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Algebraic structures

Third sheet of exercises

31. (a) Show that every group of prime order is cyclic.

(b) Show that every cyclic group is abelian.

(c) Is it true that every group with two generators is abelian? (Proof or counterexample.)

32. Prove that the list $\mathscr{L} = \{\mathbb{Z}_n \mid n \in \mathbb{N}\}$ classifies all cyclic groups, up to isomorphism.

33. Which of the following groups are isomorphic, and which are not?

$\mathbb{Z}_{100}$, $\mathbb{Z}_2 \times \mathbb{Z}_{25} \times \mathbb{Z}_2$, $\mathbb{Z}_5 \times \mathbb{Z}_4 \times \mathbb{Z}_5$, $\mathbb{Z}_4 \times \mathbb{Z}_{25}$, C_{100} , $C_2 \times C_2 \times C_{25}$, $C_{25} \times C_4$.

34. Classify all abelian groups of order 216.

35. (a) Classify all groups of order 121.

(b) Classify all groups of order 169.

36. Let G be any group. Prove that G is abelian if and only if $G/Z(G)$ is cyclic.

37. Let X be a subset of a group G. We know that $\langle X \rangle = P(X \cup X^{-1})$. Show that if G is finite, then $\langle X \rangle = P(X)$.

38. Show that if $\sigma, \tau \in \mathrm{S}_n$ have disjoint supports, then $\sigma\tau = \tau\sigma$.

39. Express the order of a permutation $\sigma \in \mathrm{S}_n$ in terms of its cycle type.

40. Consider the subset $N = \{e, (12)(34),\ (13)(24), (14)(23)\} \subset \mathrm{S}_4$. Show that $N \lhd \mathrm{S}_4$ and $N \lhd \mathrm{A}_4$.

41. Prove that S_n is generated by $\{(ij) \mid 1 \leq i < j \leq n\}$.

42. Prove that S_n is generated by $\{(12), (23), \ldots, (n-1\ n)\}$.

43. Show that every group of order 56 has a nontrivial proper normal subgroup.

44. Show that every subgroup of index 2 is normal.