Higher integrability for doubly nonlinear parabolic equations

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Nonlinear parabolic equations

- The porous medium equation/system

$$u_t - \Delta(|u|^{m-1}u) = 0, \quad 0 < m < \infty.$$

Sometimes written in the form

$$(|u|^{m-2}u)_t - \Delta u = 0, \qquad 1 < m < \infty.$$

- The parabolic p-Laplace equation/system

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$

- The doubly nonlinear equation/system

$$(|u|^{p-2}u)_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < p < \infty.$$

Sometimes called Trudinger's equation.

- All equations above are special cases of a general doubly nonlinear equation/system

$$(|u|^{m-2}u)_t - \operatorname{div}(|Du|^{p-2}Du) = 0, \quad 1 < m < \infty, 1 < p < \infty.$$

Outline of the talk

- **Goal**
  - To show that the gradient of a weak solution to

$$(|u|^{p-2}u)_t - \operatorname{div}(|Du|^{p-2}Du) = 0,$$

    is locally integrable to a better power than assumed in the definition, with a reverse Hölder inequality estimate for the gradient.
- **Motivation**
  - To extend the recent breakthroughs by Gianazza–Schwarzacher and others to cover a wider class of equations and systems.
  - To develop direct methods that only apply energy estimates, Sobolev–Poincaré inequalities and Calderón–Zygmund type covering arguments.
  - To develop methods that apply to sign-changing solutions and systems. In particular, Harnack estimates and the expansion of positivity are not applied in the argument.

Review of the elliptic case

Let $u \in W^{1,p}_{\text{loc}}(\Omega)$, with $1 < p < \infty$, be a weak solution of the stationary p-Laplace equation

$$\operatorname{div}(|Du|^{p-2}Du) = 0 \quad \text{in } \Omega \subset \mathbb{R}^n.$$

Then there exists $\varepsilon > 0$ such that

$$\left(\fint_{B(x,r)} |Du|^{p+\varepsilon}\,dy \right)^{\frac{1}{p+\varepsilon}} \le c \left(\fint_{B(x,2r)} |Du|^{p}\,dy \right)^{\frac{1}{p}}$$

for every ball $B(x,2r) \subset \Omega$. In particular,

$$u \in W^{1,p+\varepsilon}_{\text{loc}}(\Omega).$$

(Gehring 1973, Meyers and Elcrat 1975, Giaquinta and Modica 1979, Stredulinsky 1980)

A sketch of the proof

- **Step 1:** An energy (Caccioppoli) estimate

$$\left(\fint_{B(x,r)} |Du|^p \, dy\right)^{\frac{1}{p}} \leq \frac{c}{r}\left(\fint_{B(x,2r)} |u - u_{B(x,2r)}|^p \, dy\right)^{\frac{1}{p}}.$$

- **Step 2:** A Sobolev–Poincaré inequality

$$\left(\fint_{B(x,2r)} |u - u_{B(x,2r)}|^p \, dy\right)^{\frac{1}{p}} \leq cr\left(\fint_{B(x,2r)} |Du|^q \, dy\right)^{\frac{1}{q}}$$

for some $q < p$.

- **Step 3:** A reverse Hölder inequality

$$\left(\fint_{B(x,r)} |Du|^p \, dy\right)^{\frac{1}{p}} \le c\left(\fint_{B(x,2r)} |Du|^q \, dy\right)^{\frac{1}{q}},$$

  for every $B(x, 2r) \subset \Omega$ with some $q < p$.

- **Step 4:** The Gehring–Meyers–Elrcat lemma: There exists $\varepsilon > 0$ such that

$$\left(\fint_{B(x,r)} |Du|^{p+\varepsilon} \, dy\right)^{\frac{1}{p+\varepsilon}} \le c\left(\fint_{B(x,2r)} |Du|^p \, dy\right)^{\frac{1}{p}}$$

  for every $B(x, 2r) \subset \Omega$.

