Uppsala universitet
Matematiska institutionen
Ernst Dieterich

Kandidatprogrammet i matematik
Autumn term 2011
23/11/2011

Algebraic structures

Sixth sheet of exercises

72. Prove the following so-called *universal property of the field of fractions.* Every ring monomorphism $\varphi : R \to F$ from a domain R to a field F extends uniquely to a field morphism $\psi : \mathrm{frac}(R) \to F$, namely $\psi\left(\frac{a}{b}\right) = \frac{\varphi(a)}{\varphi(b)}$.

73. Let $K \subset E$ be a field extension, and $\alpha \in E$. Show that if α is transcendental over K, then $K(\alpha) \tilde{\to} K(X)$.

74. Verify for any field extension $K \subset E$ the following statements.

(a) $[E : K] \geq 1$.

(b) $[E : K] = 1$ if and only if $K = E$.

75. Show that every field extension of prime degree has precisely two intermediate fields.

76. For every prime number p find a field extension of degree $p - 1$.

77. Find the multiplication table of a field of order 9.

78. Let $K \subset E$ be a field extension, and let

$$\varepsilon_\alpha : K[X] \to E,\ \varepsilon_\alpha(f(X)) = f(\alpha)$$

be the evaluation morphism determined by an element $\alpha \in E$. If α is algebraic over K, then $\ker(\varepsilon_\alpha)$ is generated by a unique monic irreducible polynomial in $K[X]$ (see Proposition 49), called the *irreducible polynomial of* α *over* K, and denoted $\mathrm{irrpol}_K(\alpha)$.

Now consider the field extension $\mathbb{Q} \subset \mathbb{C}$, and $\alpha = -\frac{1}{2} + \frac{\sqrt{3}}{2} i \in \mathbb{C}$.

(a) Show that α is algebraic over $\mathbb{Q}$.

(b) Find $\mathrm{irrpol}_{\mathbb{Q}}(\alpha)$.

(c) Show that $(1, \alpha)$ is a $\mathbb{Q}$-basis in $\mathbb{Q}(\alpha)$.

(d) Write $\frac{1}{n+\alpha}$ in this basis, for all $n \in \mathbb{N}$.

79. Verify the product rule $(fg)' = f'g + fg'$ for all $f, g \in K[X]$, where K is any field.

80. Show that every real polynomial $f(X) \in \mathbb{R}[X]$ has splitting field either $\mathbb{R}$ or $\mathbb{C}$. Which case occurs for which real polynomials?

Please turn over!

81. Describe up to isomorphism all finite abelian groups $F = (F, +)$ for which there exists a binary operation $\cdot : F \times F \to F,\ (x, y) \mapsto xy$ such that $F = (F, +, \cdot)$ is a field.

82. Prove that $f(X) = X^4 - 10X^2 + 1$ is irreducible in $\mathbb{Q}[X]$. (Hint. Show that $f(X)$ can not be written as a product of integral polynomials of degree smaller than 4.)

83. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Show that the field extension $\mathbb{Q} \subset E$ is simple. (Hint. Find $[E : \mathbb{Q}]$ by considering $\mathbb{Q} \subset \mathbb{Q}(\sqrt{2}) \subset E$, and find $\mathrm{irrpol}_{\mathbb{Q}}(\sqrt{2} + \sqrt{3})$.)

84. Prove that every polynomial with coefficients in a field has a splitting field. (Hint. Proceed by induction on the degree of the polynomial.)

85. Find the complex roots of the complex polynomial $f(X) = X^3 - 3iX^2 - 4X + 2i$.