Stability of extremals of the Riesz potential

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(joint work with N. Fusco)

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The most "physically relevant" case is for $N \geq 3$ and $\alpha = 2$.

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Similar properties happen for a wide class of non-local energies.

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The exponent 2 is sharp.

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It would be nice to extend the stability result to the short-range interaction.

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On the bright side, this is good because a double integral is not so delicate as other terms, for instance the perimeter.
On the dark side, this is bad because there is no regularizing effect, so one cannot hope for a strategy like the Cicalese–Leonardi one (the Euler-Lagrange equation associated to the Riesz energy is not even a differential equation!)

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"Reducing to a nearly spherical set" means that for every E there is some F satisfying

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The first part contains almost all the difficulty, in particular to fix the barycenter.

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• Thus, one reduces himself to the case that $\varphi^- \cdot \varphi^+ \equiv 0$.
At this stage, a set has been found satisfying all the conditions except the barycenter one.

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Hopefully, the same idea can work also in other situations.

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Thank you