On the derivative of fractional maximal function on domains

Olli Saari, University of Bonn

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Averages

- Fix $p \in [1, \infty]$ and let $f \in L^p(\mathbb{R}^n)$.
- Let $r : \mathbb{R}^n \to (0, \infty]$ be a function. Define

$$F_r(x) = \fint_{B(x,r(x))} |f(y)|\, dy.$$

(extension to $[0, \infty]$ as $\limsup$)

Maximal function

- Define the maximal function

$$Mf(x) = \sup_{r>0} \fint_{B(x,r)} |f(x)|\, dy.$$

- There exists a function r so that

$$Mf(x) = F_r(x) = \fint_{B(x,r(x))} |f(y)|\, dy.$$

- $f \mapsto Mf$ is bounded $L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ for $p > 1$ and $L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$.

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Regularity

- "Extremal functions are more regular".
- Generic F_r possesses no smoothness; Mf is always at least lower semicontinuous.
- Unlike in the L^p case, here Mf is better than the input.

Sobolev spaces

Let $W^{1,p}(\mathbb{R}^n) = \{f \in L^p(\mathbb{R}^n) : |\nabla f| \in L^p(\mathbb{R}^n)\}$.

Theorem (Kinnunen 1997)

$M : W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ *is bounded for* $p > 1$. *The pointwise (a.e.) inequality*

$$|\nabla Mf(x)| \leq M|\nabla f|(x)$$

also holds in these cases.

Kurka–Tanaka theorem

Theorem (Kurka 2015)

Let $f \in W^{1,1}(\mathbb{R})$. Then Mf is weakly differentiable and

$$\|(Mf)'\|_{L^1(\mathbb{R})} \leq C\|f'\|_{L^1(\mathbb{R})}.$$

- This statement is an open problem in dimensions $n \geq 2$.
- Part of the challenge comes from peculiarities of L^1: maximal function is not bounded operator, the space itself is not reflexive.
- There is a variant of the problem for fractional maximal functions. It is considered more tractable.

Fractional maximal function

Let $\alpha \in [0, n)$ and for $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$, define

$$M_\alpha f(x) = \sup_{r>0} r^\alpha \fint_{B(x,r)} |f(y)|\, dy.$$

As for order α Riesz potentials, also

$$M_\alpha : L^p(\mathbb{R}^n) \to L^{\frac{pn}{n-\alpha p}}(\mathbb{R}^n)$$

boundedly. Smoothness properties are non-trivial.

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Fractional endpoint question

Question (Carneiro and Madrid (2015))

Let $\alpha \in (0,1)$ *and* $f \in W^{1,1}(\mathbb{R}^n)$*. Does it hold*

$$\|\nabla M_\alpha f\|_{L^{n/(n-\alpha)}(\mathbb{R}^n)} \leq C\|\nabla f\|_{L^1(\mathbb{R}^n)}?$$

Formula for the gradient

Theorem (Luiro, Hajłasz–Malý)

Let $f, Mf \in L^1_{loc}(\mathbb{R}^n)$ *be non-negative and weakly differentiable, and let* $r : \mathbb{R}^n \to [0, \infty)$ *be such that*

$$Mf(x) = \fint_{B(x,r(x))} f(y)\,dy.$$

Then

$$\nabla Mf(x) = \fint_{B(x,r(x))} \nabla f(y)\,dy$$

Fractional analogue, motivation

Let $r : \mathbb{R}^n \to [0, \infty)$ be such that

$$M_\alpha f(x) = r(x)^\alpha \fint_{B(x,r(x))} f(y)\, dy.$$

Suppose we had the following formula

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An optimality condition

Define the averages

$$A(x,t) = \int_{B(0,1)} f(x+ty)\,dy.$$

If t maximizes $(\cdot)^\alpha A(x,\cdot)$, then

$$0 = \partial_r r^\alpha A(x,r)|_{r=t} = \alpha t^{\alpha-1} A(x,t) + t^\alpha \int_{B(0,1)} y \cdot \nabla f(x+ty)\,dy$$

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By the divergence theorem

$$\int_{B(0,1)} y \cdot \nabla f(x+ty)\, dy$$
$$= \frac{1}{t}\left(\int_{\partial B(0,1)} f(x+yt)\, d\sigma(y) - n \int_{B(0,1)} f(x+ty)\, dy \right)$$

and so

$$t^{\alpha-1}\sigma_t * f(x) = (n-\alpha)t^{\alpha-1}A(x,t)$$

spherical = solid for optimal balls

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Let e be a unit vector.
For the derivative of the fractional maximal function

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\begin{aligned}
e \cdot \nabla M_\alpha f(x) &= t^\alpha \fint_{B(0,1)} e \cdot (\nabla f)(x + ty)\, dy \\
&= t^{\alpha-1} \int_{\partial B(0,1)} e \cdot y f(x + yt)\, dy \\
&= t^{\alpha-1} \sigma_t * (e \cdot y f)(x) \\
&\leq (n - \alpha) t^{\alpha-1} A(x, t)
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An inequality

Theorem (Kinnunen and Saksman (2003))

Let $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$ and $1 \leq \alpha \leq \frac{n}{p} - 1$. Then

$$|\nabla M_\alpha f| \leq M_{\alpha-1} f$$

so that $M_\alpha : L^p(\mathbb{R}^n) \to \dot{W}^{1, \frac{np}{n-(\alpha-1)p}}(\mathbb{R}^n)$.

