The Dominative p-Laplacian

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KTH 26.-28. August 2019

The p-Laplace Equation

$$\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right) = 0, \qquad 2 \le p < \infty.$$

Variational integral:

$$\int |\nabla u|^p \, \mathrm{d}x.$$

$$\Delta u = \operatorname{div}(\nabla u) = 0, \qquad p = 2.$$

Infinity-Laplacian:

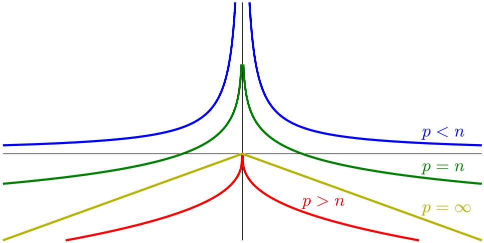
$$\Delta_\infty u := \nabla u \mathcal{H} u \nabla u^T.$$

Nonlinear when $p > 2$:

$$\Delta_p[u+v] \neq \Delta_p u + \Delta_p v.$$

The fundamental solution

$$w_{n,p}(x) := \begin{cases} -\frac{p-1}{p-n}|x|^{\frac{p-n}{p-1}}, & p \neq n, \\ -\ln|x|, & p = n, \\ -|x|, & p = \infty \text{ or } n = 1. \end{cases}$$



Satisfies $\Delta_p w_{n,p}(x) = 0$ for $x \neq 0$ in $\mathbb{R}^n$.

Ellipticity & Viscosity

Definition

A function $u\colon \Omega \to (-\infty, \infty]$ is a **(viscosity) supersolution** to the equation $\mathcal{D}w = 0$ in Ω if

- u is lower semicontinuous.
- u is finite on a dense subset of Ω.
- whenever $\phi \in C^2$ touches u from below at some point $x_0 \in \Omega$, then

$$\mathcal{D}\phi(x_0) \leq 0.$$

Definition

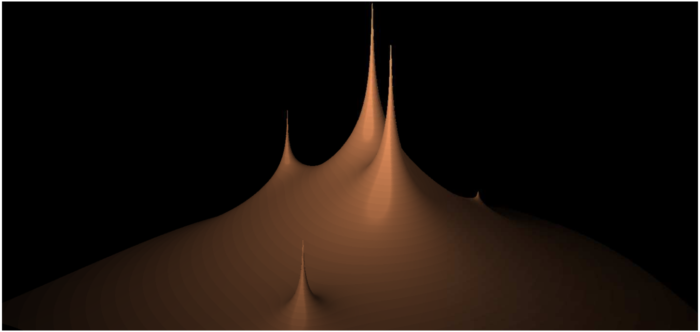
An operator $\mathcal{D}u = F(x, u, \nabla u, \mathcal{H}u)$ is (degenerate) **elliptic** if

$$X \leq Y \qquad \Rightarrow \qquad F(x, s, q, X) \leq F(x, s, q, Y)$$

for all $(x, s, q) \in \Omega \times \mathbb{R} \times \mathbb{R}^n$.

Superposition principle for the fundamental solutions

$$V(x) := \sum_{i=1}^{N} c_i w_{n,p}(x - y_i), \quad c_i \geq 0,\ y_i \in \mathbb{R}^n.$$



Theorem (Crandall/Zhang (2003))

V is p-superharmonic in $\mathbb{R}^n$.

Superposition principle for the fundamental solutions

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Theorem (Crandall/Zhang (2003))

$$\Delta_p V \leq 0.$$

Theorem (Lindqvist/Manfredi (2008))

$$\Delta_p \left[\int w_{n,p}(x - y)\rho(y)\,\mathrm{d}y \right] \leq 0, \qquad \rho \geq 0.$$

$$\Delta_p V(x) = -\tfrac{(p-2)(n+p-2)}{p-1} |\nabla V(x)|^{p-2} \sum_{i=1}^{N} \frac{c_i \sin^2 \theta_i(x)}{|x - y_i|^{\frac{n+p-2}{p-1}}},$$

$\theta_i(x)$ angle between $x - y_i$ and $\nabla V(x)$.

