Nyström-Sande and Stinga-Torrea.

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Remark

The case $a = 0$ has been studied previously by Danielli-Garofalo-Petrosyan-To [DGPT].

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- $\mathbb{Q}_r = \mathbb{B}_r \times (-r^2, 0]$ = thick parabolic cylinder in the thick space $(X, t) \in \mathbb{R}^{n+1} \times \mathbb{R}$
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- $\mathbb{B}_r^+ = \{X = (x, y) \in \mathbb{B}_r \mid y > 0\}$
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- $\mathbb{S}_r^+ = \{(X, t) \mid X \in \mathbb{R}_+^{n+1}, \quad -r^2 < t < 0\}$ = half-strip in thick space

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This is accomplished by picking a function $\zeta(X) = \zeta^{\star}(|X|) \in C_0^{\infty}(\mathbb{B}_1)$, $0 \leq \zeta \leq 1$, and then considering the new function

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Important observation: Since ζ is smooth and $\zeta(x,-y) = \zeta(x,y)$ one has

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Therefore, the function U solves the following problem in the space-time strip $\mathbb{S}_1^+$ in thick space

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We are now led to study the problem

$$\begin{cases} \mathscr{L}_a U = y^a F & \text{in } \mathbb{S}_1^+, \\ U(x,0,t) \geq 0, & \text{for } (x,t) \in S_1^+, \\ -\lim_{y\to 0^+} y^a \frac{\partial U}{\partial y}(x,y,t) \geq 0, & \text{for } (x,t) \in S_1^+, \\ \lim_{y\to 0^+} y^a \frac{\partial U}{\partial y}(x,y,t) = 0, & \text{on the set } \{(x,t) \in S_1^+ \mid U(x,0,t) > 0\} \end{cases} \tag{0.6}$$

Suppose that $\psi \in C^2_{x,t}$. Then $F \in L^\infty(\mathbb{S}_1^+)$ but also $F_t \in L^\infty(\mathbb{S}_1^+)$!

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Once we know the crucial information $U_t \in L^\infty(\mathbb{S}_1^+)$ we can bring $y^a U_t$ to the right-hand side and then, setting $F - U_t \longrightarrow F$, then at each fixed time-level $\bar{t}$, the function $u(X) = U(X, \bar{t})$ satisfies the elliptic problem

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Since our $F \in L^\infty(\mathbb{B}_1^+)$, we cannot use the results of Caffarelli, Salsa Silvestre directly.

But we can use an improved elliptic monotonicity formula due Caffarelli, De Silva and Savin (2017), which allows to obtain (0.8) when $F \in L^{\infty}(\mathbb{B}_1^+)$.

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Subsequently, using $W^{2,2}$ type estimates that we develop, dependence of such estimates only on the L^2 norm of U and the appropriate norm of the right hand side F is obtained. Such a local dependence is crucial to the blowup analysis that we do.

Classes of solutions to the parabolic thin obstacle problem

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Gaussian-Bessel spaces

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For general $a \in (-1, 1)$ some of these also appear in my recent work with Nicola on unique continuation.

The solution of the Cauchy problem for the Bessel operator $\mathscr{B}_y^{(a)}$ with Neumann boundary condition on $(\mathbb{R}^+, y^a dy)$

$$\begin{cases} \frac{\partial u}{\partial t} - \mathscr{B}_y^{(a)} u = 0 \\ \lim\limits_{y \to 0^+} y^a \partial_y u(y,0) = 0 \\ u(y,0) = \varphi(y), \end{cases}$$

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$$p(x, t) = (4\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{4t}}$$

the standard heat kernel in $\mathbb{R}^n \times (0, \infty)$

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- $\delta_\lambda(X, t) = (\lambda X, \lambda^2 t)$ = parabolic dilations
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We define two initial frequencies of U on $\mathbb{S}_1^+$ as

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Note: if U is homogeneous of degree κ, then $ZU = \kappa U$ and $N(U,r) \equiv \kappa$.

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$$I'(U,r) = \frac{1}{r^3} \int_{\mathbb{S}_r^+} (ZU)^2 \, \overline{\mathscr{G}}_a y^a + \frac{2}{r^3} \int_{\mathbb{S}_r^+} |t|(ZU)F \, \overline{\mathscr{G}}_a y^a \tag{0.16}$$
$$+ \frac{2}{r^3} \int_{\mathbb{S}_r^+} |t| UF \, \overline{\mathscr{G}}_a y^a + 2r \int_{\mathbb{R}_+^{n+1} \times \{-r^2\}} UF \, \overline{\mathscr{G}}_a y^a.$$

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$$r \mapsto \Phi_{\ell,\sigma}(U,r) \overset{def}{=} \frac{r}{2} e^{Cr^{1-\sigma}} \frac{d}{dr} \log \max \left\{ H(U,r),\ r^{2\ell-2+2\sigma} \right\} + 2(e^{Cr^{1-\sigma}} - 1), \tag{0.17}$$

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Let $r \in (0,1)$ be such that $H(r) > r^{2\ell-2+2\sigma}$. Then,

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This implies the monotonicity of $r \to e^{Cr^{1-\sigma}}(N(r)+1)$ where $H(U,r) > r^{2\ell-2+2\sigma}$ (with a proviso!)