Higher integrability results for parabolic equations 1(2)

- **Giaquinta–Struwe 1982**: Parabolic systems with a quadratic structure, i.e. $p = 2$.

- **Kinnunen–Lewis 2000**: Systems of the parabolic p-Laplacian structure

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \operatorname{div}(|F|^{p-2}F), \quad p > \tfrac{2n}{n+2}$$

- **Gianazza–Schwarzacher 2016**: Nonnegative solutions to the porous medium type equations

$$u_t - \Delta u^m = f, \quad m \geq 1.$$

Higher integrability results for parabolic equations 2(2)

- **Bögelein–Duzaar–Korte–Scheven 2017**: Systems of the porous medium type

$$u_t - \Delta\big(|u|^{m-1}u\big) = \operatorname{div} F, \quad m \geq 1.$$

- **Gianazza–Schwarzacher 2018**: Nonnegative solutions to the porous medium type equations

$$u_t - \Delta u^m = f, \quad \tfrac{(n-2)_+}{n+2} < m < 1.$$

- **Bögelein–Duzaar–Scheven 2018**: Systems of the porous medium type

$$u_t - \Delta\big(|u|^{m-1}u\big) = \operatorname{div} F, \quad \tfrac{(n-2)_+}{n+2} < m \leq 1.$$

Potential applications

- Partial regularity result for parabolic systems: Local Hölder continuity outside a small set (Misawa 2002).
- Nonlinear Calderón-Zygmund theory (Acerbi and Mingione 2007).
- Estimates up to the boundary (Parviainen 2009, Moring-Scheven-Schwarzacher-Singer 2019).
- Higher order systems (Parviainen and Bögelein 2010).
- Stability of solutions as p varies (Kinnunen and Parviainen 2010).

The parabolic Sobolev space

- Ω is an open subset of $\mathbb{R}^n$ and $0 \le t_1 < t_2 \le T$.
- The space-time cylinders are

$$\Omega_T = \Omega \times (0, T) \quad \text{and} \quad D_{t_1,t_2} = D \times (t_1, t_2),$$

  where $D \subset \Omega$ is an open set.
- The parabolic Sobolev space $L^p(0, T; W^{1,p}(\Omega))$ consists of measurable functions $u : \Omega_T \to [-\infty, \infty]$ such that $x \mapsto u(x, t)$ belongs to $W^{1,p}(\Omega)$ for almost all $t \in (0, T)$, and

$$\iint_{\Omega_T} \left(|u|^p + |Du|^p\right) dx\, dt < \infty.$$

- $u \in L^p_{\text{loc}}(0, T; W^{1,p}_{\text{loc}}(\Omega))$, if u belongs to the parabolic Sobolev space for every $D_{t_1,t_2} \Subset \Omega_T$.

Let $1 < p < \infty$. A function $u \in L^p_{\mathrm{loc}}(0, T; W^{1,p}_{\mathrm{loc}}(\Omega))$ is a weak solution to the doubly nonlinear equation

$$(|u|^{p-2}u)_t - \mathrm{div}(|Du|^{p-2}Du) = 0 \quad \text{in } \Omega_T,$$

if

$$\iint_{\Omega_T} \left(|Du|^{p-2}Du \cdot D\varphi - |u|^{p-2}u\varphi_t\right)\, dx\, dt = 0$$

for every $\varphi \in C_0^\infty(\Omega_T)$.

The Barenblatt solution

Example

The Barenblatt solution

$$u(x,t) = t^{-\frac{n}{p(p-1)}} \exp\left(-\frac{p-1}{p}\left(\frac{|x|^p}{pt}\right)^{\frac{1}{p-1}}\right),$$

where $x \in \mathbb{R}^n$ and $t > 0$, is a solution to the doubly nonlinear equation in the upper half space.

Observe: The Barenblatt solution is strictly positive for every $x \in \mathbb{R}^n$ and $t > 0$. This indicates that disturbances propagate with infinite speed.

