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Algebraic structures

Second sheet of exercises

15. Show that $\mathrm{S}_3 = \langle (1\ 2), (2\ 3) \rangle$.

16. Let $\varphi : G \to H$ be a morphism of groups, and $H = \langle Y \rangle$ for some subset $Y \subset H$. Show that φ is an epimorphism if and only if $Y \subset \mathrm{im}\varphi$.

17. Let $\mathrm{D}_n = \langle \varrho, \sigma \mid \varrho^n = \varepsilon = \sigma^2,\ \sigma\varrho\sigma^{-1} = \varrho^{-1} \rangle$ be the dihedral group of order $2n$, and let H be any group. Define a map $\varphi : \mathrm{D}_n \to H$ by choosing $x = \varphi(\varrho)$ and $y = \varphi(\sigma)$ in H freely, and setting $\varphi(\varrho^i\sigma^j) = x^i y^j$ for all $0 \le i \le n-1$ and $0 \le j \le 1$. Prove that φ is a morphism if and only if x and y satisfy the relations $x^n = e = y^2$ and $yxy^{-1} = x^{-1}$.

18. (a) Show that every automorphism $\alpha \in \mathrm{Aut}(\mathrm{D}_3)$ induces a permutation $\alpha_\iota \in \mathrm{S}_{\{\sigma, \varrho\sigma, \varrho^2\sigma\}}$.

(b) Show that the map $\varphi : \mathrm{Aut}(\mathrm{D}_3) \to \mathrm{S}_3,\ \varphi(\alpha) = \alpha_\iota$ is a monomorphism.

(c) Use exercise 17 to show that $\{(1\ 2), (2\ 3)\} \subset \mathrm{im}\varphi$.

(d) Use exercises 15 and 16 to conclude that φ is an isomorphism (cf. exercise 13).

19. Prove that if $G \tilde{\to} H$, then $\mathrm{Aut}(G) \tilde{\to} \mathrm{Aut}(H)$.

20. Every group G determines a sequence of groups $\mathrm{Aut}^n(G),\ n \in \mathbb{N}$, which is defined inductively by $\mathrm{Aut}^0(G) = G$, and $\mathrm{Aut}^n(G) = \mathrm{Aut}(\mathrm{Aut}^{n-1}(G))$ for all $n \ge 1$. Determine $\mathrm{Aut}^n(\mathrm{C}_2 \times \mathrm{C}_2)$ up to isomorphism, for all $n \in \mathbb{N}$.

21. Prove the so-called

Isomorphism Theorem for groups. *Every group morphism $\varphi : G \to H$ induces an isomorphism $\overline{\varphi} : G/\ker\varphi \tilde{\to} \mathrm{im}\varphi,\ \overline{\varphi}(x\ker\varphi) = \varphi(x)$. Moreover, $\varphi = \iota \circ \overline{\varphi} \circ \pi$, where $\pi : G \to G/\ker\varphi$ is the quotient morphism and $\iota : \mathrm{im}\varphi \to H$ is the inclusion morphism.*

22. Let $\varphi : G \to H$ be a morphism of finite groups. Show that $|\mathrm{im}\varphi|$ is a common divisor of $|G|$ and $|H|$.

23. (a) Show that for each subgroup $K < \mathrm{D}_2$ with $|K| = 2$ and for each subgroup $I < \mathrm{D}_3$ with $|I| = 2$ there is a unique morphism $\varphi : \mathrm{D}_2 \to \mathrm{D}_3$ such that $\ker\varphi = K$ and $\mathrm{im}\varphi = I$.

(b) Use (a) to describe all morphisms $\mathrm{D}_2 \to \mathrm{D}_3$ (cf. exercise 12).

Please turn over!

24. (a) Show that for each normal subgroup $K \lhd \mathrm{D}_3$ with $|K| = 3$ and for each subgroup $I < \mathrm{D}_2$ with $|I| = 2$ there is a unique morphism $\varphi : \mathrm{D}_3 \to \mathrm{D}_2$ such that $\ker\varphi = K$ and $\mathrm{im}\varphi = I$.

(b) Use (a) to describe all morphisms $\mathrm{D}_3 \to \mathrm{D}_2$ (cf. exercise 12).

25. A subgroup $H < G$ is called *characteristic* if $\alpha(H) = H$ holds for every automorphism α of G. Prove the following statements.

(a) Every characteristic subgroup is normal.

(b) For every group G, its *center* $Z(G) = \{z \in G \mid zx = xz \ \forall x \in G\}$ is a characteristic subgroup of G.

26. Find the center of D_n, for all $n \geq 2$.

27. Every element a of a group G determines a map $\kappa_a : G \to G, \ x \mapsto axa^{-1}$.

(a) Prove that κ_a is an automorphism of G, and that $\kappa : G \to \mathrm{Aut}(G), \ a \mapsto \kappa_a$ is a group morphism.

(b) Automorphisms of the form κ_a are called *inner automorphisms* of G. Prove that the inner automorphisms of G form a subgroup $\mathrm{InAut}(G) < \mathrm{Aut}(G)$ which is isomorphic to $G/Z(G)$.

28. Find all subgroups of $\mathbb{Z}$.

29. (a) Show that every finite group is finitely generated.

(b) Let $n, \ell \in \mathbb{N}$ and $p_1^{m_1}, \ldots, p_\ell^{m_\ell}$ be a sequence of prime powers. Show that the group

$$\mathbb{Z}^n \times \prod_{i=1}^{\ell} \mathbb{Z}_{p_i^{m_i}}$$

is finite if and only if $n = 0$.

(c) Use (a) and (b) to formulate the fundamental theorem for *finite* abelian groups in analogy to the fundamental theorem for *finitely generated* abelian groups, presented in lecture 4.

30. Let $\mathscr{L} = \{G_1, \ldots, G_7\}$, where

$$\begin{aligned}
G_1 &= \mathbb{Z}_5 \times \mathbb{Z}_8 \times \mathbb{Z}_9 \\
G_2 &= \mathbb{Z}_{15} \times \mathbb{Z}_{24} \\
G_3 &= \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_5 \times \mathbb{Z}_9 \\
G_4 &= \mathbb{Z}_5 \times \mathbb{Z}_6 \times \mathbb{Z}_{12} \\
G_5 &= \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_5 \times \mathbb{Z}_9 \\
G_6 &= \mathbb{Z}_5 \times \mathbb{Z}_{72} \\
G_7 &= \mathbb{Z}_{360}
\end{aligned}$$

(a) Verify that all of the groups $G_i \in \mathscr{L}$ are abelian and of order 360.

(b) Find a subset $\mathscr{L}_0 \subset \mathscr{L}$ that is irredundant.

(c) Extend $\mathscr{L}_0$ to a list $\mathscr{L}_1$ that classifies all abelian groups of order 360.