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Algebraic structures

Fourth sheet of exercises

45. Show that A_4 has no subgroup of order 6. How does that relate to the first Sylow theorem?

46. If $N \lhd G$, then the intermediate groups $N < H < G$ are in bijection to the subgroups of G/N. Describe that bijection in both directions.

47. Let G be a finite cyclic group of order n. Show that there is an order preserving bijection between the divisors of n, ordered by divisibility, and the subgroups of G, ordered by inclusion.

48. Describe the set of all subgroups of $\mathbb{Z}_{30}$, ordered by inclusion.

49. Classify all finite simple abelian groups, up to isomorphism.

50. Any pair (S, T) of subsets of a group G determines a subset $ST = \{st \mid s \in S,\ t \in T\}$ of G. Prove the following statements.

(a) If S and T are subgroups of G and one of them is normal in G, then $ST < G$.

(b) If moreover G is finite, then $|ST| = \frac{|S||T|}{|S \cap T|}$.

51. Let G be a finite group and p a prime divisor of $|G|$. Denote the set of all Sylow p-subgroups of G by $\mathscr{S}$. Choose $S \in \mathscr{S}$, and consider the action $S \times \mathscr{S} \to \mathscr{S},\ (s, S') \mapsto sS's^{-1}$.

(a) Show that this action has at least one trivial orbit.

(b) One can even show that this action has precisely one trivial orbit. Deduce from this fact the first statement of the second Sylow theorem.

52. Let $G, p, \mathscr{S}$ be as above. One can show that the action $G \times \mathscr{S} \to \mathscr{S},\ (g, S) \mapsto gSg^{-1}$ has only one orbit. Deduce from this fact the second statement of the second Sylow theorem, and the third Sylow theorem.

53. Show that there is no simple group of order 30.

54. The set of all units in a ring R is denoted by R^ι. Verify the following statements.

(a) R^ι is a group, with multiplication induced from the multiplication of R.

(b) Every ring morphism $\varphi : R \to S$ induces a group morphism $\varphi^\iota : R^\iota \to S^\iota$.

Please turn over!

55. Show that the subset

$$\mathbb{H} = \left\{ \left(\begin{array}{cc} w & -z \\ \overline{z} & \overline{w} \end{array} \right) \middle| \, w, z \in \mathbb{C} \right\} \subset \mathbb{C}^{2\times 2}$$

is a subring, which is a skew field but not a field.