More general doubly nonlinear parabolic systems

We focus on the prototype equation, but it is possible to consider solutions $u\colon \Omega_T \to \mathbb{R}^N$, $N \geq 1$, to a system

$$(|u|^{p-2}u)_t - \operatorname{div} A(x,t,u,Du) = \operatorname{div}(|F|^{p-2}F) \quad \text{in } \Omega_T$$

where $F\colon \Omega_T \to \mathbb{R}^N$ and $A\colon \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ with

$$\begin{cases} A(x,t,u,\xi)\cdot\xi \geq \alpha|\xi|^p\,, \\ |A(x,t,u,\xi)| \leq \beta|\xi|^{p-1}, \end{cases}$$

for almost every $(x,t) \in \Omega_T$ and every $u \in \mathbb{R}^N$, $\xi \in \mathbb{R}^{Nn}$ with $0 < \alpha \leq \beta < \infty$.

Structural properties

- The equation is nonlinear: The sum of two solutions is not a solution, in general.
- Solutions can be scaled.
- Constants cannot be added to solutions. Thus the boundary values cannot be perturbed in a standard way by adding an epsilon.
- In the natural geometry a scaling by r in the spatial variable corresponds to r^p in the time direction. For $p = 2$, this works for the heat equation.

The Cole–Hopf transformation

Consider a nonnegative solution of

$$(|u|^{p-2}u)_t - \operatorname{div}(|Du|^{p-2}Du) = 0.$$

The transformation $v = \log u$ leads to the diffusive Hamilton–Jacobi equation

$$v_t - \tfrac{1}{p-1}\operatorname{div}(|Dv|^{p-2}Dv) = -|Dv|^p,$$

with $Dv = \frac{Du}{u}$.

Note: In the elliptic case $\log u$ is a subsolution to the p-Laplace equation, but for the doubly nonlinear equation the equation changes.

A change of variables

Consider a solution of

$$(|u|^{p-2}u)_t - \operatorname{div}(|Du|^{p-2}Du) = 0.$$

The transformation $v = |u|^{p-2}u$ leads to

$$v_t = \operatorname{div}\left(\left(\frac{|Dv|}{|v|}\right)^{p-2} Dv\right).$$

Takeaway: The quotient $\frac{|Du|}{|u|}$ appears again. This indicates that it should play some role in the intrinsic geometry.

Some known results

- Existence (Ivanov–Mkrtychyan–Jäger 1997, Sturm 2017)
- Asymptotic behavior (Manfredi-Vespri 1994, Savaré–Vespri 1993, Tedeev–Vespri 2015)
- Nonnegative weak solutions satisfy a scale and location invariant parabolic Harnack inequality in the space-time cylinders $Q_{r,r^p}(x,t) = B(x,r) \times (t - r^p, t + r^p)$. (Fornaro–Sosio–Vespri 2015, Gianazza–Vespri 2006, Kinnunen–Kuusi 2007, Trudinger 1968)
- Nonnegative weak solutions and are locally Hölder continuous. (Ivanov 1994, Kuusi–Lageoglu–Siljander–Urbano 2012, Porzio-Vespri 1993, Vespri 1992)
- The weak gradient of a positive weak solution is locally bounded. (Siljander 2010)

Unexpected features

One might expect that the doubly nonlinear equation would have a similar behaviour as the heat equation, but this is not clear.

- The parabolic Harnack inequality does not immediately give local Hölder continuity, because constants cannot be added to solutions.
- The role of the intrinsic geometry is not clear. For example, the intrinsic geometry is not needed in Harnack estimates, but it is used in regularity theory.
- The question about uniqueness for the Dirichlet boundary value problem seems to be unsettled without additional assumptions. (Ivanov 1997, Lindgren-Lindqvist 2019)
- Very little is known for sign-changing solutions. For example, continuity of a weak solution (and its gradient) seems to be an open question.

