Boundary Regularity for the Free Boundary in the One-phase Problem

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Joint work with Ovidiu Savin (Columbia University)

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A talk about fountains

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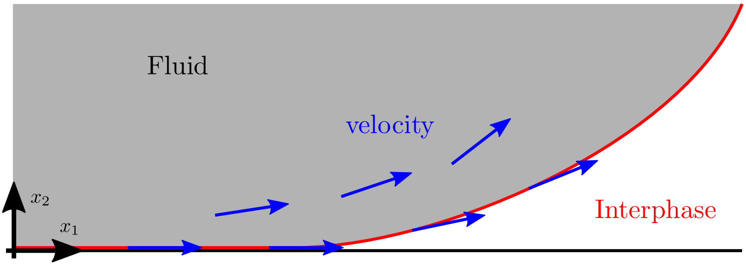
Goal: Describe how the jet separates.

Jets



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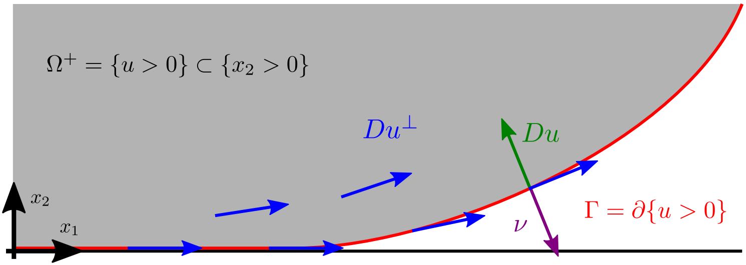

Incompressibility: Exists stream function $u \geq 0$ such that

$$\text{velocity} = Du^\perp = (\partial_2 u, -\partial_1 u)$$

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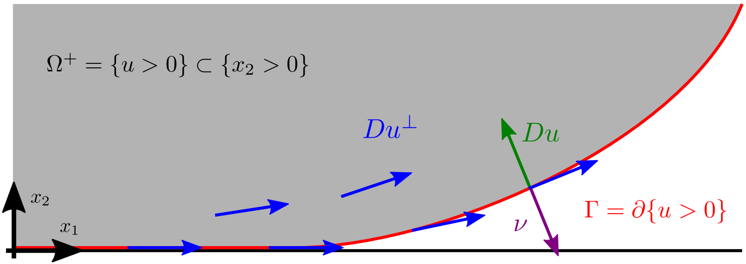

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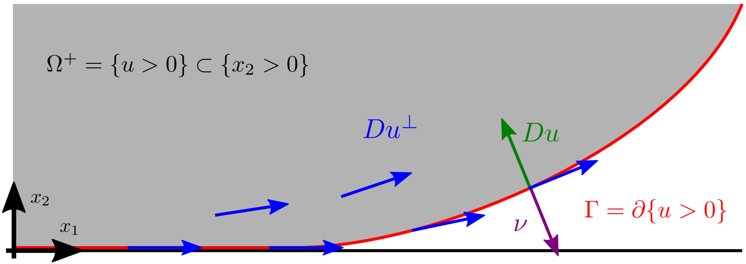

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Irrotationality: $\Delta u = 0$

Bernoulli's law: $|Du| = 1$ ($Du = -\nu$ or $\partial_\nu u = -1$) over Γ^+

One-phase problem

$u \geq 0$ satisfies

$$\begin{cases} \Delta u = 0 \text{ in } \Omega^+ = \{u > 0\} \cap \Omega \\ |Du| = 1 \text{ on } \Gamma^+ = \partial\{u > 0\} \cap \Omega \end{cases}$$

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- $u \in C^{0,1}_{loc}$
- Ω^+ has local finite perimeter
- Γ^+ is $C^{1,\alpha}$ regular provided a flatness hypothesis

Jets and cavities

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Slip condition: $u = 0$ over $Z = \{x_2 > 0\}$

How does Γ detach from Z?

