On the spatial decay of the Boltzmann equation with hard potentials

Haitao WANG (王海涛)

Joint with Yu-Chu Lin and Kung-Chien Wu

Swedish Summer PDEs, KTH, Stockholm
August 26-28, 2019

Outline

Boltzmann equation

Two decompositions

Regularization estimate

Conclusion and nonlinear theory

Outline

Boltzmann equation

Two decompositions

Regularization estimate

Conclusion and nonlinear theory

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*) \right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V, \theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \leq C |\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f)\,, \quad f(0, x, \xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle\xi\rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + K f$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span} \sqrt{\mathcal{M}} \{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)\right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V, \theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \le C\,|\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f)\,, \quad f(0, x, \xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle\xi\rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + Kf$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} [F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)]\, B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V,\theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \le C\, |\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle\xi\rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + Kf$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F,F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)\right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V,\theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \le C\,|\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle\xi\rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + Kf$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)\right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V,\theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \leq C\,|\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle\xi\rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + Kf$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)\right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V,\theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \le C\,|\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle \xi \rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + Kf$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*)\right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V, \theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \leq C\,|\cos\theta|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2}\right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f)\,, \quad f(0, x, \xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle\xi\rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + K f$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Boltzmann with cutoff $(0 < \gamma < 1)$

- $\partial_t F + \xi \cdot \nabla_x F = Q(F, F)$

$$Q(F,F)(\xi) = \int_{\mathbb{R}^3} \int_{\mathbf{S}^2_+} \left[F(\xi')F(\xi'_*) - F(\xi)F(\xi_*) \right] B(|\xi - \xi_*|, \theta) d\Omega d\xi_*,$$

- Angular cutoff: $B(V, \theta) = |V|^\gamma b(\theta)$, $0 < b(\theta) \le C \left|\cos\theta\right|$
- Equilibrium: $\mathcal{M}(\xi) = \frac{1}{(2\pi)^{3/2}} \exp\left(\frac{-|\xi|^2}{2} \right)$.
- Around equilibrium $\mathcal{M}$: $F = \mathcal{M} + \sqrt{\mathcal{M}} f$.
- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f)\,, \quad f(0, x, \xi) = f_0$
- f_0 cpt. supp. in x and polynomial ξ weight $\langle \xi \rangle^\beta$ $(\beta = (3/2)^+)$
- $Lf = -\nu(\xi) f + K f$.
- $\nu(\xi) \sim (1 + |\xi|)^\gamma$, K: integral operator.
- $\mathrm{Ker} L = \mathrm{span}\sqrt{\mathcal{M}}\{1, \xi_i, |\xi|^2\}$: 5 dimensional

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to$ Ker L
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$

$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t(P_0 f) + \partial_x(P_0 \xi_1 P_0 f) = -\partial_x(P_0 \xi_1 P_1 f) = -\partial_x^2\Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$

$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t (P_0 f) + \partial_x (P_0 \xi_1 P_0 f) = -\partial_x (P_0 \xi_1 P_1 f) = -\partial_x^2 \Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$

$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t (P_0 f) + \partial_x (P_0 \xi_1 P_0 f) = -\partial_x (P_0 \xi_1 P_1 f) = -\partial_x^2 \Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$

$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t (P_0 f) + \partial_x (P_0 \xi_1 P_0 f) = -\partial_x (P_0 \xi_1 P_1 f) = -\partial_x^2 \Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$

$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t (P_0 f) + \partial_x (P_0 \xi_1 P_0 f) = -\partial_x (P_0 \xi_1 P_1 f) = -\partial_x^2 \Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$

$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t (P_0 f) + \partial_x (P_0 \xi_1 P_0 f) = -\partial_x (P_0 \xi_1 P_1 f) = -\partial_x^2 \Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t f + \xi_1 \partial_x f = Lf$$

- Micro-Macro decomposition:
  - Macroscopic: P_0 : orthogonal projection $L^2_\xi \to \text{Ker } L$
  - Microscopic: $P_1 = Id - P_0$
  - $P_0 L = L P_0 = 0, \quad P_1 L = L P_1 = L, \quad P_0 P_1 = 0$
- Apply P_0: $\partial_t P_0 f + \partial_x P_0 \xi_1 P_0 f + \partial_x P_0 \xi_1 P_1 f = 0$
- Apply P_1: $\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f = L P_1 f$
- L invertible on Range P_1

$$P_1 f = L^{-1}\Big[\partial_t P_1 f + \partial_x P_1 \xi_1 P_0 f + \partial_x P_1 \xi_1 P_1 f\Big]$$
$$\underset{P_1 f \ll P_0 f}{\sim} L^{-1}\Big[\partial_x P_1 \xi_1 P_0 f\Big]$$

Therefore we obtain the equation for $P_0 f$ (3 dimensional)

$$\partial_t (P_0 f) + \partial_x (P_0 \xi_1 P_0 f) = -\partial_x (P_0 \xi_1 P_1 f) = -\partial_x^2 \Big[P_0 \xi_1 L^{-1} P_1 \xi_1 P_0 f\Big]$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t(P_0 f) + \partial_x(P_0\xi_1 P_0 f) = \partial_x^2\Big[\underbrace{P_0\xi_1(-L)^{-1}P_1\xi_1}_{\text{viscosity matrix}} P_0 f\Big]$$

- $P_0 f$ is 3 dimensional, in terms of the basis, the above equation is a viscous conservation law system.

- Diagonalize the system to give 3 eigenvalues

$$\Big\{\lambda_1 = -\sqrt{5/3},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{5/3}\Big\}$$

$\sqrt{5/3}$ sound speed, 0 is the background velocity. The numerical values are due to our linearization $M = (2\pi)^{-3/2}\exp(-|\xi|^2/2)$. In general

$$\{\lambda_1 = u - c,\ \lambda_2 = u,\ \lambda_3 = u + c\}$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t(P_0 f) + \partial_x(P_0\xi_1 P_0 f) = \partial_x^2\Big[\underbrace{P_0\xi_1(-L)^{-1}P_1\xi_1}_{\text{viscosity matrix}} P_0 f\Big]$$

- $P_0 f$ is 3 dimensional, in terms of the basis, the above equation is a viscous conservation law system.
- Diagonalize the system to give 3 eigenvalues

$$\Big\{\lambda_1 = -\sqrt{5/3},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{5/3}\Big\}$$

$\sqrt{5/3}$ sound speed, 0 is the background velocity. The numerical values are due to our linearization $M = (2\pi)^{-3/2}\exp(-|\xi|^2/2)$. In general

$$\{\lambda_1 = u - c,\ \lambda_2 = u,\ \lambda_3 = u + c\}$$

Heuristic: Chapman-Enskog Expansion

$$\partial_t(P_0 f) + \partial_x(P_0\xi_1 P_0 f) = \partial_x^2\Big[\underbrace{P_0\xi_1(-L)^{-1}P_1\xi_1}_{\text{viscosity matrix}} P_0 f\Big]$$