Remark: The original proof is different from the previous argument.

Open sets $\Omega \subset \mathbb{R}^n$

Let $\Omega \subset \mathbb{R}^n$ be open and define

$$M_\alpha^\Omega f(x) = \sup_{0<r<\operatorname{dist}(x,\Omega^c)} r^\alpha \fint_{B(x,r)} |f(y)|\, dy.$$

Problem

Let $\alpha \geq 0$ and $f \in W^{1,1}(\Omega)$. Does

$$\|\nabla M_\alpha f\|_{L^{n/(n-\alpha)}(\Omega)} \leq C\|\nabla f\|_{W^{1,1}(\Omega)}$$

hold?

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Failure of smoothing inequality

- The inequality $|\nabla M_\alpha^\Omega f| \lesssim M_{\alpha-1}^\Omega f$ does not hold (Heikkinen–Kinnunen–Korvenpää–Tuominen)!
- Instead, they show that

$$|\nabla M_\alpha^\Omega f| \lesssim M_{\alpha-1}^\Omega f + S_{\alpha-1} f$$

where $S_{\alpha-1}$ is the fractional spherical maximal function

$$S_{\alpha-1} f(x) = \sup_{r>0} r^{\alpha-1} (\sigma_r * |f|)(x).$$

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Very simple cases, $\alpha \in (1, n)$

From now on assume Ω is an open set so that the Sobolev embedding

$$W^{1,1}(\Omega) \hookrightarrow L^{n/(n-1)}(\Omega)$$

is valid (e.g. Lipschitz domain).

Then for $\alpha > 1$

$$\|\nabla M_\alpha^\Omega f\|_{L^{n/(n-\alpha)}} \lesssim \|M_{\alpha-1}^\Omega f + S_{\alpha-1} f\|_{L^{n/(n-\alpha)}} \overset{\text{Schlag–Sogge}}{\lesssim} \|f\|_{L^{n/(n-1)}} \overset{\text{Sobolev}}{\lesssim} \|f\|_{W^{1,1}(\Omega)}.$$

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Relatively simple case $\alpha = 1$

- The strategy with Sobolev embedding is still valid.
- The problem with smoothing inequality becomes a problem: S_0 is not bounded in $L^{n/(n-1)}$, and the previous computation breaks down.
- Pointwise estimate can be replaced by good enough L^p inequality.

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Endpoint result on domains

Theorem (Ramos–S–Weigt (work in progress))

Let Ω be a domain with $W^{1,1}(\Omega) \hookrightarrow L^{n/(n-1)}(\Omega)$. Then there is a constant C only depending on the domain and the dimension so that

$$\|\nabla M_1^{\Omega} f\|_{L^{n/(n-1)}(\Omega)} \leq \|f\|_{W^{1,1}(\Omega)}$$

for all $f \in W^{1,1}(\Omega)$.

(A version) of the main tool

Theorem (Ramos–S–Weigt (work in progress))

- *If $\alpha = 1$, assume that Ω is a bounded Lipschitz domain satisfying a (uniform) interior ball condition.*
- *If $\alpha > 1$ assume that Ω is merely bounded.*

Let $p > 1$. Then there is a constant $C = C(n, \Omega, \alpha)$ so that

$$\|\nabla M_\alpha^\Omega f\|_{L^p(\Omega)} \leq C \|f\|_{L^p(\Omega)}$$

for all $f \in L^p(\Omega)$.

Remarks:

- The estimate has bad scaling properties and it is necessary to have Ω bounded for such an inequality.
- The estimate proves weak differentiability of $M_\alpha^\Omega f$ for $L^p(\Omega)$ with $p \in (1, n/(n-1)]$ which was previously not known.

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Idea of the proof

Define

$$\delta(x) := \mathrm{dist}(x, \Omega^c)$$
$$A_\alpha f(x) := \delta(x)^\alpha \fint_{B(x,\delta(x))} f(y)\, dy$$

A point x is

- *constrained*, if $M_\alpha^\Omega f(x) = A_\alpha f(x)$
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Idea of the proof, II

Unconstrained points are dealt with argument similar to the case of $\mathbb{R}^n$.

Constrained points give rise to an operator

$$Bf(x) := \int_{\partial B(x,\delta(x))} \frac{|b_x - y|}{\delta(x)} f(y) \, \mathcal{H}^{n-1}(y)$$

where b_x is a point on $\partial\Omega$ that is closest to x.

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Final part consist in showing that B is significantly better behaved than the spherical maximal function:

- It can only zoom in to points on the boundary of the domain (not everywhere).
- The weight dampens the measure near the boundary of the domain.
- The rough idea is to prove bounds in L^1 and L^∞ and interpolate.

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Thank you for your attention!