Regularity of Γ up to the (fixed) boundary

Theorem (C-Savin)

Let $\Omega \subseteq \mathbb{R}^n$ be a domain with a $C^{1,\alpha}$ boundary portion $Z \subseteq \partial\Omega$ with $\alpha > 1/2$. Let $u \geq 0$ be a (viscosity) solution of

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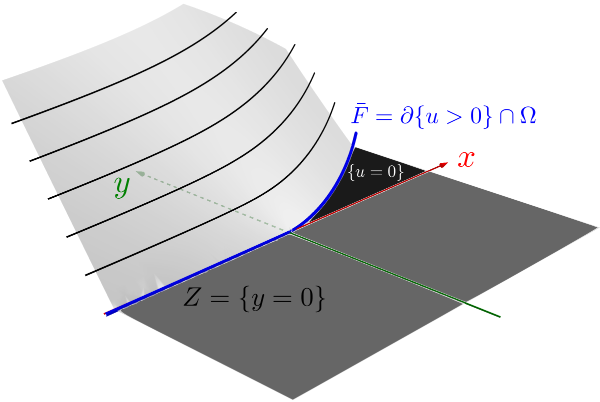

$(\bar{F} = \Gamma)$

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For a domain variation $u_\varepsilon(x + \varepsilon\eta(x)) = u(x)$

$$Ju_\varepsilon = Ju + \varepsilon \int_\Gamma (1 - |Du|^2)\eta \cdot \nu + o(\varepsilon)$$

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We are also allow to perform inward deformations around Z

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Also in this setting we recover

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$Du = -\nu$ means that

$$\partial_n w = \frac{1}{\varepsilon}(1 + \nu_n) = \frac{\varepsilon}{2}|Dw|^2 \text{ on } \Gamma^+.$$

Additionally, $|Du| \geq 1$ on Γ_0 says that

$$\partial_n w \geq 0 \text{ on } \Gamma_0.$$

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As $\varepsilon \to 0$, we expect $\Omega^+ \to B_1^+$, and w to solve

$$\begin{cases} \Delta w = 0 \text{ in } B_1^+ \\ w \geq 0 \text{ on } B_1' \\ \partial_n w = 0 \text{ on } \{w > 0\} \cap B_1' \\ \partial_n w \leq 0 \text{ on } B_1' \end{cases}$$

The **Signorini** or **thin obstacle problem**!

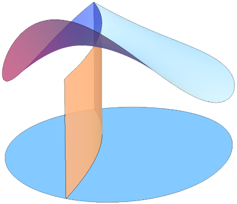

Figure: Image by Arshak Petrosyan

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Our result says that $\Gamma = \{x_n = \varepsilon w\}$ inherits the optimal regularity of the Signorini problem.

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From now on $\Omega = B_1^+$, $Z = B_1'$ and u satisfies

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\begin{cases}
\Delta u = 0 \text{ in } \Omega^+ = \{u > 0\} \cap B_1^+ \\
|Du| = 1 \text{ on } \Gamma^+ = \partial\{u > 0\} \cap B_1^+ \\
u = 0 \text{ on } B_1' \\
|Du| \geq 1 \text{ on } \Gamma_0 = \partial\Omega^+ \cap B_1'
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Preliminaries: Free flatness

Linear behavior at regular boundary points:

$v \in C(B_r)$, nonnegative, harmonic over $\Omega^+ = \{v > 0\}$, and $0 \in \partial\Omega^+$. An exterior ball condition for Ω^+ at 0 implies

$$v(x) = \alpha(x \cdot \nu)_+ + o(|x|) \text{ as } x \in \Omega^+ \to 0 \text{ nontangentially}$$

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If Γ detaches from Z at 0, then after a rescaling we can assume

$$\boxed{(1+\varepsilon)x_n \geq u \geq x_n - \varepsilon \text{ in } B_1^+}$$

for $\varepsilon > 0$ as small as we wish.