- $P_0 f$ is 3 dimensional, in terms of the basis, the above equation is a viscous conservation law system.
- Diagonalize the system to give 3 eigenvalues

$$\Big\{\lambda_1 = -\sqrt{5/3},\ \lambda_2 = 0,\ \lambda_3 = \sqrt{5/3}\Big\}$$

$\sqrt{5/3}$ sound speed, 0 is the background velocity. The numerical values are due to our linearization $M = (2\pi)^{-3/2}\exp(-|\xi|^2/2)$. In general

$$\{\lambda_1 = u - c,\ \lambda_2 = u,\ \lambda_3 = u + c\}$$

Outline

Boltzmann equation

Two decompositions

Regularization estimate

Conclusion and nonlinear theory

Long wave-short wave decomposition

$$\begin{cases} \partial_t f + \xi \cdot \nabla_x f = Lf \,, & (t,x,\xi) \in \mathbb{R}^+ \times \mathbb{R}^3 \times \mathbb{R}^3 \,, \\ f(0,x,\xi) = f_0(x,\xi) \end{cases}$$

$$f(t,x,\xi) = \int_{\mathbb{R}^3} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0\,(\eta,\xi)\,d\eta$$

$$f_L(t,x,\xi) = \int_{|\eta|\leq 1} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0\,(\eta,\xi)\,d\eta$$

$$f_S(t,x,\xi) = \int_{|\eta|>1} e^{i\eta x + (-i\xi\cdot\eta + L)t} \hat{f}_0\,(\eta,\xi)\,d\eta$$

Fluid structure $0 \le \gamma < 1$ (Liu-Yu)

Let $\mathbf{c} = \sqrt{5/3}$ be the sound speed associated with normalized global Maxwellian

$$|f_L|_{L^2_\xi} \le C \left[(1+t)^{-2} \left(1 + \frac{(|x| - \mathbf{c}t)^2}{1+t} \right)^{-N} + (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-N} + \mathbf{1}_{\{|x| \le \mathbf{c}t\}} (1+t)^{-3/2} \left(1 + \frac{|x|^2}{1+t} \right)^{-3/2} + e^{-ct} \right] \|f_0\|_{L^1_x L^2_\xi}.$$

$$\|f_S\|_{L^2_{x,\xi}} \lesssim e^{-ct} \|f_0\|_{L^2_{x,\xi}}$$

Singular-Regular decomposition

$$\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f = Kf.$$

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0,x,\xi) = f_0(x,\xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0,x,\xi) = 0 \end{cases}$$

Intuition: $h^{(j)}$ becomes more and more regular.
We can define the singular part and the regular part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L\mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0,x,\xi) = 0 \end{cases}$$

Singular-Regular decomposition

$$\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f = K f.$$

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0,x,\xi) = f_0(x,\xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0,x,\xi) = 0 \end{cases}$$

Intuition: $h^{(j)}$ becomes more and more regular.

We can define the singular part and the regular part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L\mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0,x,\xi) = 0 \end{cases}$$

Singular-Regular decomposition

$$\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f = Kf.$$

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0, x, \xi) = f_0(x, \xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0, x, \xi) = 0 \end{cases}$$

Intuition: $h^{(j)}$ becomes more and more regular.
We can define the singular part and the regular part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L\mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0, x, \xi) = 0 \end{cases}$$

Singular-Regular decomposition

$$\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f = K f.$$

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0, x, \xi) = f_0(x, \xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0, x, \xi) = 0 \end{cases}$$

Intuition: $h^{(j)}$ becomes more and more regular.

We can define the singular part and the regular part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L \mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0, x, \xi) = 0 \end{cases}$$

Singular-Regular decomposition

$$\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f = K f.$$

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0, x, \xi) = f_0(x, \xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0, x, \xi) = 0 \end{cases}$$

Intuition: $h^{(j)}$ becomes more and more regular.
We can define the singular part and the regular part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L\mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0, x, \xi) = 0 \end{cases}$$

Singular-Regular decomposition

$$\partial_t f + \xi \cdot \nabla_x f + \nu(\xi) f = K f.$$

$$\begin{cases} \partial_t h^{(0)} + \xi \cdot \nabla_x h^{(0)} + \nu(\xi) h^{(0)} = 0\,, \\ h^{(0)}(0,x,\xi) = f_0(x,\xi) \end{cases}$$

The j^{th} order approximation $h^{(j)}$, $j \geq 1$

$$\begin{cases} \partial_t h^{(j)} + \xi \cdot \nabla_x h^{(j)} + \nu(\xi) h^{(j)} = K h^{(j-1)}\,, \\ h^{(j)}(0,x,\xi) = 0 \end{cases}$$

Intuition: $h^{(j)}$ becomes more and more regular.
We can define the singular part and the regular part:

$$W^{(9)} = \sum_{j=0}^{9} h^{(j)}\,, \quad \mathcal{R}^{(9)} = f - W^{(9)}$$

$$\begin{cases} \partial_t \mathcal{R}^{(9)} + \xi \cdot \nabla_x \mathcal{R}^{(9)} = L\mathcal{R}^{(9)} + K h^{(9)}\,, \\ \mathcal{R}^{(9)}(0,x,\xi) = 0 \end{cases}$$

Singular-Regular decomposition

- Damped transport operator: $h(t) = \mathbb{S}^t h_0$
$\partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0$, with $h(0, x, \xi) = h_0$

Let $\alpha \geq 0$. Then for $0 \leq \gamma < 1$ and any positive number σ,

$$\left| \mathbb{S}^t \langle \xi \rangle^{-\alpha} h_0 \right|_{L^\infty_\xi} \leq C \sup_y e^{-\frac{\nu_0}{3} t} (1 + |x-y|)^{-\frac{\alpha}{1-\gamma}} \frac{t^{-\sigma}}{t^{-\sigma} + (1 + |x-y|)^{\frac{\sigma\gamma}{1-\gamma}}} |h_0(y, \cdot)|_{L^\infty_\xi} .$$

Singular-Regular decomposition

- Damped transport operator: $h(t) = \mathbb{S}^t h_0$
$\partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0$, with $h(0,x,\xi) = h_0$

Let $\alpha \geq 0$. Then for $0 \leq \gamma < 1$ and any positive number σ,

$$\left| \mathbb{S}^t \langle \xi \rangle^{-\alpha} h_0 \right|_{L^\infty_\xi} \leq C \sup_y e^{-\frac{\nu_0}{3} t} \left(1 + |x-y|\right)^{-\frac{\alpha}{1-\gamma}} \frac{t^{-\sigma}}{t^{-\sigma} + \left(1+|x-y|\right)^{\frac{\sigma\gamma}{1-\gamma}}} \left| h_0(y,\cdot) \right|_{L^\infty_\xi} .$$

Singular-Regular decomposition

- Damped transport operator: $h(t) = \mathbb{S}^t h_0$
$\partial_t h + \xi \cdot \nabla_x h + \nu(\xi) h = 0$, with $h(0, x, \xi) = h_0$

Let $\alpha \geq 0$. Then for $0 \leq \gamma < 1$ and any positive number σ,

$$\left| \mathbb{S}^t \langle \xi \rangle^{-\alpha} h_0 \right|_{L^\infty_\xi} \leq C \sup_y e^{-\frac{\nu_0}{3} t} (1 + |x-y|)^{-\frac{\alpha}{1-\gamma}} \frac{t^{-\sigma}}{t^{-\sigma} + (1 + |x-y|)^{\frac{\sigma\gamma}{1-\gamma}}} |h_0(y, \cdot)|_{L^\infty_\xi} .$$

