Analyzing a special case of the Hele-Shaw flow using integro-differential operators

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The Law of the Instrument

"I suppose it is tempting, if the only tool you have is a hammer, to treat everything as if it were a nail."

– thanks, wikipedia

A theme to keep in mind

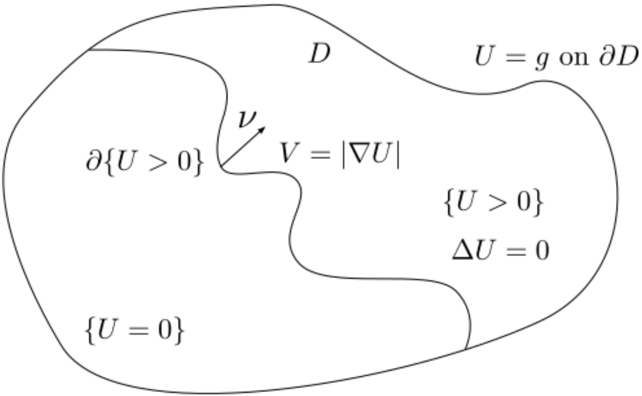
There are a few instances where regularity is shown to occur for Hele-Shaw.

Is it regular**izing**?

Where does it **come from**?

Hele-Shaw

Let us recall what the one phase Hele-Shaw problem looks like



Hele-Shaw

A few references for existence, uniqueness, regularity:

- Escher-Simonett 1997
- Kim 2003, 2006
- Jerison-Kim 2005
- Choi-Jerison-Kim 2007
- Chang Lara - Guillen 2016

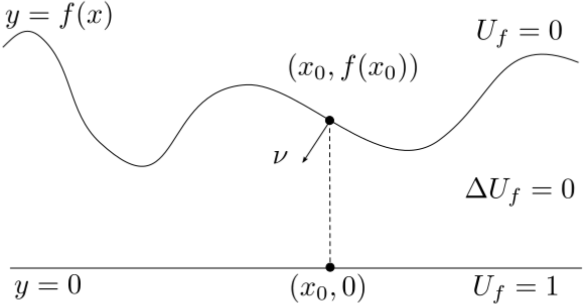
Hele-Shaw, special case for the interface

Then, $U : \mathbb{R}_+^{d+1} \times \mathbb{R}_+ \to \mathbb{R}$ is a non-negative function solving

$$
\text{(HS)} \quad \begin{cases} \Delta U = 0 & \text{in } \{U > 0\}, \\ U = 1 & \text{on } \{y = 0\}, \\ V = |\nabla U| & \text{on } \partial\{U > 0\}. \end{cases}
$$

V denoting the normal velocity of the free boundary $\partial\{U > 0\}$.

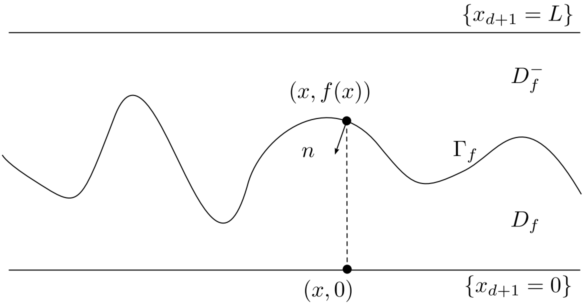
Hele-Shaw, special case for the interface



$$D_f = \{(x, x_{d+1}) \in \mathbb{R}^{d+1} \ : \ 0 < x_{d+1} < f(x)\} \quad \text{and} \quad \Gamma_f = \text{graph}(f)$$

Hele-Shaw, special case for the interface

These methods actually apply to a two-phase version of Hele-Shaw



$$V = G(\partial_n^+ U_f^+, \partial_n^- U_f^-) \quad (\text{e.g. } G = \left|\partial_n^+ U_f^+\right|^2 - \left|\partial_n^- U_f^-\right|^2)$$

Hele-Shaw, the goal

So, the goal of studying the problem is to produce a function U and describe its properties.

Thus, in our special case, this is equivalent to producing and describing the function, f.

