A fundamental inequality for the p-Laplacian and the ∞-Laplacian

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Denote by Δ and Δ_∞ the Laplacian and ∞-Laplacian, respectively, in $\mathbb{R}^n$ with $n \geq 2$, i.e.

$$\Delta v = \mathrm{div}(Dv) \quad \text{and} \quad \Delta_\infty v = D^2 v Dv \cdot Dv \quad \forall v \in C^\infty.$$

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The equation

$$\Delta_\infty u = 0$$

is introduced by Aronsson in 1960's (assuming $u \in C^2(\Omega)$) as the Euler-Lagrange's equation while absolutely minimizing the L^∞-functional

$$\mathcal{F}(u, \Omega) = \operatorname*{ess\,sup}_{\Omega} |Du|^2$$

A function $u \in W^{1,\infty}_{\mathrm{loc}}(\Omega)$ is an absolute minimizer in Ω if for any $V \subset\subset \Omega$ we have

$$\mathcal{F}(u, V) \leq \mathcal{F}(v, V)$$

provided that $v \in W^{1,\infty}_{\mathrm{loc}}(V)$ and $v = u$ on ∂V.

However, not every ∞-harmonic is C^2: Aronsson gives an example

$$w(x_1, x_2) = x_1^{4/3} - x_2^{4/3}$$

in the plane which is only $C^{1,\,1/3}$. This function is usually called the Aronsson function.

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Later, Lu and Wang considered the inhomogeneous ∞-Laplace equation

$$-\Delta_\infty u = f, \ u = g \in C(\partial\Omega) \tag{1}$$

in Ω in the viscosity sense, where $f \in C(\Omega)$ is always assumed. They proved the existence and uniqueness of such an equation under the assumption that f is bounded and $|f| > 0$. However, when f changes sign, they gave a counter-example to the uniqueness of (1). The uniqueness for the case where $f \geq 0$ or $f \leq 0$ is still open.

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Recall that $u_p \in W^{1,p}(\Omega)$ is called a p-harmonic function if it minimizes the Dirichlet p-energy

$$\int_\Omega |\nabla u_p|^p \, dx \leq \int_\Omega |\nabla v|^p \, dx$$

whenever $u_p - v \in W_0^{1,p}(\Omega)$. Equivalently,

$$-\Delta_p u_p := -\mathrm{div}(|\nabla u_p|^{p-2} \nabla u_p) = 0$$

in Ω in the weak sense or viscosity sense. It is clear that, by the additivity of L^p-integral, we automatically have that u_p's are also absolute minimizers.

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The regularity of p-harmonic functions has been widely studied and is understood quite well. (Uraltseva, Lewis, Dibenedetto, Evans, Uhlenbeck, Iwaniec, Manfredi, Lindqvist, Fusco, Kinnunen...)

By the standard energy estimate, there exists a subsequence $p_i \to \infty$ and $u \in W^{1,\infty}(\Omega)$ so that

$$u = \lim_{p_i \to \infty} u_{p_i}$$

weakly in $\cap_{q>1} W^{1,q}(\Omega)$ and u absolutely minimizing the L^∞-functional. Therefore, u is infinity harmonic.

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Moreover, a formal calculation gives the following: For a p-harmonic function,

$$\Delta_p u = (p-2)|\nabla u|^{p-4} \left(\frac{|\nabla u|^2}{p-2} \Delta u + \Delta_\infty u \right) = 0.$$

In particular, we have

$$\frac{|\nabla u|^2}{p-2} \Delta u + \Delta_\infty u = 0.$$

By letting $p \to \infty$, we obtain $\Delta_\infty u = 0$.

We call

$$\Delta_p^N u := \frac{1}{p-2}\Delta u + |\nabla u|^{-2}\Delta_\infty u$$

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Indeed, there is a rigorous way to interpret this relation via probability model. Recall that harmonic functions is related to Brownian motion. Peres et al. introduced a notion of tug-of-war and use it to model the (normalized) infinity Laplacian $|Du|^{-2}\Delta_\infty$ and then the (normalized) p-Laplacian.

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Lemma 1

Let $n \geq 2$ and U be a domain of $\mathbb{R}^n$. For any $v \in C^\infty(U)$, we have

$$\left| |D^2 v Dv|^2 - \Delta v \Delta_\infty v - \frac{1}{2}[|D^2 v|^2 - (\Delta v)^2]|Dv|^2 \right| \leq \frac{n-2}{2}[|D^2 v|^2 |Dv|^2 - |D^2 v Dv|^2] \quad \text{in } U.$$

Based on this inequality, we are able to show the following results for euqations involving p-Laplacian:

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Theorem 1

Let $n \geq 2$, $p \in (1,2) \cup (2,\infty)$ and $\gamma < \gamma_{n,p}$, where $\gamma_{n,p} := \min\{p + \frac{n}{n-1}, 3 + \frac{p-1}{n-1}\}$. For any weak/viscosity solution u to

$$\Delta_p u = 0 \quad in\ \Omega,$$

we have $|Du|^{\frac{p-\gamma}{2}} Du \in W^{1,2}_{\text{loc}}(\Omega)$ and, for any $B = B(z,r) \subset 2B \subset\subset \Omega$

$$\int_B |D[|Du|^{\frac{p-\gamma}{2}} Du]|^2 \, dx \leq C(n,p,\gamma) \frac{1}{r^2} \int_{2B} |Du|^{p-\gamma+2} \, dx.$$

Theorem 1 improves the earlier result by Bojarski and Iwaniec, where the convexity and the monotonicity of the p-Laplacian were heavily used in their proof.