Singular-Regular decomposition

- $0 < \gamma < 1$, with polynomial weight $\langle \xi \rangle^{\alpha}$

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \leq C \sup_y e^{-\frac{\nu_0}{3}t} \left(1 + |x-y|\right)^{-\frac{\alpha}{1-\gamma}}$$

$$\frac{t^{-\sigma}}{t^{-\sigma} + \left(1+|x-y|\right)^{\frac{\sigma\gamma}{1-\gamma}}} \left|h_0(y,\cdot)\right|_{L^\infty_\xi(\langle\xi\rangle^\alpha)} .$$

- $0 < \gamma < 1$, with exponential weight $e^{\alpha|\xi|^p}$ (Lin-Wang-Wu, JSP 2018)

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \leq \sup_y e^{-c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x-y|^{\frac{p}{p+1-\gamma}}\right)} \left|h_0(y,\cdot)\right|_{L^\infty_\xi\left(e^{\alpha|\xi|^p}\right)}$$

- $\gamma = 1$, (Liu-Yu, CPAM 2004)

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \leq \sup_y e^{-c_0(t+|x-y|)} \left|h_0(y,\cdot)\right|_{L^\infty_\xi}$$

Singular-Regular decomposition

- $0 < \gamma < 1$, with polynomial weight $\langle \xi \rangle^{\alpha}$

$$\left| \mathbb{S}^t h_0 \right|_{L^\infty_\xi} \leq C \sup_y e^{-\frac{\nu_0}{3} t} \left(1 + |x-y|\right)^{-\frac{\alpha}{1-\gamma}} \frac{t^{-\sigma}}{t^{-\sigma} + \left(1 + |x-y|\right)^{\frac{\sigma\gamma}{1-\gamma}}} \left| h_0 (y, \cdot) \right|_{L^\infty_\xi (\langle \xi \rangle^{\alpha})} .$$

- $0 < \gamma < 1$, with exponential weight $e^{\alpha|\xi|^p}$ (Lin-Wang-Wu, JSP 2018)

$$\left| \mathbb{S}^t h_0 \right|_{L^\infty_\xi} \leq \sup_y e^{-c_0 \left(t + \alpha^{\frac{1-\gamma}{p+1-\gamma}} |x-y|^{\frac{p}{p+1-\gamma}} \right)} \left| h_0 (y, \cdot) \right|_{L^\infty_\xi \left(e^{\alpha|\xi|^p} \right)}$$

- $\gamma = 1$, (Liu-Yu, CPAM 2004)

$$\left| \mathbb{S}^t h_0 \right|_{L^\infty_\xi} \leq \sup_y e^{-c_0 (t + |x-y|)} \left| h_0 (y, \cdot) \right|_{L^\infty_\xi}$$

Singular-Regular decomposition

- $0 < \gamma < 1$, with polynomial weight $\langle\xi\rangle^{\alpha}$

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \le C \sup_y e^{-\frac{\nu_0}{3}t}\left(1+|x-y|\right)^{-\frac{\alpha}{1-\gamma}} \frac{t^{-\sigma}}{t^{-\sigma}+\left(1+|x-y|\right)^{\frac{\sigma\gamma}{1-\gamma}}}\left|h_0(y,\cdot)\right|_{L^\infty_\xi(\langle\xi\rangle^\alpha)}\,.$$

- $0 < \gamma < 1$, with exponential weight $e^{\alpha|\xi|^p}$ (Lin-Wang-Wu, JSP 2018)

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \le \sup_y e^{-c_0\left(t+\alpha^{\frac{1-\gamma}{p+1-\gamma}}|x-y|^{\frac{p}{p+1-\gamma}}\right)}\left|h_0(y,\cdot)\right|_{L^\infty_\xi\left(e^{\alpha|\xi|^p}\right)}$$

- $\gamma = 1$, (Liu-Yu, CPAM 2004)

$$\left|\mathbb{S}^t h_0\right|_{L^\infty_\xi} \le \sup_y e^{-c_0(t+|x-y|)}\left|h_0(y,\cdot)\right|_{L^\infty_\xi}$$

Singular-Regular decomposition

- Let $j \geq 1$, $\beta = (3/2)^+$, $0 \leq \gamma < 1$,

$$\left|h^{(j)}\right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{\gamma+\alpha_j+\beta_j}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$$

where $\alpha_j = \frac{j(2-\gamma)}{j+1}$, $\beta_j = \frac{\beta}{j+1}$.

- Key fact: $|Kg(\xi)| \leq \langle \xi \rangle^{-(2-\gamma)} |g|_{L^\infty_\xi}$.
- Note: $\gamma + \alpha_j + \beta_j > 3/2$.
- Why $W^{(9)}$?
- How to estimate $\mathcal{R}^{(9)}$

Singular-Regular decomposition

- Let $j \geq 1$, $\beta = (3/2)^+$, $0 \leq \gamma < 1$,

$$\left| h^{(j)} \right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{\gamma+\alpha_j+\beta_j}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$$

where $\alpha_j = \frac{j(2-\gamma)}{j+1}$, $\beta_j = \frac{\beta}{j+1}$.

- Key fact: $|Kg(\xi)| \leq \langle \xi \rangle^{-(2-\gamma)} |g|_{L^\infty_\xi}$.

- Note: $\gamma + \alpha_j + \beta_j > 3/2$.

- Why $W^{(9)}$?
- How to estimate $\mathcal{R}^{(9)}$

Singular-Regular decomposition

- Let $j \geq 1$, $\beta = (3/2)^+$, $0 \leq \gamma < 1$,

$$\left| h^{(j)} \right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{\gamma+\alpha_j+\beta_j}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$$

where $\alpha_j = \frac{j(2-\gamma)}{j+1}$, $\beta_j = \frac{\beta}{j+1}$.

- Key fact: $|Kg(\xi)| \leq \langle \xi \rangle^{-(2-\gamma)} |g|_{L^\infty_\xi}$.

- Note: $\gamma + \alpha_j + \beta_j > 3/2$.

- Why $W^{(9)}$?
- How to estimate $\mathcal{R}^{(9)}$

Singular-Regular decomposition

- Let $j \geq 1$, $\beta = (3/2)^+$, $0 \leq \gamma < 1$,

$$\left| h^{(j)} \right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{\gamma+\alpha_j+\beta_j}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$$

where $\alpha_j = \frac{j(2-\gamma)}{j+1}$, $\beta_j = \frac{\beta}{j+1}$.

- Key fact: $|Kg(\xi)| \leq \langle \xi \rangle^{-(2-\gamma)} |g|_{L^\infty_\xi}$.
- Note: $\gamma + \alpha_j + \beta_j > 3/2$.
- Why $W^{(9)}$?
- How to estimate $\mathcal{R}^{(9)}$

Singular-Regular decomposition

- Let $j \geq 1$, $\beta = (3/2)^+$, $0 \leq \gamma < 1$,

$$\left| h^{(j)} \right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{\gamma+\alpha_j+\beta_j}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$$

where $\alpha_j = \frac{j(2-\gamma)}{j+1}$, $\beta_j = \frac{\beta}{j+1}$.