Integro-Differential Equations

$$\partial_t f - L(f,x) = g(x,t) \;\text{ in }\; \mathbb{R}^d \times (0,T] \;\text{ and }\; f(\cdot,0) = f_0.$$

Where $L(f,x)$ has the form

$$L(f,x) = b(x)\cdot\nabla f(x) + \int_{\mathbb{R}^d} \left(f(x+h) - f(x) - \mathbb{1}_{B_1}(h)\nabla f(x)\cdot h \right) \mu(x,dh),$$

with

b bounded, and $\mu(x,\cdot) \geq 0$ is a measure, possibly singular at $h = 0$.

NOTATION: $\delta_h f(x) := f(x+h) - f(x) - \mathbb{1}_{B_1}(h)\nabla f(x)\cdot h$

Integro-Differential Equations

The (arguably) simplest and most canonical case is for some $\alpha \in (0, 2)$

$$b = 0 \quad \text{and} \quad \mu(x, dh) = C_{d,\alpha} \, |h|^{-d-\alpha} \, dh,$$

giving

$$L(f, x) = -(-\Delta)^{\alpha/2} f(x)$$

Integro-Differential Equations, a powerful tool

Theorem (Krylov-Safonov)

Under (ASSUMPTIONS, listed below), there exists a universal γ, and C, so that any (appropriately defined) solution to

$$\partial_t f - L(f,x) = g(x) \quad \text{in} \quad B_1 \times (-1,0],$$

enjoys the estimate

$$[f]_{C^\gamma(B_{1/2}\times(-1/2,0])} \leq C\left(\|f\|_{L^\infty(\mathbb{R}^d\times(-1,0])} + \|g\|_{L^\infty}\right),$$

where

$$[f]_{C^\gamma(B_{1/2}\times(-1/2,0])} = \sup_{0<|x-y|<1/2,\ 0<|t-s|<1/2} \frac{|f(x,s)-f(y,t)|}{\left(|x-y|+|s-t|^{1/\alpha}\right)^\gamma}$$

Some Results for Kyrlov-Safonov

all results assume a density: $\mu(x, dh) = k(x, h)dh$

symmetry: $K(x, -h) = K(x, h)$,

LB: $c_1(2-\alpha)\,|h|^{-d-\alpha} \leq K(x, h) \leq c_2(2-\alpha)\,|h|^{-d-\alpha}$:UB

Some Results for Kyrlov-Safonov

- Bass-Levin (2002): elliptic; $b \equiv 0$; symmetry, LB, UB (not robust)
- Bass-Kassmann (2004): elliptic; $b \equiv 0$; variable α; symmetry, LB, UB (not robust)
- Silvestre (2006): elliptic; $b \equiv 0$; variable α; slightly relaxed symmetry, LB, UB (not robust)
- Caffarelli-Silvestre (2009): elliptic; $b \equiv 0$; symmetry, LB, UB (robust)
- Chang Lara (2012): elliptic; nontrivial b; non-symmetric, LB, UB (robust)
- Change Lara - Davila (2016): parabolic; non-trivial b; non-symmetric; LB, UB (robust)
- Silvestre (2014): parabolic; non-trivial b; non-symmetric; LB, UB (robust)
- Schwab-Silvestre (2016): parabolic; non-trivial b; non-symmetric, relaxed LB, UB only in integral sense (robust)

For Later Use – Result for Kyrlov-Safonov

$$\partial_t f - \left(b(x) \cdot \nabla + \int_{\mathbb{R}^d} \delta_y f(x) K(x,y) dy \right) = g(x,t)$$

Theorem (Chang Lara - Davila 2016, also Chang Lara 2012 elliptic)

