15. Show that $S_3 = \langle (1 \ 2), (2 \ 3) \rangle$.

16. Let $\varphi : G \to H$ be a morphism of groups, and $H = \langle Y \rangle$ for some subset $Y \subset H$. Show that $\varphi$ is an epimorphism if and only if $Y \subset \text{im}\varphi$.

17. Let $D_n = \langle \varrho, \sigma | \varrho^n = e = \sigma^2, \sigma \varrho \sigma^{-1} = \varrho^{-1} \rangle$ be the dihedral group of order $2n$, and let $H$ be any group. Define a map $\varphi : D_n \to H$ by choosing $x = \varphi(\varrho)$ and $y = \varphi(\sigma)$ in $H$ freely, and setting $\varphi(\varrho^i \sigma^j) = x^i y^j$ for all $0 \leq i \leq n-1$ and $0 \leq j \leq 1$. Prove that $\varphi$ is a morphism if and only if $x$ and $y$ satisfy the relations $x^n = e = y^2$ and $yxy^{-1} = x^{-1}$.

18. (a) Show that every automorphism $\alpha \in \text{Aut}(D_3)$ induces a permutation $\alpha_i \in S\{\sigma, \varrho \sigma, \varrho^2 \sigma\}$.
(b) Show that the map $\varphi : \text{Aut}(D_3) \to S_3, \varphi(\alpha) = \alpha_i$ is a monomorphism.
(c) Use exercise 17 to show that $\{(1 \ 2), (2 \ 3)\} \subset \text{im}\varphi$.
(d) Use exercises 15 and 16 to conclude that $\varphi$ is an isomorphism (cf. exercise 13).

19. Prove that if $G \to H$, then $\text{Aut}(G) \to \text{Aut}(H)$.

20. Every group $G$ determines a sequence of groups $\text{Aut}^n(G), n \in \mathbb{N}$, which is defined inductively by $\text{Aut}^0(G) = G$, and $\text{Aut}^n(G) = \text{Aut}(\text{Aut}^{n-1}(G))$ for all $n \geq 1$. Determine $\text{Aut}^n(C_2 \times C_2)$ up to isomorphism, for all $n \in \mathbb{N}$.

21. Prove the so-called

**Isomorphism Theorem for groups.** Every group morphism $\varphi : G \to H$ induces an isomorphism $\overline{\varphi} : G/\ker\varphi \to \text{im}\varphi$, $\overline{\varphi}(x \ker\varphi) = \varphi(x)$. Moreover, $\varphi = \iota \circ \overline{\varphi} \circ \pi$, where $\pi : G \to G/\ker\varphi$ is the quotient morphism and $\iota : \text{im}\varphi \to H$ is the inclusion morphism.

22. Let $\varphi : G \to H$ be a morphism of finite groups. Show that $|\text{im}\varphi|$ is a common divisor of $|G|$ and $|H|$.

23. (a) Show that for each subgroup $K < D_2$ with $|K| = 2$ and for each subgroup $I < D_3$ with $|I| = 2$ there is a unique morphism $\varphi : D_2 \to D_3$ such that $\ker \varphi = K$ and $\text{im}\varphi = I$.
(b) Use (a) to describe all morphisms $D_2 \to D_3$ (cf. exercise 12).
24. (a) Show that for each normal subgroup \( K \triangleleft D_3 \) with \( |K| = 3 \) and for each subgroup \( I \triangleleft D_2 \) with \( |I| = 2 \) there is a unique morphism \( \varphi : D_3 \rightarrow D_2 \) such that \( \ker \varphi = K \) and \( \text{im} \varphi = I \).

(b) Use (a) to describe all morphisms \( D_3 \rightarrow D_2 \) (cf. exercise 12).

25. A subgroup \( H < G \) is called **characteristic** if \( \alpha(H) = H \) holds for every automorphism \( \alpha \) of \( G \). Prove the following statements.

(a) Every characteristic subgroup is normal.

(b) For every group \( G \), its **center** \( Z(G) = \{ z \in G \mid zx = xz \ \forall x \in G \} \) is a characteristic subgroup of \( G \).

26. Find the center of \( D_n \), for all \( n \geq 2 \).

27. Every element \( a \) of a group \( G \) determines a map \( \kappa_a : G \rightarrow G, x \mapsto axa^{-1} \).

(a) Prove that \( \kappa_a \) is an automorphism of \( G \), and that \( \kappa : G \rightarrow \text{Aut}(G), a \mapsto \kappa_a \) is a group morphism.

(b) Automorphisms of the form \( \kappa_a \) are called **inner automorphisms** of \( G \). Prove that the inner automorphisms of \( G \) form a subgroup \( \text{InAut}(G) < \text{Aut}(G) \) which is isomorphic to \( G/Z(G) \).

28. Find all subgroups of \( \mathbb{Z} \).

29. (a) Show that every finite group is finitely generated.

(b) Let \( n, \ell \in \mathbb{N} \) and \( p_1^{m_1}, \ldots, p_\ell^{m_\ell} \) be a sequence of prime powers. Show that the group

\[
\mathbb{Z}^n \times \prod_{i=1}^{\ell} \mathbb{Z}_{p_i^{m_i}}
\]

is finite if and only if \( n = 0 \).

(c) Use (a) and (b) to formulate the fundamental theorem for **finite** abelian groups in analogy to the fundamental theorem for **finitely generated** abelian groups, presented in lecture 4.

30. Let \( \mathcal{L} = \{ G_1, \ldots, G_7 \} \), where

\[
\begin{align*}
G_1 &= \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_9 \\
G_2 &= \mathbb{Z}_{15} \times \mathbb{Z}_{24} \\
G_3 &= \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_9 \\
G_4 &= \mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_{12} \\
G_5 &= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_9 \\
G_6 &= \mathbb{Z}_5 \times \mathbb{Z}_{72} \\
G_7 &= \mathbb{Z}_{360}
\end{align*}
\]

(a) Verify that all of the groups \( G_i \in \mathcal{L} \) are abelian and of order 360.

(b) Find a subset \( \mathcal{L}_0 \subset \mathcal{L} \) that is irredundant.

(c) Extend \( \mathcal{L}_0 \) to a list \( \mathcal{L}_1 \) that classifies all abelian groups of order 360.