- Key fact: $|Kg(\xi)| \leq \langle \xi \rangle^{-(2-\gamma)} |g|_{L^\infty_\xi}$.

- Note: $\gamma + \alpha_j + \beta_j > 3/2$.

- Why $W^{(9)}$?
- How to estimate $\mathcal{R}^{(9)}$

Time-like region: $|x| < Mt$

- $f = \underbrace{f_L}_{\text{long wave}} + \underbrace{f_S}_{\text{short wave}} = \underbrace{W^{(9)}}_{\text{singular}} + \underbrace{\mathcal{R}^{(9)}}_{\text{regular}}$
- $f = f_L + W^{(9)} + f_R$
- Tail part $f_R = \mathcal{R}^{(9)} - f_L = f_S - W^{(9)}$
- $\|f_R\|_{L^2} = \|f_S - W^{(9)}\|_{L^2} \le e^{-Ct}\|f_0\|_{L^2}$ $(0 \le \gamma < 1)$
- $\|\nabla_x^2 \mathcal{R}^{(9)}\|_{L^2} \le \int_0^t \left\|\mathbb{G}^{t-s}\nabla_x^2 h^{(9)}(s)\right\|_{L^2} ds$
- Goal: $\|\nabla_x^2 h^{(9)}\|_{L^2}$

Time-like region: $|x| < Mt$

- $f = \underbrace{f_L}_{\text{long wave}} + \underbrace{f_S}_{\text{short wave}} = \underbrace{W^{(9)}}_{\text{singular}} + \underbrace{\mathcal{R}^{(9)}}_{\text{regular}}$
- $f = f_L + W^{(9)} + f_R$
- Tail part $f_R = \mathcal{R}^{(9)} - f_L = f_S - W^{(9)}$
- $\|f_R\|_{L^2} = \|f_S - W^{(9)}\|_{L^2} \le e^{-Ct}\|f_0\|_{L^2}$ $(0 \le \gamma < 1)$
- $\|\nabla_x^2 \mathcal{R}^{(9)}\|_{L^2} \le \int_0^t \left\|\mathbb{G}^{t-s}\nabla_x^2 h^{(9)}(s)\right\|_{L^2} ds$
- Goal: $\|\nabla_x^2 h^{(9)}\|_{L^2}$

Time-like region: $|x| < Mt$

- $f = \underbrace{f_L}_{\text{long wave}} + \underbrace{f_S}_{\text{short wave}} = \underbrace{W^{(9)}}_{\text{singular}} + \underbrace{\mathcal{R}^{(9)}}_{\text{regular}}$

- $f = f_L + W^{(9)} + f_R$

- Tail part $f_R = \mathcal{R}^{(9)} - f_L = f_S - W^{(9)}$

- $\|f_R\|_{L^2} = \|f_S - W^{(9)}\|_{L^2} \le e^{-Ct}\|f_0\|_{L^2}$ $(0 \le \gamma < 1)$

- $\|\nabla_x^2 \mathcal{R}^{(9)}\|_{L^2} \le \int_0^t \left\|\mathbb{G}^{t-s}\nabla_x^2 h^{(9)}(s)\right\|_{L^2} ds$

- Goal: $\|\nabla_x^2 h^{(9)}\|_{L^2}$

Time-like region: $|x| < Mt$

- $f = \underbrace{f_L}_{\text{long wave}} + \underbrace{f_S}_{\text{short wave}} = \underbrace{W^{(9)}}_{\text{singular}} + \underbrace{\mathcal{R}^{(9)}}_{\text{regular}}$
- $f = f_L + W^{(9)} + f_R$
- Tail part $f_R = \mathcal{R}^{(9)} - f_L = f_S - W^{(9)}$
- $\|f_R\|_{L^2} = \|f_S - W^{(9)}\|_{L^2} \le e^{-Ct}\|f_0\|_{L^2}$ $(0 \le \gamma < 1)$
- $\|\nabla_x^2 \mathcal{R}^{(9)}\|_{L^2} \le \int_0^t \left\|\mathbb{G}^{t-s}\nabla_x^2 h^{(9)}(s)\right\|_{L^2} ds$
- Goal: $\|\nabla_x^2 h^{(9)}\|_{L^2}$

Time-like region: $|x| < Mt$

- $f = \underbrace{f_L}_{\text{long wave}} + \underbrace{f_S}_{\text{short wave}} = \underbrace{W^{(9)}}_{\text{singular}} + \underbrace{\mathcal{R}^{(9)}}_{\text{regular}}$
- $f = f_L + W^{(9)} + f_R$
- Tail part $f_R = \mathcal{R}^{(9)} - f_L = f_S - W^{(9)}$
- $\|f_R\|_{L^2} = \|f_S - W^{(9)}\|_{L^2} \leq e^{-Ct}\|f_0\|_{L^2}$ $(0 \leq \gamma < 1)$
- $\|\nabla_x^2 \mathcal{R}^{(9)}\|_{L^2} \leq \int_0^t \left\|\mathbb{G}^{t-s}\nabla_x^2 h^{(9)}(s)\right\|_{L^2} ds$
- Goal: $\|\nabla_x^2 h^{(9)}\|_{L^2}$

Time-like region: $|x| < Mt$

- $f = \underbrace{f_L}_{\text{long wave}} + \underbrace{f_S}_{\text{short wave}} = \underbrace{W^{(9)}}_{\text{singular}} + \underbrace{\mathcal{R}^{(9)}}_{\text{regular}}$
- $f = f_L + W^{(9)} + f_R$
- Tail part $f_R = \mathcal{R}^{(9)} - f_L = f_S - W^{(9)}$
- $\|f_R\|_{L^2} = \|f_S - W^{(9)}\|_{L^2} \leq e^{-Ct}\|f_0\|_{L^2}$ $(0 \leq \gamma < 1)$
- $\|\nabla_x^2 \mathcal{R}^{(9)}\|_{L^2} \leq \int_0^t \left\|\mathbb{G}^{t-s}\nabla_x^2 h^{(9)}(s)\right\|_{L^2} ds$
- Goal: $\|\nabla_x^2 h^{(9)}\|_{L^2}$

Space-like region: $|x| > Mt$

- $f = W^{(9)} + \mathcal{R}^{(9)}$, we have accurate description of $W^{(9)}$.
- (Weighted energy estimate for $\mathcal{R}^{(9)}$, $0 < \gamma < 1$)

$$w(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{2}{1-\gamma}}(1-\chi) + \left[(1-\chi)\left[\delta\left(\langle x\rangle - Mt\right)\right]\langle\xi\rangle_D^{\gamma+1} + 3\langle\xi\rangle_D^2\right]\chi.$$

where $\langle\xi\rangle_D = (D + |\xi|^2)^{1/2}, \chi = \chi\left(\frac{\delta(\langle x\rangle - Mt)}{\langle\xi\rangle_D^{1-\gamma}}\right)$ and $\mu(x,\xi) = w(0,x,\xi)$.