Higher integrability for the doubly nonlinear system

Theorem (Bögelein, Duzaar, Kinnunen and Scheven 2018)

Let

$$\max\left\{\tfrac{2n}{n+2}, 1\right\} < p < \tfrac{2n}{(n-2)_+}$$

and assume that u is a weak solution to the doubly nonlinear equation in Ω_T. There exists $\varepsilon = \varepsilon(n, p) > 0$ such that

$$Du \in L^{p(1+\varepsilon)}_{loc}(\Omega_T).$$

A Reverse Hölder inequality for the gradient

The result comes with a uniform estimate

$$\fint_{Q_{r,r^p}} |Du|^{(1+\varepsilon)p}\,dx\,dt$$

$$\leq c\left[1 + \fint_{Q_{2r,(2r)^p}} \left(\frac{|u|^p}{(2r)^p} + |Du|^p\right) dx\,dt\right]^{\varepsilon} \fint_{Q_{2r,(2r)^p}} |Du|^p\,dx\,dt$$

for every cylinder $Q_{2r,(2r)^p} \subset \Omega_T$.

Notation: $A \subset \mathbb{R}^{n+1}$, $0 < |A| < \infty$,

$$\fint_A u\,dx\,dt = \frac{1}{|A|}\int_A u(x,t)\,dx\,dt.$$

The range of p

- $n \in \{1,2\}$: $1 < p < \infty$.
- $n \geq 3$: $\frac{2n}{n+2} < p < \frac{2n}{n-2}$.
- For the parabolic p-Laplace equation, the critical exponent is

$$p > \frac{2n}{n+2}.$$

- For the porous medium equation, the critical exponent is

$$m > \frac{(n-2)_+}{n+2} \quad \text{and} \quad m \sim \frac{1}{p-1} \quad \Longleftrightarrow \quad p < \frac{2n}{(n-2)_+}.$$

Open questions

- Does the result hold true in the entire range $1 < p < \infty$ for $n \geq 3$?
- Is the reverse Hölder inequality true in the form

$$\left(\fint_{Q_{r,r^p}} |Du|^{p+\varepsilon} \, dx \, dt \right)^{\frac{1}{p+\varepsilon}} \leq c \left(\fint_{Q_{2r,(2r)^p}} |Du|^p \, dx \, dt \right)^{\frac{1}{p}}$$

  for every cylinder $Q_{2r,(2r)^p} \subset \Omega_T$?
- Is the intrinsic geometry is really needed in the argument?

New features

- The argument is self-contained, transparent and flexible.
- A new type of intrinsic geometry that depends both on u and Du is applied in the argument.
- Estimates for a power of a weak solution.
- A regularity result for the gradient.
- Result covers sign-changing solutions and doubly nonlinear systems.
- Proof can be extended to more general doubly nonlinear systems.

Strategy of the proof

- **Step 1:** An energy estimate on intrinsic cylinders.
- **Step 2:** A Sobolev inequality on intrinsic cylinders.
- **Step 3:** A reverse Hölder inequality for the gradient on intrinsic cylinders.
- **Step 4:** A Calderón–Zygmund type stopping time argument to construct a collection of intrinsic cylinders $\{Q_{r,s}\}$ with a Vitali covering property.
- **Step 5:** A covering of the distribution set $\{z : |Du|(z) > \lambda\}$ by intrinsic cylinders $Q_{r,s}$ as constructed above satisfying

$$\lambda^p \approx \fint_{Q_{r,s}} |Du|^p \, dx \, dt.$$

- **Step 6:** Cavalieri's principle gives an estimate for

$$\fint_{Q_{r,r^p}} |Du|^{(1+\varepsilon)p} \, dx \, dt.$$

Intrinsic geometry 1(3)

We discuss a formal motivation for an appropriate intrinsic geometry.

