Improved regularity for elliptic equations in the double-divergence form

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General overview

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3. Regularity in Hölder spaces;
4. Regularity transmission by approximation methods.

The double-divergence setting

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2. Uniform ellipticity - there exist contants $0 < \lambda < \Lambda$ such that

$$\lambda Id \leq (a^{ij}(x))_{i,j=1}^d \leq \Lambda Id,$$

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where $g := I + (D^2u)^T D^2 u$ is the induced metric.

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$\longrightarrow$ **Were $a(x)$ discontinuous, so would be $v(x)$.**

Our program and main results

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Are there **gains of regularity**, as solutions approach their zero level-sets?

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Import information from the well-understood non-divergence problem

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The regularity of the coefficients is an upper bound for the regularity of the solutions 'in the large'. Therefore, we look for regularity improvements at $x_0 \in \{u = 0\}$.

Hölder continuity

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where $(\overline{a}^{ij})^d_{i,j=1}$ is a constant matrix.

Then, $u \in \mathcal{C}^{1-}_{loc}(B_1 \cap \partial\{u > 0\})$ and for every $\alpha \in (0,1)$ there exists $C_\alpha > 0$ such that

$$\sup_{B_r(x_0)} |u(x) - u(x_0)| \leq C_\alpha r^\alpha,$$

for every $0 < r \ll 1/2$ and $x_0 \in \partial\{u > 0\}$.

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Our results extend to equations involving lower-order terms

$$\partial^2_{x_i x_j}\left(a^{ij}(x)u(x)\right) + \partial_{x_i}\left(b^i(x)u(x)\right) + c(x)u(x) = 0 \quad \text{in} \quad B_1$$

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provided b^i, $c : B_1 \to \mathbb{R}$ are well-prepared.

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Main ingredients are:

$\longrightarrow$ Preliminary (uniform) compactness;

$\longrightarrow$ Approximation results preserving zero level-sets;

$\longrightarrow$ Iteration mechanism through scaling techniques.

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Moreover,

$$h(x_0) = 0$$

for every $x_0 \in \partial\{u > 0\}$.

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Let $u \in L^1_{loc}(B_1)$ be a weak solution to the double-divergence equation. Suppose $a^{ij} \in \mathcal{C}^{\beta}_{loc}(B_1)$.

For every $\alpha \in (0,1)$ there exists $\varepsilon > 0$ and $\rho \in (0, 1/2)$ such that, if

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Hölder regularity of the gradient

Impose further conditions on the coefficients to unlock a similar analysis at the level of the gradient Du; in principle, to ask for

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Consider a suitable zero level-set in this context:

$$\{u = |Du| = 0\}.$$

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Then, $u \in \mathcal{C}^{1,1-}_{loc}(B_1 \cap \partial\{u > 0\})$ and there exists $C_\alpha > 0$ such that

$$\sup_{B_r(x_0)} |Du(x) - Du(x_0)| \leq C_\alpha r^\alpha,$$

for every $0 < r \ll 1/2$, and $x_0 \in \partial\{u > 0\} \cap \partial\{|Du| > 0\}$ and $\alpha \in (0, 1)$.

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Moreover,

$$h(x_0) = 0 \quad \text{and} \quad Dh(x_0) = \mathbf{0}$$

for every $x_0 \in \partial\{u > 0\} \cap \partial\{|Du| > 0\}$.

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   $\longrightarrow$ preliminary compactness.

Thank you very much!