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$$\Phi_{\ell,\sigma}(U,0^+) = \lim\, N(U,r) = \lim_{r\to 0^+} \tilde{N}(U,r).$$

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$$H(U_r, 1) = 1, \; N(U_r, \rho) = N(U, r\rho).$$

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(iii) U_0 is parabolically homogeneous of degree κ and is a global solution of the thin obstacle problem, i.e.,

$$\begin{cases} \mathscr{L}_a U_0 = 0 \quad \text{in } \mathbb{S}_\infty^+ \\ U_0 \geq 0, \ \lim\limits_{y \to 0^+} y^a \partial_y U_0 \leq 0, \ U_0 \left(\lim\limits_{y \to 0^+} y^a \partial_y U_0 \right) = 0. \end{cases} \tag{0.20}$$

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$$\Phi_{\ell,\sigma}(U, 0^+) = 1 + s, \quad \text{or} \quad \Phi_{\ell,\sigma}(U, 0^+) \geq 2.$$

An important consequence of the gap theorem is that the set of free boundary points (remember: these points live in the thin space!) which have minimal frequency $\Phi_\sigma(U, 0^+) = 1 + s$ is a relatively open subset of the free boundary.

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Theorem: The regular free boundary is locally a $\mathbb{H}^{1+\alpha,1+\alpha/2}$ hypersurface. By this we mean that if e.g. 0 is a regular free boundary point, then (after a suitable rotation in the thin space) in a sufficiently small neighborhood the free boundary is given by

$$x_n = f(x', t), \qquad \text{with} \quad f \in \mathbb{H}^{1+\alpha,1+\alpha/2}_{x',t}.$$

Our strategy for proving the previous theorem is based on the elliptic epiperimetric inequality previously established by N. Garofalo, A. Petrosyan, C. A. Pop & M. Smit Vega Garcia: (also independently by Focardi-Spadaro)

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Once the problem of the smoothness of the regular free boundary is out of the way we turn to the issue of the singular free boundary points

Definition (Singular points)

Let $v \in \mathfrak{S}_F(\mathbb{S}_1^+)(\mathbb{Q}_1^+)$ with $\psi \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 2$. We say that a free boundary point $X_0 = (x_0, 0, t_0)$ is singular if

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A word of warning: the whole free boundary can be made exclusively of singular points!

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Let $u \in \mathfrak{S}_F(\mathbb{S}_1^+)$ with F satisfying $|F(X,t)| \leq M\|(X,t)\|^{\ell-2}$ in $\mathbb{S}_1^+$ and $|\nabla_X F(X,t)| \leq L\|(X,t)\|^{\ell-3}$ in $\mathbb{Q}_{1/2}^+$, $\ell \geq 3$ and $0 \in \Gamma_\kappa^{(\ell)}(u)$ with $\kappa < \ell - 1 + \sigma$.

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(iii) $\kappa = 2m$, $m \in \mathbb{N}$.

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In particular, with $C = C'/2\sigma$ the function $r \longrightarrow \mathscr{W}_\kappa(U,r) + Cr^{2\sigma}$, is monotonically nondecreasing in $(0,1)$, and therefore the limit exists

$$\mathscr{W}_\kappa(U,0^+) \overset{def}{=} \lim_{r\to 0^+} \mathscr{W}_\kappa(U,r).$$

Using previous theorem we can prove the main tool to analyze singular points

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$$\frac{d}{dr}\mathscr{M}_\kappa(U, p_\kappa, r) \geq -C'' \left(1 + ||U||_{L^2(\mathbb{S}_1^+, \overline{\mathscr{G}}_a y^a)} + ||p_\kappa||_{L^2(\mathbb{S}_1^+, \overline{\mathscr{G}}_a y^a)}\right) r^{-1+\sigma}.$$

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In particular, with $C = C''/\sigma$ the function $r \to \mathscr{M}_\kappa(U, p_\kappa, r) + C r^\sigma$ is monotonically nondecreasing on $(0,1)$.

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$$d_\kappa^{x_0,t_0} = \dim[\xi \in \mathbb{R}^n : \xi.\nabla_x \partial_x^\alpha \partial_t^j q_{x_0,t_0}^\kappa = 0, \text{for all } |\alpha| + 2j = \kappa - 1]$$

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Then we let

$$\Sigma_\kappa^d = \{(x_0, t_0) \in \Sigma_\kappa : d_\kappa^{x_0,t_0} = d\}$$

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Let $U \in \mathfrak{S}_F(\mathbb{Q}_1^+)$ with $F \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 4$. Then for every $d = 0, 1, \ldots, n-1$, the set $\Sigma_\kappa^d(U)$ is contained in a countable union of $(d+1)$-dimensional space-like $C^{1,0}$ manifolds

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Let $U \in \mathfrak{S}_F(\mathbb{Q}_1^+)$ with $F \in H^{\ell,\ell/2}(Q_1)$, $\ell \geq 4$. Then for every $d = 0, 1, \ldots, n-1$, the set $\Sigma_\kappa^d(U)$ is contained in a countable union of $(d+1)$-dimensional space-like $C^{1,0}$ manifolds and $\Sigma_\kappa^n(v)$ is contained in a countable union of n-dimensional time-like C^1 manifolds.

Relevant papers

- (joint with Donatella Danielli, Nicola Garofalo and Arshak Petrosyan), The structure of the singular set in the thin obstacle problem for degenerate parabolic equations. arXiv:1902.07457
- (joint with Donatella Danielli, Nicola Garofalo and Arshak Petrosyan) The regular free boundary in the thin obstacle problem for degenerate parabolic equations, arXiv:1906.06885

Thank you all for your kind attention.