Preliminaries.

The Hessian matrix:

$$\mathcal{H}u := \left(\frac{\partial^2 u}{\partial x_i \partial x_j}\right)_{i,j=1}^{n}.$$

n real eigenvalues:

$$\lambda_1 \leq \cdots \leq \lambda_n, \qquad \lambda_i = \lambda_i(\mathcal{H}u).$$

Bounds on the Rayleigh quotient:

$$\lambda_1 \leq \frac{\xi^T \mathcal{H}u \xi}{|\xi|^2} \leq \lambda_n, \qquad 0 \neq \xi \in \mathbb{R}^n.$$

The Dominative p-Laplacian

$$\begin{aligned}\Delta_p u &= \operatorname{div}(|\nabla u|^{p-2}\nabla u)\\ &= |\nabla u|^{p-2}\left((p-2)\frac{\nabla u \mathcal{H} u \nabla u^T}{|\nabla u|^2} + \Delta u\right).\end{aligned}$$

$$\mathcal{D}_p u := (p-2)\lambda_n(\mathcal{H}u) + \Delta u, \qquad 2 \le p < \infty.$$

Domination:

$$\mathcal{D}_p u \le 0 \qquad \Rightarrow \qquad \Delta_p u \le |\nabla u|^{p-2}\mathcal{D}_p u \le 0.$$

The Dominative p-Laplacian

Sublinearity:

$$\mathcal{D}_p[u+v] \le \mathcal{D}_p u + \mathcal{D}_p v,$$
$$\mathcal{D}_p[cu] = c\mathcal{D}_p u, \qquad \text{for } c \ge 0.$$

$$A_p\colon \mathbb{R}^n \to S(n), \qquad A_p(\xi) := (p-2)\xi\xi^T + I,$$

$$\mathcal{D}_p u = \max_{|\xi|=1} \operatorname{tr}(A_p(\xi)\mathcal{H}u),$$

$$\mathcal{D}_p u \le 0,\ \mathcal{D}_p v \le 0 \qquad \Rightarrow \qquad \mathcal{D}_p[u+v] \le 0.$$

The Dominative p-Laplacian

Equivalence:

The gradient $\nabla w_{n,p}$ is an eigenvector to the Hessian $\mathcal{H}w_{n,p}$ corresponding to the largest eigenvalue $\lambda_n(\mathcal{H}w_{n,p})$.

$$\frac{\nabla w_{n,p}\mathcal{H}w_{n,p}\nabla w_{n,p}^T}{|\nabla w_{n,p}|^2} = \lambda_n(\mathcal{H}w_{n,p}).$$

$$\mathcal{D}_p w_{n,p} = 0.$$

An explanation of the superposition principle

$$
\begin{aligned}
\Delta_p w_{n,p} = 0 \quad & \text{and} \quad \mathcal{D}_p w_{n,p} = 0, && \text{(Equivalence)}\\
\mathcal{D}_p u \leq 0,\ \mathcal{D}_p v \leq 0 \quad & \Rightarrow \quad \mathcal{D}_p[u+v] \leq 0, && \text{(Sublinearity)}\\
\mathcal{D}_p w \leq 0 \quad & \Rightarrow \quad \Delta_p w \leq 0. && \text{(Domination)}
\end{aligned}
$$

We immediately get

$$\Delta_p \left[\sum_{i=1}^{N} c_i w_{n,p}(x - y_i) \right] \leq 0.$$

Taking it further

New questions.

When is a generic sum $\sum_{i=1}^N u_i$ p-superharmonic?