Assume $\alpha \in [1,2)$. Krylov-Safonov and $C^{1,\gamma}$ for the class of equations where the pair (b,K) satisfies

$$\sup_{r \in (0,1)} r^{\alpha-1} \left| b + \int_{B_1 \setminus B_r} y K(x,y) dy \right| \leq C$$

and

$$c_1 |y|^{-d-\alpha} \leq K(x,y) \leq c_2 |y|^{-d-\alpha}.$$

Krylov-Safonov, fully nonlinear

Note, Krylov-Safonov holds for fully nonlinear equations, for f a viscosity solution of

$$\partial_t f - F(f, x) = g(x, t) \quad \text{in} \quad B_1 \times (-1, 0],$$

where F is an operator that enjoys the structure, for some family of L^{ij} as above,

$$F(f, x) = \min_i \max_j L^{ij}(f, x).$$

(There is much more to say, but not enough time)

$C^{1,\gamma}$, translation invariant

If furthermore, F is translation invariant, i.e. $F(f(\cdot + z), x) = F(f, x + z)$ (or, concretely, b^{ij} and μ^{ij} are independent of x), Krylov-Safonov implies higher regularity

Theorem ($C^{1,\gamma}$, Assume F is translation invariant)

There is a universal γ and C (depending upon the assumptions on F, above) so that if f is a viscosity solution of $\partial_t f - F(f, x) = 0$ in $B_1 \times (-1, 0]$, then

$$(*)[\partial_t f(x, \cdot)]_{C^\gamma((-1/2,0])} + [\nabla f(\cdot, t)]_{C^\gamma(B_{1/2})} \leq C(\|u\|_{L^\infty(\mathbb{R}^d)\times(-1,0]} + (**))$$

Sometimes $(*)$ is present and sometimes it is not, sometimes $(**)$ contains an extra term for the time behavior of u, and sometimes it doesn't, depending upon the particular result.

See: Chang Lara - Davila 2016 and Serra 2015 (also Kriventsov 2013)

Operators with the GCP

Definition

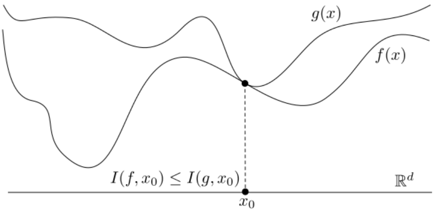
$$I : D \subset \mathbb{R}^X \to \mathbb{R}^X$$

is said to have the **global comparison property** *(GCP) if*

$$f, g \in D \text{ and } g \text{ touches } f \text{ from above at } x_0 \Rightarrow I(f, x_0) \leq I(g, x_0)$$

$$f(x) \leq g(x) \; \forall x \in X$$

$$f(x_0) = g(x_0)$$



Structure from GCP

Theorem (Guillen-Schwab 2016 and 2019)

(Generalizes to a complete, d-dimensional manifold)
If $I : C^2(\mathbb{R}^d) \to C^0(\mathbb{R}^d)$ *is Lipschitz, with the GCP, then*

$$\forall\ u \in C^2,\ x \in \mathbb{R}^d, \quad I(u,x) = \min_i \max_j \{f_{ij}(x) + L_{ij}(u,x)\}$$

where, for each pair of indices ij*, we have*

- $f_{ij}(x) \in C^0(\mathbb{R}^d)$ *(uniformly)*

$$\begin{aligned} L_{ij}(u,x) =& \mathrm{Tr}(A_{ij}(x)D^2u) + B_{ij}(x)\cdot\nabla u + C_{ij}(x)u \\ &+ \int_{\mathbb{R}^d} (u(x+y) - u(x) - y\cdot\nabla u(x)\mathbb{1}_{B_1}(y))\mu_{ij}(x,dy) \end{aligned}$$

Structure from GCP

Theorem (Guillen-Schwab 2016 and 2019)

Furthermore if $I : C^{1,\gamma}(\mathbb{R}^d) \to C(\mathbb{R}^d)$ is Lipschitz and satisfies the GCP, then

$$
\begin{aligned}
L_{ij}(u,x) = {} & C_{ij}(x)u(x) + B_{ij}(x)\cdot\nabla u \\
& + \int_{\mathbb{R}^d} u(x+y) - u(x) - \nabla u(x)\cdot y \mathbb{1}_{B_1(0)}\, \mu_{ij}(x,dy)
\end{aligned}
$$

and

$$\sup_{ij}\sup_{x}\int \min\{|y|^{1+\gamma},1\}\,\mu_{ij}(x,dy) < \infty.$$

What is the connection?