Harnack-type inequality

Lemma

There exist $\varepsilon_0, \theta \in (0,1)$ *such that if for* $\varepsilon \in (0, \varepsilon_0)$

$$(1+\varepsilon)x_n \geq u \geq x_n - \varepsilon \ \textit{in} \ B_1^+$$

Then in $B_{1/2}^+$ *either*

$$x_n \geq u \qquad \textit{or} \qquad u \geq x_n - (1-\theta)\varepsilon$$

Harnack-type inequality

Proof based on

Lemma (De Silva 2011)

Let v be a viscosity ***solution*** *of the one-phase problem in B_1. There exist $\varepsilon_0, \theta \in (0,1)$ such that if for $a, b \in (0, \varepsilon_0)$,*

$$x_n + a \geq v \geq x_n - b \text{ in } B_1$$

then in $B_{1/2}$ either

$$x_n + a - \theta c \geq v \qquad \text{or} \qquad v \geq x_n - b + \theta c \qquad (c = (a+b)/2)$$

Compactness

Consider for $\varepsilon_k \to 0$ a sequence of solutions $\{u_k\}$ satisfying

$$(1+\varepsilon_k)x_n \geq u \geq x_n - \varepsilon_k \text{ in } B_1^+$$

In the following statement

$$w_k = \frac{x_n - u_k}{\varepsilon_k}, \qquad G_k = \{(x,y) \in D_k \times \mathbb{R} : y = w_k(x)\}$$

where $D_k = (\Omega_k^+ \cup F_k) \cap B_{1/2}$.

Corollary

There exists $w \in C(B_{1/2}^+ \cup B'_{1/2})$ a solution of the Signorini problem such that a subsequence from $\{G_k\}$ converges to $\{(x,y) \in (B_{1/2}^+ \cup B'_{1/2}) \times \mathbb{R} : y = w(x)\}$ with respect to the Hausdorff distance.

Almost optimal regularity

Lemma

Given $\beta \in (0,1/2)$, *there exist* $\varepsilon_0, \mu \in (0,1)$ *such that if* $0 \in \Lambda$ *and for* $\varepsilon \in (0,\varepsilon_0)$

$$(1+\varepsilon)x_n \geq u \geq x_n - \varepsilon \ \textit{in} \ B_1^+$$

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$$u \geq x_n - \varepsilon\mu^{1+\beta} \ \textit{in} \ B_\mu^+$$

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This implies a $C^{1,\beta}$ modulus of continuity for Γ at 0

Almgren-type monotonicity formula

Assume $0 \in \Gamma_0$ and let $w = x_n - u$. Our goal, showing that $\Gamma = \{x_n = w\} \in C^{1,1/2}(0)$ will follow from

$$\boxed{H(r) := \left(\fint_{\partial B_r \cap \Omega^+} w^2\right)^{1/2} \leq C r^{3/2}}$$

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If $H(r) = Cr^k$ we can capture the **frequency** k from

$$k = N(r) := r \frac{d}{dr} \ln H(r) = \frac{r \frac{d}{dr} \fint_{\partial B_r \cap \Omega^+} w^2}{2 \fint_{\partial B_r \cap \Omega^+} w^2}$$

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Using $\Delta w = 0$ and integration by parts

$$\begin{aligned} \frac{d}{dr} \fint_{\partial B_r \cap \Omega^+} w^2 &= \fint_{\partial B_r \cap \Omega^+} 2w Dw \cdot \nu + error \\ &= 2r \fint_{B_r \cap \Omega^+} |Dw|^2 + error \end{aligned}$$

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In general, if $H(r) \le Cr^{3/2}$ it makes sense to expect

$$\lim_{r\to 0^+} N(r) = \frac{r\frac{d}{dr}\fint_{\partial B_r\cap\Omega^+} w^2}{2\fint_{\partial B_r\cap\Omega^+} w^2} \ge 3/2$$

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Lemma

There exist $\varepsilon, \eta > 0$ such that $(1 + Cr^{\varepsilon})\widetilde{N}(r)$ is non-decreasing where

$$\widetilde{N}(r) = r\frac{d}{dr}\ln\max(\widetilde{H}(r), r^{3/2+\varepsilon})$$

$$\widetilde{H}(r) = \int_0^r 2\rho\fint_{B_\rho\cap\Omega^+} |Dw|^2 d\rho = H(r) + O(r^{3/2+\eta})$$

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Here the almost optimal regularity is crucial in order to justify the computations.

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Thanks!