As a byproduct, we reprove the following higher integrability of $D^2 u$, which was shown earlier by using the Cordes condition.

Corollary 2

Let $n \geq 2$ and $p \in (1,2) \cup (2, 3 + \frac{2}{n-2})$. There exists $\delta_{n,p} \in (0,1)$ such that for any weak/viscosity solution u to

$$\Delta_p u = 0 \quad \text{in } \Omega,$$

we have $D^2 u \in L^q_{\text{loc}}(\Omega)$ for any $q < 2 + \delta_{n,p}$ and, for any $B = B(z,r) \subset 2B \subset\subset \Omega$,

$$\left(⨍_B |D^2 u|^q \, dx \right)^{1/q} \leq C(n,p,q) \frac{1}{r} \left(⨍_{2B} |Du|^2 \, dx \right)^{1/2}$$

Similar results also hold in the parabolic case, and some of them were completely open problems. Write $Q_r(z,s) := (s-r^2, s) \times B(z,r)$.

Theorem 2

Let $n \geq 2$ and $p \in (1,2) \cup (2, 3+\frac{2}{n-2})$. There exists $\delta_{n,p} \in (0,1)$ such that for any viscosity solution $u = u(x,t)$ to

$$u_t - \Delta_p^N u = 0 \quad \text{in } \Omega_T := \Omega \times (0,T),$$

we have $u_t, D^2u \in L^q_{\text{loc}}(\Omega)$ for any $q < 2 + \delta_{n,p}$, and for every $Q_r = Q_r(z,s) \subset Q_{2r} \subset\subset \Omega_T$.

$$\left(\fint_{Q_r} [|u_t|^q + |D^2u|^q] \, dx \right)^{1/q} \leq C(n,p,q) \frac{1}{r} \left(\fint_{Q_{2r}} |Du|^2 \, dx \right)^{1/2}.$$

Theorem 3

Let $n \geq 1$. For any weak/viscosity solution $u = u(x,t)$ to

$$u_t - \Delta_p u = 0 \quad \text{in } \Omega_T,$$

the following results hold.

(i) *For $p \in (1,2) \cup (2,\infty)$, we have $u_t \in L^2_{\mathrm{loc}}(\Omega_T)$ and, for any $Q_r = Q_r(z,s) \subset Q_{2r} \subset\subset \Omega_T$,*

$$\int_{Q_r} (u_t)^2 \, dx \, dt \leq \frac{C}{r^2} \int_{Q_{2r}} |Du|^p + |Du|^{2p-2} \, dx \, dt$$

(ii) *For $p \in (1,2) \cup (2,3)$, we have $D^2 u \in L^2_{\mathrm{loc}}(\Omega_T)$ and, for any $Q_r = Q_r(z,s) \subset Q_{2r} \subset\subset \Omega_T$,*

$$\int_{Q_r} |D^2 u|^2 \, dx \, dt \leq C(n,p) \frac{1}{r^2} \int_{Q_{2r}} |Du|^2 + |Du|^{4-p} \, dx \, dt.$$

The range of p (including $p = 2$ from the classical result) here is sharp for the $W^{2,2}_{\mathrm{loc}}$-regularity.

Unlike the p-Laplacian, lots of problems on infinity Laplace equations are open. One of the main problems for infinity Laplace equations is to show the continuity of it gradient. Here is a list of known results on the interior regularity.

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$\Delta_\infty u = 0$ in $\mathbb{R}^n$:

- Crandall and Evans, 2001, linear approximation property;
- Savin, 2005, C^1_{loc}-regularity when $n = 2$;
- Evans and Savin, 2008, $C^{1,\alpha}_{\text{loc}}$-regularity when $n = 2$;
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- Lu and Wang, 2008, $W^{1,\infty}_{\text{loc}}$-regularity;
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The difficulty to show the regularity comes from the following fact observed by Evans in 1993: For any smooth infinity harmonic function in $\mathbb{R}^n$, one has a uniform $C^{1,1}_{\mathrm{loc}}$-estimate in terms of $\|u\|_{C^{0,1}}$. Therefore, one cannot approximate every infinity harmonic function via smooth ones.

Here is the main result of the joint work with H. Koch and Y. Zhou.