Why are these three topics related?

A heuristic answer is that the D-to-N on half space is the $-(-\Delta)^{1/2}$ operator.

So you can think after flattening the domain, the D-to-N is like a $-(-\Delta)^{1/2}$ that depends in a nonlinear fashion on f.

Analysis of Hele-Shaw

There is a more direct, but less obvious way to proceed. (Thanks to Hector!)

Level-set formulation

First, let's remind ourselves of the level-set interpretation of Hele-Shaw flow
Let $\Omega(t) \subset \mathbb{R}^N$ be a generic set which is said to have a boundary motion dictated by

$$\text{normal velocity}(x) = V(x)n(x) \quad \text{on} \quad \partial\Omega(t),$$

where $n(x)$ is the outward normal to $\Omega(t)$ at x, and $V(x)$ is a scalar.
Assume Φ is some function so that

$$\Omega(t) = \{\Phi(\cdot, t) > 0\} \quad \text{and} \quad \partial\Omega(t) = \{\Phi(\cdot, t) = 0\}.$$

so

$$n(x) = \frac{-\nabla\Phi(x, t)}{|\nabla\Phi|}.$$

Level-set formulation

Assume that $\gamma : (0,1) \to \mathbb{R}^N$ such that $\forall\ t,\ \gamma(t) \in \partial\Omega(t)$. Hence

$$0 = \partial_t (\Phi \circ \gamma) = \partial_t \Phi + \nabla \Phi \cdot \dot{\gamma},$$

and since we are assuming that $(\dot{\gamma})_n = V$,

$$0 = \partial_t \Phi + (-n(x) |\nabla \Phi|) \cdot \dot{\gamma} = \partial_t \Phi - V |\nabla \Phi| ,$$

so

$$\partial_t \Phi = V |\nabla \Phi| .$$

Choices for Φ

Now, let's go back to Hele-Shaw.

The first (most obvious) choice of Φ is $\Phi = U$, which gives

$$\partial_t U_f = (\partial_n U_f)\,|\nabla U_f| \quad \text{on} \quad \partial\{U_f(\cdot, t) > 0\}$$

The drawbacks are that:

- not obviously an equation for f, so no reduction of complexity,
- the domain is a manifold and is not fixed in time!

Choices for Φ

A different choice of Φ is $\Phi = x_{d+1} - f(x,t)$, so that

$$\partial_t f(x,t) = (\partial_n U_f(x, f(x,t)))\sqrt{1 + |\nabla f(x,t)|^2} \quad \text{on} \quad \mathbb{R}^d \times (0, T].$$

This is already an improvement, but still not perfect

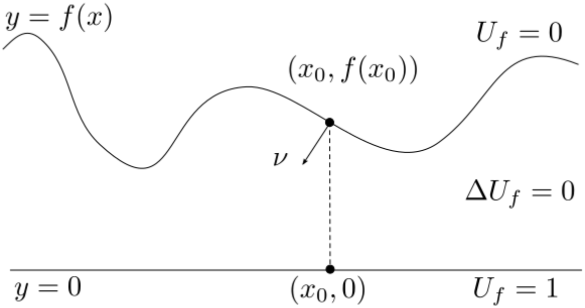
- CLOSER to an equation on only f
- takes place on the nice, fixed domain, $\mathbb{R}^d$, so a reduction of variables!

Hele-Shaw as a parabolic integro-differential equation

Again, **why** is this related to the **integro-differential equations**?!?!?

Focus on the following map (thanks, Hector!)