Sufficient conditions:

$$\mathcal{D}_p u_i \leq 0 \qquad \Rightarrow \qquad \Delta_p \left[\sum_{i=1}^N u_i\right] \leq 0.$$

Necessary conditions?:

$u \in C^2$ is **addable** if

1. $\Delta_p[u + l] \leq 0$ for every linear function $l(x) = a^T x$.
2. $\Delta_p[u + c w_{n,p} \circ T] \leq 0$ for every $c \geq 0$ and every translation $T(x) = x - x_0$.
3. $\Delta_p[u + u \circ T] \leq 0$ for every isometry T.

1-3 is **equivalent.**

1-3 is equivalent to u being **dominative p-superharmonic.**

Cylindrical functions

Next question: What are the addable p-**harmonic** functions?

- Addable = dominative p-superharmonic.
- $0 = \Delta_p u \leq |\nabla u|^{p-2}\mathcal{D}_p u \leq 0$.

The double equation:

$$\mathcal{D}_p u = 0 = \Delta_p u.$$

A function f is **radial** if there exists a one-variable function F so that

$$f(x) = F(|x|).$$

Cylindrical functions

Definition

A function f in $\mathbb{R}^n$ is **cylindrical** (or k-cylindrical) if there exists a one-variable function F, an integer $1 \leq k \leq n$, and an $n \times k$ matrix Q with orthonormal columns, i.e. $Q^TQ = I_k$, so that

$$f(x) = F\Big(|Q^T(x - x_0)|\Big)$$

for some $x_0 \in \mathbb{R}^n$. We say that a function w in $\mathbb{R}^n$ is a **cylindrical fundamental solution** (to the p-Laplace Equation) if w is in the form

$$w(x) = C_1 w_{k,p}\Big(Q^T(x - x_0)\Big) + C_2, \qquad C_1 \geq 0,$$

for some k, Q and x_0 as above.

$$\mathcal{D}_p w = 0 = \Delta_p w.$$

Cylindrical functions

Theorem

Let $2 < p < \infty$ and let $u \in C^2(\Omega)$. If

$$\Delta_p u = 0 = \mathcal{D}_p u \qquad \text{in } \Omega,$$

then u is locally a cylindrical fundamental solution.

The addable p-harmonic functions are exactly the cylindrical fundamental solutions.

Isoparametric functions

A nonconstant smooth function $u\colon M \to \mathbb{R}$ on a Riemannian manifold M is called **isoparametric** if there exists functions f and g so that

$$|\nabla u| = f(u) \qquad \text{and} \qquad \Delta u = g(u).$$

A regular level-set of an isoparametric function is called an **isoparametric hypersurface**.

Theorem (Segre 1938)

A connected isoparametric hypersurface in $\mathbb{R}^n$ is, upon scaling and an Euclidean motion, an open part of one of the following hypersurfaces:

1. *a hyperplane $\mathbb{R}^{n-1}$,*
2. *a sphere S^{n-1},*
3. *a generalized cylinder $S^{k-1} \times \mathbb{R}^{n-k}$, $k = 2, \ldots, n-1$.*

Isoparametric functions

$$0 = |\nabla u|^{p-2}\left((p-2)\frac{\nabla u \mathcal{H}u \nabla u^T}{|\nabla u|^2} + \Delta u\right),$$
$$0 = (p-2)\lambda_n + \Delta u.$$

$$\frac{\nabla u \mathcal{H}u \nabla u^T}{|\nabla u|^2} = \lambda_n, \qquad \mathcal{H}u\nabla u^T = \lambda_n \nabla u^T.$$

$\mathbf{c}$ curve in a level set of u:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}|\nabla u(\mathbf{c}(t))|^2 = \nabla u \mathcal{H}u\mathbf{c}' = \lambda_n \nabla u \mathbf{c}' = 0.$$

$$f(u) = \frac{1}{2}|\nabla u|^2, \quad f'(u)\nabla u = \nabla u \mathcal{H}u = \lambda_n \nabla u$$

$$\Delta u = -(p-2)\lambda_n = -(p-2)f'(u) =: g(u).$$

Corollary

Let $2 < p < \infty$. The largest family, $\mathcal{F}$, of p-harmonic functions containing the affine functions, in which every sum of the members,

$$x \mapsto \sum_{i=1}^{N} u_i(x), \qquad u_i \in \mathcal{F},\ N \in \mathbb{N},$$

is p-superharmonic, is the set of cylindrical fundamental solutions.