- Let $Q_{r,s} = Q_{r,s}(z_0) = B(x_0, r) \times (t_0 - s, t_0 + s) \subset \Omega_T$ with $z_0 = (x_0, t_0)$.
- Consider $v(y, t) = u(ry, s\tau)$ for $(y, t) \in B(0, 1) \times (-1, 1)$.
- Infinitesimally

$$|u| \approx \left(\fint_{Q_{r,s}} |u|^p \, dx \, dt \right)^{\frac{1}{p}} \quad \text{and} \quad |Du| \approx \left(\fint_{Q_{r,s}} |Du|^p \, dx \, dt \right)^{\frac{1}{p}}.$$

Intrinsic geometry 2(3)

$$\partial_\tau v = s\partial_t u \approx \frac{s(|u|^{p-2}u)_t}{\left(\fint_{Q_{r,s}} |u|^p \, dx\, dt\right)^{\frac{p-2}{p}}} = \frac{s \operatorname{div}(|Du|^{p-2}Du)}{\left(\fint_{Q_{r,s}} |u|^p \, dx\, dt\right)^{\frac{p-2}{p}}}$$

$$\approx \frac{s\left(\fint_{Q_{r,s}} |Du|^p \, dx\, dt\right)^{\frac{p-2}{p}}}{\left[\fint_{Q_{r,s}} |u|^p \, dx\, dt\right]^{\frac{p-2}{p}}} \Delta_x u = \frac{s\left(\fint_{Q_{r,s}} |Du|^p \, dx\, dt\right)^{\frac{p-2}{p}}}{r^2\left(\fint_{Q_{r,s}} |u|^p \, dx\, dt\right)^{\frac{p-2}{p}}} \Delta_y v.$$

Intrinsic geometry 3(3)

We choose the scaling so that the factor in front of $\Delta_y v$ is one so that the equation looks like the heat equation in its own geometry. This gives

$$s = \frac{r^2 \left(\fint_{Q_{r,s}} |u|^p \, dx \, dt \right)^{\frac{p-2}{p}}}{\left(\fint_{Q_{r,s}} |Du|^p \, dx \, dt \right)^{\frac{p-2}{p}}} = \frac{r^p \left(\fint_{Q_{r,s}} \frac{|u|^p}{r^p} \, dx \, dt \right)^{\frac{p-2}{p}}}{\left(\fint_{Q_{r,s}} |Du|^p \, dx \, dt \right)^{\frac{p-2}{p}}}$$

and thus

$$\frac{s}{r^p} = \mu^{p-2} \quad \text{with} \qquad \mu^p = \frac{\fint_{Q_{r,s}} \frac{|u|^p}{r^p} \, dx \, dt}{\fint_{Q_{r,s}} |Du|^p \, dx \, dt}.$$

Takeaway: This indicates homogeneous behavior on intrinsic cylinders of the form $Q_{r,s}$ with $\frac{s}{r^p} = \mu^{p-2}$.

An energy estimate

Notation: $u^\alpha = |u|^{\alpha-1}u$, $\alpha > 0$ $(0^\alpha = 0)$.

For every $a \in \mathbb{R}$, we have

$$\sup_{t\in(t_0-s,t_0+s)} \fint_{B(x_0,r)} \frac{|u(x,t)^{\frac{p}{2}} - a^{\frac{p}{2}}|^2}{S}\,dx + \fint_{Q_{r,s}(z_0)} |Du|^p\,dx\,dt$$
$$\le c \fint_{Q_{R,S}(z_0)} \left(\frac{|u^{\frac{p}{2}} - a^{\frac{p}{2}}|^2}{S-s} + \frac{|u-a|^p}{(R-r)^p} \right) dx\,dt,$$

where $Q_{R,S}(z_0) = B(x_0,R) \times (t_0 - S, t_0 + S) \subset \Omega_T$ with $R, S > 0$, and $r \in \left[\frac{R}{2}, R\right)$, $s \in \left[\frac{S}{2^p}, S\right)$.

Takeaway: The energy estimate is for a power of a solution. The power appears in the boundary term in the proof of the energy estimate.