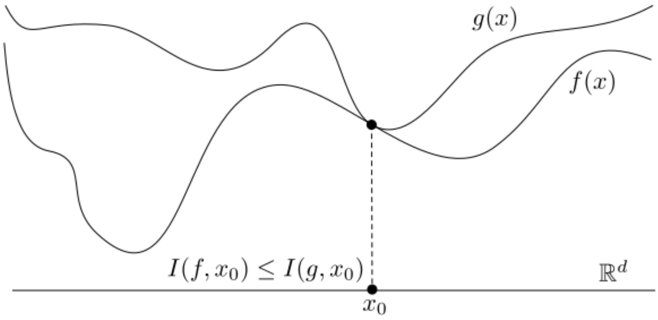
$$f \mapsto \partial_n U_f, \quad I(f,x) := \partial_n U_f(x, f(x)).$$



Hele-Shaw as a parabolic integro-differential equation

Lemma

The operator $I(f,x) = \partial_n U_f(x, f(x))$ *has the GCP.*



Hele-Shaw as a parabolic integro-differential equation

The other property of $H(f) = I(f)\sqrt{1+|\nabla f|^2}$ we need is the following.

Lemma (Chang Lara - Guillen - Schwab 2019)

For each $\gamma \in (0,1)$, H is Lipschitz continuous as a map from $C_b^{1,\gamma}(\mathbb{R}^d)$ to $C_b^0(\mathbb{R}^d)$.

Hence... using the structure of GCP...

Theorem (Chang Lara - Guillen - Schwab 2019)

There exists a family, a^{ij}, c^{ij}, b^{ij}, $\mu^i j$, so that f is the unique viscosity solution of

$$\partial_t f = \min_i \max_j \; a^{ij} + c^{ij} f(x) + b^{ij} \cdot \nabla f(x) + \int_{\mathbb{R}^d} \delta_h f(x) \mu^{ij}(dh).$$

Alert! min-max depends on the class of functions

place-holder, hopefully return to this later

Regularity????

The previous theorem is only half of the battle!

Theorem (Abedin-Schwab, forthcoming)

Define

$$\mathcal{K}(\delta, m, \rho) = \{f \in C^1 : \delta < f < m, |\nabla f| \leq m \text{ and } \nabla f \text{ is } \rho - Dini\}.$$

H is Lipschitz on $\mathcal{K}$, the previous min-max remains intact, and with b^{ij}, μ^{ij} as above, for R_0 depending on δ, m, ρ

$$\mu^{ij}(dh) = K^{ij}(h)dh, \text{ and } \forall\, |h| \leq R_0, \quad c_1 |h|^{-d-1} \leq K^{ij}(h) \leq c_2 |h|^{-d-1},$$

with

$$\forall r \in (0, R_0), \left| b^{ij} + \int_{B_{R_0} \setminus B_r} hK^{ij}(h)dh \right| \leq C$$

Improvement of regularity

Consequence (via Chang Lara- Davila 2016)

Theorem (Abedin-Schwab, forthcoming)

There exists a universal γ *so that if for all* t, $\partial_t U \in C^0$, $\partial\{U_f(\cdot,t)\} = \text{graph}(f(\cdot,t))$, *and* $f(\cdot,t) \in \mathcal{K}$, *then*

$$\partial\{U(\cdot,t) > 0\} \text{ is a } C^{1,\gamma} \text{ graph.}$$

Key Lemma: strict monotonicty

The main lemma is a strict monotonicity in direction of positive perturbations:

Lemma

If support$(\psi) \subset B_{R_0}$ *and* $|\psi(x)| \leq C\,|x|\,\rho(|x|)$*, then for all* $f \in \mathcal{K}$*,*

$$tc_1 \int_{\mathbb{R}^d} \frac{\psi(y)}{|y|^{d+1}} dy \leq I(f + t\psi, 0) - I(f, 0) \leq tc_2 \int_{\mathbb{R}^d} \frac{\psi(y)}{|y|^{d+1}} dy.$$

place-holder

Clarke differential versus finite dimensional approximation.
Key step in min-max... the MVT. with which sets can you get the MVT?!?!?!

Some questions, next steps

- remove the *global* graph assumption... only locally a graph
- higher regularity
- other operators
- include gravity (see a recent work of Alarard - Meunier - Smets 2019 using water waves techniques)
- non-translation invariant situations
- other free boundary problems?

The End

Thanks!