Let $u = w\mathcal{R}^{(9)}$, then u solves the equation

$$\partial_t u + \xi\cdot\nabla_x u - (\partial_t w + \xi\cdot\nabla_x w)w^{-1}u - wL\left(w^{-1}u\right) = wKh^{(9)}.$$

Space-like region: $|x| > Mt$

- $f = W^{(9)} + \mathcal{R}^{(9)}$, we have accurate description of $W^{(9)}$.
- (Weighted energy estimate for $\mathcal{R}^{(9)}$, $0 < \gamma < 1$)

$$w(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{2}{1-\gamma}}(1-\chi) + \left[(1-\chi)\left[\delta\left(\langle x\rangle - Mt\right)\right]\langle \xi\rangle_D^{\gamma+1} + 3\langle \xi\rangle_D^2\right]\chi.$$

where $\langle \xi\rangle_D = (D+|\xi|^2)^{1/2}, \chi = \chi\left(\frac{\delta(\langle x\rangle - Mt)}{\langle \xi\rangle_D^{1-\gamma}}\right)$ and $\mu(x,\xi) = w(0,x,\xi)$.

Let $u = w\mathcal{R}^{(9)}$, then u solves the equation

$$\partial_t u + \xi\cdot\nabla_x u - (\partial_t w + \xi\cdot\nabla_x w)w^{-1}u - wL\left(w^{-1}u\right) = wKh^{(9)}.$$

Space-like region: $|x| > Mt$

- $f = W^{(9)} + \mathcal{R}^{(9)}$, we have accurate description of $W^{(9)}$.
- (Weighted energy estimate for $\mathcal{R}^{(9)}$, $0 < \gamma < 1$)

$$w(t,x,\xi) = 5\left(\delta\left(\langle x\rangle - Mt\right)\right)^{\frac{2}{1-\gamma}}(1-\chi) + \left[(1-\chi)\left[\delta\left(\langle x\rangle - Mt\right)\right]\langle\xi\rangle_D^{\gamma+1} + 3\langle\xi\rangle_D^2\right]\chi.$$

where $\langle\xi\rangle_D = (D + |\xi|^2)^{1/2}, \chi = \chi\left(\frac{\delta(\langle x\rangle - Mt)}{\langle\xi\rangle_D^{1-\gamma}}\right)$ and $\mu(x,\xi) = w(0,x,\xi)$.

Let $u = w\mathcal{R}^{(9)}$, then u solves the equation

$$\partial_t u + \xi\cdot\nabla_x u - (\partial_t w + \xi\cdot\nabla_x w)w^{-1}u - wL\left(w^{-1}u\right) = wKh^{(9)}.$$

Space-like region: $|x| > Mt$

Define $wLw^{-1} = L_w$ and
$H_+ = \{(x,\xi) : \delta(\langle x\rangle - Mt) > 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_0 = \{(x,\xi) : \langle\xi\rangle_D^{1-\gamma} \le \delta(\langle x\rangle - Mt) \le 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_- = \{(x,\xi) : \delta(\langle x\rangle - Mt) < \langle\xi\rangle_D^{1-\gamma}\}$.

- $\int \langle u, L_w u\rangle_\xi \, dx \le \int \langle u, Lu\rangle_\xi \, dx + CD^{-2} \int \langle\xi\rangle^\gamma |\mathrm{P}_1 u|^2 \, d\xi dx$
$+CD^{-\frac{3}{2}-\frac{\gamma}{2}} \left[\int_{H_+} [\delta(\langle x\rangle - Mt)]^{-1} |\mathrm{P}_0 u|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 u|^2 \, d\xi dx\right].$

- The red part can be controlled by $\int_{\mathbb{R}^3} \left\langle u, (\partial_t w) w^{-1} u\right\rangle_\xi dx$.

- $\frac{d}{dt}\|u\|^2_{H^2_x L^2_\xi} \lesssim \|u\|_{H^2_x L^2_\xi} \|wKh^{(9)}\|_{H^2_x L^2_\xi} + \|\mathcal{R}^{(9)}\|^2_{H^2_x L^2_\xi}$

- Goal: Estimate $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)}$

Space-like region: $|x| > Mt$

Define $wLw^{-1} = L_w$ and
$H_+ = \{(x,\xi) : \delta(\langle x\rangle - Mt) > 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_0 = \{(x,\xi) : \langle\xi\rangle_D^{1-\gamma} \le \delta(\langle x\rangle - Mt) \le 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_- = \{(x,\xi) : \delta(\langle x\rangle - Mt) < \langle\xi\rangle_D^{1-\gamma}\}$.

- $\int \langle u, L_w u\rangle_\xi \, dx \le \int \langle u, Lu\rangle_\xi \, dx + CD^{-2}\int \langle\xi\rangle^\gamma |\mathrm{P}_1 u|^2 \, d\xi dx$
$+CD^{-\frac{3}{2}-\frac{\gamma}{2}}\left[\int_{H_+} [\delta(\langle x\rangle - Mt)]^{-1} |\mathrm{P}_0 u|^2 \, d\xi dx + \int_{H_0\cup H_-} |\mathrm{P}_0 u|^2 \, d\xi dx\right].$

- The red part can be controlled by $\int_{\mathbb{R}^3} \langle u, (\partial_t w)w^{-1}u\rangle_\xi \, dx$.

- $\frac{d}{dt}\|u\|^2_{H^2_x L^2_\xi} \lesssim \|u\|_{H^2_x L^2_\xi}\|wKh^{(9)}\|_{H^2_x L^2_\xi} + \|\mathcal{R}^{(9)}\|^2_{H^2_x L^2_\xi}$

- Goal: Estimate $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)}$

Space-like region: $|x| > Mt$

Define $wLw^{-1} = L_w$ and
$H_+ = \{(x,\xi) : \delta(\langle x\rangle - Mt) > 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_0 = \{(x,\xi) : \langle\xi\rangle_D^{1-\gamma} \le \delta(\langle x\rangle - Mt) \le 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_- = \{(x,\xi) : \delta(\langle x\rangle - Mt) < \langle\xi\rangle_D^{1-\gamma}\}$.

- $\int \langle u, L_w u\rangle_\xi \, dx \le \int \langle u, Lu\rangle_\xi \, dx + CD^{-2} \int \langle\xi\rangle^\gamma |\mathrm{P}_1 u|^2 \, d\xi dx$
$+CD^{-\frac{3}{2}-\frac{\gamma}{2}} \left[\int_{H_+} [\delta(\langle x\rangle - Mt)]^{-1} |\mathrm{P}_0 u|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 u|^2 \, d\xi dx\right]$.

- The red part can be controlled by $\int_{\mathbb{R}^3} \left\langle u, (\partial_t w) w^{-1} u\right\rangle_\xi dx$.

- $\frac{d}{dt}\|u\|^2_{H^2_x L^2_\xi} \lesssim \|u\|_{H^2_x L^2_\xi} \|wKh^{(9)}\|_{H^2_x L^2_\xi} + \|\mathcal{R}^{(9)}\|^2_{H^2_x L^2_\xi}$

- Goal: Estimate $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)}$

Space-like region: $|x| > Mt$

Define $wLw^{-1} = L_w$ and
$H_+ = \{(x,\xi) : \delta(\langle x\rangle - Mt) > 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_0 = \{(x,\xi) : \langle\xi\rangle_D^{1-\gamma} \le \delta(\langle x\rangle - Mt) \le 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_- = \{(x,\xi) : \delta(\langle x\rangle - Mt) < \langle\xi\rangle_D^{1-\gamma}\}$.