An energy estimate on intrinsic cylinders

By choosing $S = \mu^{p-2} R^p$ and $s = \mu^{p-2} r^p$, we have

$$\sup_{t \in (t_0 - s, t_0 + s)} \fint_{B(x_0, r)} \frac{\mu^{2-p} |u(x,t)^{\frac{p}{2}} - a^{\frac{p}{2}}|^2}{R^p} \, dx + \fint_{Q_{r,s}(z_0)} |Du|^p \, dx \, dt$$
$$\leq c \fint_{Q_{R,S}(z_0)} \left(\frac{\mu^{2-p} |u^{\frac{p}{2}} - a^{\frac{p}{2}}|^2}{R^p - r^p} + \frac{|u - a|^p}{(R - r)^p} \right) dx \, dt$$

with

$$\mu^p \approx \frac{\displaystyle \fint_{Q_{r,s}(z_0)} \frac{|u|^p}{r^p} \, dx \, dt}{\displaystyle \fint_{Q_{r,s}(z_0)} |Du|^p \, dx \, dt}.$$

Takeaway: The energy estimate becomes homogeneous in the intrinsic geometry.

A gluing lemma

Let $Q_{R,S}(z_0) = B(x_0, R) \times (t_0 - S, t_0 + S) \subset \Omega_T$ with $R, S > 0$. There exists $r \in \left[\frac{R}{2}, R\right)$ such that

$$\left| \fint_{B(x_0,r)} u(x, t_2)^{p-1}\,dx - \fint_{B(x_0,r)} u(x, t_1)^{p-1}\,dx \right| \le c\frac{S}{R} \fint_{Q_{R,S}(z_0)} |Du|^{p-1}\,dx\,dt$$

for every $t_1, t_2 \in (t_0 - S, t_0 + S)$.

Takeaway: The gluing lemma enables us to pass from slice-wise integral averages in Sobolev–Poincaré inequalities to space-time integral averages in energy estimates. It compensates the lack of differentiability with respect to time.

A Sobolev-Poincaré inequality

Let $p > \frac{2n}{n+2}$. There exists $q < p$ such that

$$\fint_{Q_{r,s}(z_0)} \left(\frac{\mu^{2-p} |u^{\frac{p}{2}} - a^{\frac{p}{2}}|^2}{r^p} + \frac{|u - a|^p}{r^p} \right) dx\, dt \le c \left(\fint_{Q_{r,s}(z_0)} |Du|^q \, dx\, dt \right)^{\frac{p}{q}} + \text{other terms}$$

for every intrinsic cylinder $Q_{r,s}(z_0)$.

Takeaway: This version of the Sobolev-Poincaré inequality holds for weak solutions of the doubly nonlinear equation. The gluing lemma is used in the argument.

A reverse Hölder inequality

A combination of the energy estimate and the Sobolev-Poincaré inequality gives a reverse Hölder inequality. Let $p > \frac{2n}{n+2}$. Then there exists $q < p$ such that

$$\fint_{Q_{r,s}(z_0)} |Du|^p \, dx \, dt \le c \left(\fint_{Q_{r,s}(z_0)} |Du|^q \, dx \, dt \right)^{\frac{p}{q}}$$

for every intrinsic cylinder $Q_{r,s}(z_0)$.

Takeaway: The reverse Hölder inequality holds in the intrinsic geometry.

The proof completed

- The Calderón–Zygmund type stopping time argument to construct a collection of cylinders $\{Q_{r,s}(z)\}$ with a Vitali covering property and a covering of the distribution set $\{z : |Du|(z) > \lambda\}$ by intrinsic cylinders $Q_{r,s}(z)$ is a modification of the argument by Gianazza-Schwarzacher.
- The fact that Cavalieri's principle gives an estimate for

$$\fint_{Q_{r,r^p}} |Du|^{(1+\varepsilon)p} \, dx \, dt$$

is rather standard.

Extensions and possible future developments

- Higher integrability up to the boundary.
- General doubly nonlinear systems with different powers.
- Stability of solutions with respect to p.
- Very weak solutions.