That is, the family of functions on the form

$$x \mapsto C_1 W_{k,p}\Big(|Q^T(x - x_0)|\Big) + C_2, \qquad C_1 \geq 0,$$

where $k \in \{1, \ldots, n\}$, $Q \in \mathbb{R}^{n\times k}$ with $Q^T Q = I_k$, and where

$$W_{k,p}(r) := \begin{cases} -\frac{p-1}{p-k} r^{\frac{p-k}{p-1}}, & p \neq k, \\ -\ln r, & p = k. \end{cases}$$

Fundamental properties of the dominative p-Laplacian

1. **Domination:** A dominative p-superharmonic function is p-superharmonic.
2. **Cylindrical equivalence:** The cylindrical dominative- and ordinary p-superharmonic functions are the same.
3. **Sublinearity:** If u and v are dominative p-superharmonic, then so is $u + v$.
4. **Nesting property:**
   - If u is dominative p-superharmonic, then it is dominative q-superharmonic for every $q \in [2, p]$.
   - If u is dominative p-subharmonic, then it is dominative q-subharmonic for every $q \in [p, \infty]$.
5. **Rotational invariance:**

$$(\mathcal{D}_p u) \circ T = \mathcal{D}_p[u \circ T]$$

for all isometries $T\colon \mathbb{R}^n \to \mathbb{R}^n$.

The doubly nonlinear diffusion operator

$$\mathcal{L}_{p,q}u := \operatorname{div}\left(u^{q-1}|\nabla u|^{p-2}\nabla u\right)$$
$$= u^{q-1}|\nabla u|^{p-2}\left(\Delta_p^h u + (q-1)\frac{|\nabla u|^2}{u}\right)$$

$$q = 1: \ p\text{-Laplace}, \qquad p = 2: \ \text{Porous Medium}.$$

Superposition principles for the fundamental solution in the cases $2 \le p < n, q \ge 1$ and $p > n, 0 < q \le 1$ (J.Tyson 2016).

Definition (The dominative (p, q)-Laplacian)

$$\mathcal{D}_{p,q}u := \mathcal{D}_p u + (q-1)\frac{|\nabla u|^2}{u}, \qquad 2 \le p < \infty,\ q \ge 1,\ u > 0.$$

$$\mathcal{L}_{p,q}u \le u^{q-1}|\nabla u|^{p-2}\mathcal{D}_{p,q}u.$$

The dominative (p,q)-Laplacian

$\mathcal{D}_{p,q}u = \mathcal{D}_p u + (q-1)\frac{|\nabla u|^2}{u}$ is sublinear for positive smooth functions since

$$\frac{|\nabla u|^2}{u} = \max_{c \le -\frac{1}{4}|\mathbf{b}|^2} \left(\nabla u \mathbf{b} + cu\right).$$

$2 \le p < n$, $q \ge 1$.

- Fundamental solution $w_{n,p,q}(x) := |x|^{\frac{p-n}{p-2+q}} > 0$.
- $\lambda_{\max}(\mathcal{H}w_{n,p,n}) = \Delta_\infty^h w_{n,p,q}$ which implies

$$\mathcal{D}_{p,q}w_{n,p,q} = 0 = \mathcal{L}_{p,q}w_{n,p,q}.$$

$$V(x) := \sum_{i=1}^N c_i w_{n,p,q}(x - x_i), \qquad c_i > 0$$

is (p,q)-superharmonic in $\mathbb{R}^n$ since

$$\mathcal{L}_{p,q}V \le V^{q-1}|\nabla V|^{p-2}\mathcal{D}_{p,q}V \le 0, \qquad \text{ptw.} \;\; x \ne x_i.$$

No test functions from below at $x = x_i$.