Algebraic structures

Fifth sheet of exercises

56. If $K$ is a field and $q(X) \in \text{irr}(K[X])$, then $E = K[X]/(q)$ is a field extension of $K$. Show that $E = K[\alpha] = K(\alpha)$, where $\alpha = X + (q) \in E$.

57. Find all ideals in $\mathbb{R}^{n\times n}$, for any $n \in \mathbb{N}\{0\}$.

58. We know that if $K = (K, +, \cdot)$ is a field, then both $(K, +)$ and $(K \setminus \{0\}, \cdot)$ are abelian groups. Suppose conversely that a set $K$ is equipped with an abelian group structure $+$ on $K$ and an abelian group structure $\cdot$ on $K \setminus \{0\}$. Which additional requirements must $(K, +, \cdot)$ satisfy to make sure that $K = (K, +, \cdot)$ is a field?

59. Show that a commutative ring is a field if and only if it has precisely two ideals.

60. Show that every field is a principal ideal domain which has no irreducible elements. Is every field a unique factorization domain?

61. Let $R$ be a domain. Prove the following statements.
   (a) $R[X]$ is a domain.
   (b) $R^e = R[X]^e$.
   (c) $\text{irr}(R) \subset \text{irr}(R[X])$.

62. Let $n = \prod_{i=1}^\ell p_i$, where $p_1, \ldots, p_\ell$ are distinct prime numbers. Show that the real number $\sqrt[n]{m}$ is not rational, for all $m \geq 2$.

63. Let $K$ be a field of characteristic not 2. Show that the polynomial $X^2 + Y^2 - 1$ is irreducible in $K[X,Y]$.

64. Prove the following so-called universal property of the polynomial ring. Let $\varphi : R \rightarrow S$ be a morphism of commutative rings, and let $s \in S$. Then $\varphi$ extends uniquely to a ring morphism $\varphi_s : R[X] \rightarrow S$ with $\varphi_s(X) = s$, namely $\varphi_s(\sum a_i X^i) = \sum \varphi(a_i)s^i$.

65. Let $R$ be a commutative ring, and let $a \in R$. Show that the substitution map $\sigma_a : R[X] \rightarrow R[X]$, $\sigma_a(f(X)) = f(X + a)$ is a ring automorphism.

Please turn over!
66. Let $R$ be a commutative ring, let $f(X) \in R[X]$ and $a \in R$. Prove that $f(X) \in \text{irr}(R[X])$ if and only if $f(X + a) \in \text{irr}(R[X])$.

67. Decide whether or not the following polynomials are irreducible in $\mathbb{Z}[X]$.
   (a) $f(X) = 1 + X^4$.
   (b) $g(X) = 1 + X^3 + X^6$.

68. Let $p$ be a prime number. Prove that $p$ divides \( \binom{p}{i} \) for all $0 < i < p$.

69. Let $K$ be a field of prime characteristic $p$. Prove that the following identities hold for all $a, b \in K$ and all $n \in \mathbb{N}$.
   (a) $(a + b)^p = a^p + b^p$.
   (b) $(a + b)^{p^n} = a^{p^n} + b^{p^n}$.
   (c) $(a - b)^{p^n} = a^{p^n} - b^{p^n}$.

70. Prove that for each prime number $p$ the $p$-th cyclotomic polynomial

$$
\Phi_p(X) = 1 + X + X^2 + \ldots + X^{p-1}
$$

is irreducible in $\mathbb{Z}[X]$ and in $\mathbb{Q}[X]$.

71. Show that $[\mathbb{R} : \mathbb{Q}] = \infty$. 