- $\int \langle u, L_w u\rangle_\xi \, dx \le \int \langle u, Lu\rangle_\xi \, dx + CD^{-2}\int \langle\xi\rangle^\gamma |\mathrm{P}_1 u|^2 \, d\xi dx$
$+CD^{-\frac{3}{2}-\frac{\gamma}{2}}\left[\int_{H_+} [\delta(\langle x\rangle - Mt)]^{-1} |\mathrm{P}_0 u|^2 \, d\xi dx + \int_{H_0\cup H_-} |\mathrm{P}_0 u|^2 \, d\xi dx\right]$.

- The red part can be controlled by $\int_{\mathbb{R}^3} \left\langle u, (\partial_t w) w^{-1} u\right\rangle_\xi dx$.

- $\frac{d}{dt}\|u\|^2_{H^2_x L^2_\xi} \lesssim \|u\|_{H^2_x L^2_\xi}\|wKh^{(9)}\|_{H^2_x L^2_\xi} + \|\mathcal{R}^{(9)}\|^2_{H^2_x L^2_\xi}$

- Goal: Estimate $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)}$

Space-like region: $|x| > Mt$

Define $wLw^{-1} = L_w$ and
$H_+ = \{(x,\xi) : \delta(\langle x\rangle - Mt) > 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_0 = \{(x,\xi) : \langle\xi\rangle_D^{1-\gamma} \le \delta(\langle x\rangle - Mt) \le 2\langle\xi\rangle_D^{1-\gamma}\}$,
$H_- = \{(x,\xi) : \delta(\langle x\rangle - Mt) < \langle\xi\rangle_D^{1-\gamma}\}$.

- $\int \langle u, L_w u\rangle_\xi \, dx \le \int \langle u, Lu\rangle_\xi \, dx + CD^{-2} \int \langle\xi\rangle^\gamma |\mathrm{P}_1 u|^2 \, d\xi dx$
$+CD^{-\frac{3}{2}-\frac{\gamma}{2}} \left[\int_{H_+} [\delta(\langle x\rangle - Mt)]^{-1} |\mathrm{P}_0 u|^2 \, d\xi dx + \int_{H_0 \cup H_-} |\mathrm{P}_0 u|^2 \, d\xi dx \right]$.

- The red part can be controlled by $\int_{\mathbb{R}^3} \left\langle u, (\partial_t w) w^{-1} u \right\rangle_\xi dx$.

- $\frac{d}{dt}\|u\|^2_{H^2_x L^2_\xi} \lesssim \|u\|_{H^2_x L^2_\xi} \|wKh^{(9)}\|_{H^2_x L^2_\xi} + \|\mathcal{R}^{(9)}\|^2_{H^2_x L^2_\xi}$

- Goal: Estimate $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)}$

Outline

Boltzmann equation

Two decompositions

Regularization estimate

Conclusion and nonlinear theory

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t] \times [0,s_1] \times [0,s_2] \times [0,s_3] \times [0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} K g(\cdot, s_5) dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim t e^{-c_0 t} \|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t} \left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2} \right)$
- Gualdani, Mischler and Mouhot.

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t] \times [0,s_1] \times [0,s_2] \times [0,s_3] \times [0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} K g(\cdot, s_5) dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim t e^{-c_0 t} \|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t} \left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2} \right)$
- Gualdani, Mischler and Mouhot.

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t] \times [0,s_1] \times [0,s_2] \times [0,s_3] \times [0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} K g(\cdot, s_5) dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim te^{-c_0 t} \|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t} \left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2} \right)$
- Gualdani, Mischler and Mouhot.

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t]\times[0,s_1]\times[0,s_2]\times[0,s_3]\times[0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} Kg(\cdot, s_5)\, dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi\cdot\nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim te^{-c_0 t}\|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t}\left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2}\right)$
- Gualdani, Mischler and Mouhot.

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t]\times[0,s_1]\times[0,s_2]\times[0,s_3]\times[0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} Kg(\cdot, s_5)\, dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi\cdot\nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim t e^{-c_0 t}\|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t}\left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2}\right)$
- Gualdani, Mischler and Mouhot.

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t] \times [0,s_1] \times [0,s_2] \times [0,s_3] \times [0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} K g(\cdot, s_5) dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim te^{-c_0 t} \|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t} \left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2} \right)$
- Gualdani, Mischler and Mouhot.

Regularization: Estimate of $Kh^{(9)}$

Mixture operator: $\mathbb{T}_4 = [0,t] \times [0,s_1] \times [0,s_2] \times [0,s_3] \times [0,s_4]$
$dS_4 = ds_1 ds_2 ds_3 ds_4 ds_5$.

$$\mathbb{M}^t g = \int_{\mathbb{T}_4} K\mathbb{S}^{t-s_1} K\mathbb{S}^{s_1-s_2} K\mathbb{S}^{s_2-s_3} K\mathbb{S}^{s_3-s_4} K\mathbb{S}^{s_4-s_5} K g(\cdot, s_5) dS_4 \,.$$

- $Kh^{(9)} = \mathbb{M}^t h^{(4)}$.
- Key observation: Good properties of K.
- Key observation: $\mathcal{D}_t = t\nabla_x + \nabla_\xi$, we have $[\mathcal{D}_t, \partial_t + \xi \cdot \nabla_x] = 0$
- Key observation: $\|\mathcal{D}_t \mathbb{S}^t g_0\|_{L^2(\mu)} \lesssim t e^{-c_0 t} \|g_0\|_{L^2_x H^1_\xi(\mu)}$
- $\mathbb{M}^t$:regularization. $h^{(4)}$:cancel the weight μ.
- $\|Kh^{(9)}\|_{H^2_x L^2_\xi(\mu)} \lesssim t^7 (1+t)^2 e^{-c_0 t} \left(\|f_0\|_{L^\infty_\xi L^2_x} + \|f_0\|_{L^2} \right)$
- Gualdani, Mischler and Mouhot.

Outline

Boltzmann equation

Two decompositions

Regularization estimate

Conclusion and nonlinear theory

Conclusion

Theorem

Let f be a solution to the linearized Boltzmann equation with initial data compactly supported in the x-variable and bounded in $L^\infty_{\xi,\beta}$ space,

$\beta = (3/2)^+$, *then* $\left|W^{(9)}\right|_{L^\infty_\xi} \leq e^{-c_0 t} \langle x \rangle^{-\frac{3/2}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}$.