Theorem 4

Let $\Omega \subset \mathbb{R}^2$ be a domain and u be an ∞-harmonic function in Ω. For each $\alpha > 0$, we have $|Du|^\alpha \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and

$$(|Du|^\alpha)_i u_i = 0 \quad \text{almost everywhere in } \Omega. \tag{2}$$

Moreover, the distributional determinant $-\det D^2 u dx$ is a Radon measure satisfying

$$-\det D^2 u \geq |D|Du||^2$$

where $=$ holds when $u \in C^2(\Omega)$. Quantitative estimates are also given.

Remark 5

1. The result above is sharp when $\alpha \to 0$, i.e. for the Aronsson function $\log|Dw| \notin W^{1,2}_{\rm loc}(\Omega)$
2. Note that the function $u(x) = |x| \in W^{1,\infty}_{\rm loc}(\mathbb{R}^2)$ satisfies $|Du|^2 \in W^{1,2}_{\rm loc}(\mathbb{R}^2)$ and (2), but is not ∞-harmonic.
3. In the plane for p-harmonic functions we have $D^2u \in L^{\gamma}_{\rm loc}$ for $\gamma < 3$, while for the Aronsson function $D^2w \in L^{\beta}_{\rm loc}$ with $\beta < \frac{3}{2}$. However, $\frac{|D^2wDw|}{|Dw|} \in L^{\gamma}_{\rm loc}$.

Theorem 6

Suppose $\Omega \subset \mathbb{R}^2$ is a bounded domain and $f \in BV_{\mathrm{loc}}(\Omega) \cap C^0(\Omega)$ with $|f| > 0$ in Ω. Let $u \in C^0(\Omega)$ be a viscosity solution to

$$-\Delta_\infty u = f \quad \text{in } \Omega.$$

Then we have:

- (i) *For $\alpha > 3/2$, we have $|Du|^\alpha \in W^{1,2}_{\mathrm{loc}}(\Omega)$, which is (asymptotic) sharp when $\alpha \to 3/2$.*
- (ii) *For $\alpha \in (0, 3/2]$ and $p \in [1, 3/(3-\alpha))$, we have $|Du|^\alpha \in W^{1,p}_{\mathrm{loc}}(\Omega)$, which is sharp when $p \to 3/(3-\alpha)$.*
- (iii) *For $\epsilon > 0$, we have $|Du|^{-3+\epsilon} \in L^1_{\mathrm{loc}}(\Omega)$, which is sharp when $\epsilon \to 0$.*
- (iv) *For $\alpha > 0$, we have*

$$-(|Du|^\alpha)_i u_i = 2\alpha |Du|^{\alpha-2} f \quad \textit{almost everywhere in } \Omega.$$

Some quantative bounds are given.

Remark 7

1. The function $w(x_1, x_2) = -x_1^{4/3}$ as a viscosity solution to $-\Delta_\infty w = \frac{4^3}{3^4}$ in $\mathbb{R}^2$, clarifies the above sharpness: $|Dw|^{-3} \notin L^1_{\mathrm{loc}}(\mathbb{R}^2)$;$|Dw|^\alpha \notin W^{1,3/(3-\alpha)}_{\mathrm{loc}}(\mathbb{R}^2)$ whenever $\alpha \in (0, 3/2]$; and for any $p > 2$, $|Dw|^\alpha \notin W^{1,p}_{\mathrm{loc}}(\mathbb{R}^2)$ whenever $\alpha \in (3/2, 3 - 3/p)$.
2. For the solutions u to the infinity Laplace equations in both of the cases above, we *conjecture* that, there exists $\epsilon > 0$ so that

$$|Du|^2 \in W^{1,\,2+\epsilon}_{\mathrm{loc}}(\Omega).$$

Here is a sketch proof of the first theorem on the infinity Laplacian.

1. For $\epsilon \in (0,1]$, consider $u^\epsilon \in C^\infty(U) \cap C(\overline{U})$ satisfies

$$-\Delta_\infty u^\epsilon - \epsilon \Delta u^\epsilon = 0 \text{ in } \Omega, \quad u^\epsilon = u \text{ on } \partial\Omega.$$

It is known that $u^\epsilon \to u$ locally uniformly as $\epsilon \to 0$.

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2. Observe that, for any smooth function v in the plane,
$$-\det D^2 v = -\frac{1}{2}\operatorname{div}(\Delta v Dv - D^2 v Dv)$$
and
$$(-\det D^2 v)|Dv|^2 = |D^2 v Dv|^2 - \Delta v \Delta_\infty v.$$

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4. Integral by parts to obtain a uniform L^2-flatness estimate (roughly speaking, $|Du^\epsilon|^2 - \langle Du^\epsilon \cdot DP \rangle|$ is small in the L^2-average sense) and conclude $u^\epsilon \to u$ strongly in $W^{1,p}_{\mathrm{loc}}$ for any $1 < p < \infty$.

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5. Let $\epsilon \to 0$.


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