(a) $|x| < Mt$

$$\left|\mathcal{R}^{(9)}\right|_{L^2_\xi} \leq C_N \left[\begin{array}{l} (1+t)^{-2}\left(1+\frac{(|x|-\mathbf{c}t)^2}{1+t}\right)^{-N} \\ +(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ +\mathbf{1}_{\{|x|\leq \mathbf{c}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2} \end{array}\right] \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}} .$$

(b) $|x| > Mt$

$$\left|\mathcal{R}^{(9)}\right|_{L^2_\xi} \leq (1+t)^{-N/4}(t+|x|)^{-\frac{3/2}{1-\gamma}} \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}} .$$

Conclusion

Bootstrap from L^2_ξ to L^∞_ξ:

$$f = \mathbb{S}^t f_0 + \int_0^t \mathbb{S}^{t-s} K(W^{(9)} + \mathcal{R}^{(9)})(s)ds = W^{(10)} + \int_0^t \mathbb{S}^{t-s} K\mathcal{R}^{(9)}(s)ds\,.$$

Theorem (L^∞_ξ estimate)

Let f be a solution to the linearized Boltzmann equation with initial data compactly supported in the x-variable and bounded in $L^\infty_{\xi,\beta}$ space, $\beta = (3/2)^+$, then

$$|f|_{L^\infty_\xi} \le C_N \left[\begin{array}{l} (1+t)^{-2}\left(1+\frac{(|x|-\mathbf{c}t)^2}{1+t}\right)^{-\frac{3/2}{1-\gamma}} \\ +(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ +\mathbf{1}_{\{|x|\le \mathbf{c}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2} \\ +(1+t)^{-3/2}(t+|x|)^{-\frac{3/2}{1-\gamma}} \end{array} \right] \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}\,.$$

Conclusion

Bootstrap from L^2_ξ to L^∞_ξ:

$$f = \mathbb{S}^t f_0 + \int_0^t \mathbb{S}^{t-s} K(W^{(9)} + \mathcal{R}^{(9)})(s)ds = W^{(10)} + \int_0^t \mathbb{S}^{t-s} K\mathcal{R}^{(9)}(s)ds\,.$$

Theorem (L^∞_ξ estimate)

Let f be a solution to the linearized Boltzmann equation with initial data compactly supported in the x-variable and bounded in $L^\infty_{\xi,\beta}$ space, $\beta = (3/2)^+$, then

$$|f|_{L^\infty_\xi} \le C_N \left[\begin{array}{l} (1+t)^{-2}\left(1+\frac{(|x|-\mathbf{c}t)^2}{1+t}\right)^{-\frac{3/2}{1-\gamma}} \\ +(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-N} \\ +\mathbf{1}_{\{|x|\le \mathbf{c}t\}}(1+t)^{-3/2}\left(1+\frac{|x|^2}{1+t}\right)^{-3/2} \\ +(1+t)^{-3/2}(t+|x|)^{-\frac{3/2}{1-\gamma}} \end{array} \right] \|f_0\|_{L^\infty_x L^\infty_{\xi,\beta}}\,.$$

Nonlinear theorem

- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f, f)\,, \quad f(0, x, \xi) = f_0$ small
- $f_0 \in H^N_x L^2_\xi(\langle\xi\rangle^p)$, $N \geq 4$, $p \geq 2$.
- Existence, uniqueness and optimal time decay:

Ukai, Kawashima, Y. Guo, Strain, T. Yang, R-J. Duan, etc.

Define
$\mathcal{E}(f)(t) = \sum_{|\alpha|\leq N} \|\partial^\alpha_x f\|^2_{L^2} + \sum_{|\alpha|\leq N} \|\,\langle\xi\rangle^p\,\partial^\alpha_x f\|^2_{L^2}\,.$
$\mathcal{D}(f)(t) = \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha|\leq N} \|\,\langle\xi\rangle^p\,\partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2_\sigma}\,.$

If we assume $\mathcal{E}(f)(0) \leq \eta$, then

- $\mathcal{E}(f)(t) + \int_0^t \mathcal{D}(f)(s)ds \leq \mathcal{E}(f)(0)\,.$
- $\|f\|^2_{L^2} = \|\mathrm{P}_0 f\|^2_{L^2} + \|\mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-3/2}\,,$
- $\sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha|\leq N} \|\,\langle\xi\rangle^p\,\partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-5/2}\,.$

Nonlinear theorem

- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$ small

- $f_0 \in H_x^N L_\xi^2(\langle\xi\rangle^p)$, $N \geq 4$, $p \geq 2$.

- Existence, uniqueness and optimal time decay:

Ukai, Kawashima, Y. Guo, Strain, T. Yang, R-J. Duan, etc.

Define

$\mathcal{E}(f)(t) = \sum_{|\alpha|\leq N} \|\partial_x^\alpha f\|_{L^2}^2 + \sum_{|\alpha|\leq N} \|\langle\xi\rangle^p \partial_x^\alpha f\|_{L^2}^2\,.$

$\mathcal{D}(f)(t) = \sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \mathrm{P}_0 f\|_{L^2}^2 + \sum_{|\alpha|\leq N} \|\langle\xi\rangle^p \partial_x^\alpha \mathrm{P}_1 f\|_{L_\sigma^2}^2\,.$

If we assume $\mathcal{E}(f)(0) \leq \eta$, then

- $\mathcal{E}(f)(t) + \int_0^t \mathcal{D}(f)(s)ds \leq \mathcal{E}(f)(0)\,.$

- $\|f\|_{L^2}^2 = \|\mathrm{P}_0 f\|_{L^2}^2 + \|\mathrm{P}_1 f\|_{L^2}^2 \leq \eta(1+t)^{-3/2}\,,$

- $\sum_{1\leq|\alpha|\leq N} \|\partial_x^\alpha \mathrm{P}_0 f\|_{L^2}^2 + \sum_{|\alpha|\leq N} \|\langle\xi\rangle^p \partial_x^\alpha \mathrm{P}_1 f\|_{L^2}^2 \leq \eta(1+t)^{-5/2}\,.$

Nonlinear theorem

- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$ small

- $f_0 \in H^N_x L^2_\xi(\langle\xi\rangle^p)$, $N \geq 4$, $p \geq 2$.

- Existence, uniqueness and optimal time decay:

Ukai, Kawashima, Y. Guo, Strain, T. Yang, R-J. Duan, etc.

Define

$\mathcal{E}(f)(t) = \sum_{|\alpha|\leq N} \|\partial^\alpha_x f\|^2_{L^2} + \sum_{|\alpha|\leq N} \|\langle\xi\rangle^p \partial^\alpha_x f\|^2_{L^2}\,.$

$\mathcal{D}(f)(t) = \sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha|\leq N} \|\langle\xi\rangle^p \partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2_\sigma}\,.$

If we assume $\mathcal{E}(f)(0) \leq \eta$, then

- $\mathcal{E}(f)(t) + \int_0^t \mathcal{D}(f)(s)ds \leq \mathcal{E}(f)(0)\,.$

- $\|f\|^2_{L^2} = \|\mathrm{P}_0 f\|^2_{L^2} + \|\mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-3/2}\,,$

- $\sum_{1\leq|\alpha|\leq N} \|\partial^\alpha_x \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha|\leq N} \|\langle\xi\rangle^p \partial^\alpha_x \mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-5/2}\,.$

Nonlinear theorem

- $\partial_t f + \xi \cdot \nabla_x f = Lf + \Gamma(f,f)\,, \quad f(0,x,\xi) = f_0$ small
- $f_0 \in H^N_x L^2_\xi(\langle \xi \rangle^p)$, $N \geq 4$, $p \geq 2$.
- Existence, uniqueness and optimal time decay:

Ukai, Kawashima, Y. Guo, Strain, T. Yang, R-J. Duan, etc.

Define

$\mathcal{E}(f)(t) = \sum_{|\alpha| \leq N} \|\partial_x^\alpha f\|^2_{L^2} + \sum_{|\alpha| \leq N} \| \langle \xi \rangle^p \, \partial_x^\alpha f\|^2_{L^2}\,.$

$\mathcal{D}(f)(t) = \sum_{1 \leq |\alpha| \leq N} \|\partial_x^\alpha \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha| \leq N} \| \langle \xi \rangle^p \, \partial_x^\alpha \mathrm{P}_1 f\|^2_{L^2_\sigma}\,.$

If we assume $\mathcal{E}(f)(0) \leq \eta$, then

- $\mathcal{E}(f)(t) + \int_0^t \mathcal{D}(f)(s) ds \leq \mathcal{E}(f)(0)\,.$
- $\|f\|^2_{L^2} = \|\mathrm{P}_0 f\|^2_{L^2} + \|\mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-3/2}\,,$
- $\sum_{1 \leq |\alpha| \leq N} \|\partial_x^\alpha \mathrm{P}_0 f\|^2_{L^2} + \sum_{|\alpha| \leq N} \| \langle \xi \rangle^p \, \partial_x^\alpha \mathrm{P}_1 f\|^2_{L^2} \leq \eta(1+t)^{-5/2}\,.$

Nonlinear theorem

Let $u = wf$, then u solves the equation

$$\partial_t u + \xi \cdot \nabla_x u - (\partial_t w + \xi \cdot \nabla_x w) w^{-1} u - wL\left(w^{-1}u\right) = w\Gamma(w^{-1}u, f)\,.$$

$$\begin{aligned}
&\left|\left\langle u, w\Gamma(w^{-1}u, f)\right\rangle_\xi - \left\langle u, \Gamma(u, f)\right\rangle_\xi\right| \\
&\leq D^{-2}\left|\mathrm{P}_1 u\right|_{L^2_\sigma}\left(|u|_{L^2_\sigma}\left|\langle\xi\rangle^p f\right|_{L^2_\xi} + |u|_{L^2_\xi}\left|\langle\xi\rangle^p f\right|_{L^2_\sigma}\right) \\
&\quad + D^{-2}\left|\mathrm{P}_0 u\right|_{L^2_\xi}\left(|u|_{L^2_\xi}\left|\langle\xi\rangle^p f\right|_{L^2_\xi} + |u|_{L^2_\xi}\left|\langle\xi\rangle^p f\right|_{L^2_\xi}\right)
\end{aligned}$$

Nonlinear theorem

Let $u = wf$, then u solves the equation

$$\partial_t u + \xi \cdot \nabla_x u - (\partial_t w + \xi \cdot \nabla_x w) w^{-1} u - wL\left(w^{-1}u\right) = w\Gamma(w^{-1}u, f)\,.$$

$$\begin{aligned}
&\left|\left\langle u, w\Gamma(w^{-1}u, f)\right\rangle_\xi - \langle u, \Gamma(u,f)\rangle_\xi\right| \\
&\le D^{-2}\left|\mathrm{P}_1 u\right|_{L^2_\sigma}\left(|u|_{L^2_\sigma}\left|\langle\xi\rangle^p f\right|_{L^2_\xi} + |u|_{L^2_\xi}\left|\langle\xi\rangle^p f\right|_{L^2_\sigma}\right) \\
&\quad + D^{-2}\left|\mathrm{P}_0 u\right|_{L^2_\xi}\left(|u|_{L^2_\xi}\left|\langle\xi\rangle^p f\right|_{L^2_\xi} + |u|_{L^2_\xi}\left|\langle\xi\rangle^p f\right|_{L^2_\xi}\right)
\end{aligned}$$

Nonlinear theorem

$$
\begin{aligned}
&\sum_{j=0}^{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\nabla_x^j u\right)\nabla_x^j\left(w\Gamma(w^{-1}u,f)\right)d\xi dx\\
&\leq \eta^{1/2}(1+t)^{-9/8}\left(\|\mathrm{P}_1 u\|^2_{H^2_x L^2_\sigma}+\|u\|^2_{H^2_x L^2_\xi}\right)\\
&\quad + D^{-2}\eta^{1/2}(1+t)^{-9/8}\left(\|\mathrm{P}_1 u\|^2_{H^2_x L^2_\sigma}+\|u\|^2_{H^2_x L^2_\xi}\right)\\
&\quad + D^{-2}\|\mathrm{P}_1 u\|^2_{H^2_x L^2_\sigma}+D^{-2}\|u\|^2_{H^2_x L^2_\xi}\mathcal{D}(f)\\
&\quad + D^{-2}\eta^{1/2}(1+t)^{-9/8}\|u\|^2_{H^2_x L^2_\xi}\ .
\end{aligned}
$$

Theorem (Energy estimate for fully nonlinear equation)

Let f be a solution to the Boltzmann equation with small initial data $f_0 \in H^N_x L^2_\xi(\langle\xi\rangle^p)$, $N \geq 4$, and compactly supported in x, if $|x| > Mt$ for some M large, then

$$
|f|_{L^2_\xi} \leq C(1+|x|)^{-\frac{p}{1-\gamma}}\|f_0\|_{H^4_x L^2_\xi(\langle\xi\rangle^p)}\ .
$$

Nonlinear theorem

$$
\begin{aligned}
&\sum_{j=0}^{2}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\left(\nabla_x^j u\right)\nabla_x^j\left(w\Gamma(w^{-1}u,f)\right)d\xi dx \\
\le\, &\eta^{1/2}(1+t)^{-9/8}\left(\|\mathrm{P}_1 u\|^2_{H_x^2 L_\sigma^2}+\|u\|^2_{H_x^2 L_\xi^2}\right) \\
&+D^{-2}\eta^{1/2}(1+t)^{-9/8}\left(\|\mathrm{P}_1 u\|^2_{H_x^2 L_\sigma^2}+\|u\|^2_{H_x^2 L_\xi^2}\right) \\
&+D^{-2}\|\mathrm{P}_1 u\|^2_{H_x^2 L_\sigma^2}+D^{-2}\|u\|^2_{H_x^2 L_\xi^2}\mathcal{D}(f) \\
&+D^{-2}\eta^{1/2}(1+t)^{-9/8}\|u\|^2_{H_x^2 L_\xi^2}\,.
\end{aligned}
$$

Theorem (Energy estimate for fully nonlinear equation)

Let f be a solution to the Boltzmann equation with small initial data $f_0 \in H_x^N L_\xi^2(\langle\xi\rangle^p)$, $N \ge 4$, and compactly supported in x, if $|x| > Mt$ for some M large, then

$$
|f|_{L_\xi^2} \le C(1+|x|)^{-\frac{p}{1-\gamma}}\|f_0\|_{H_x^4 L_\xi^2(\langle\xi\rangle^p)}\,.
$$

End

THANK YOU