Algebraic structures

Seventh sheet of exercises

86. As an application of the theory of finite abelian groups one can prove the following

**Theorem.** Every finite subgroup of the unit group of a field is cyclic.

Use this theorem to show that every finite extension of a finite field is simple.

87. Show that the real number \( \alpha = 2 + \sqrt[4]{5} \) is algebraic, and find \( \text{irrpol}_Q(\alpha) \).

88. The \( n \)-th cyclotomic polynomial is defined for all \( n \in \mathbb{N} \setminus \{0\} \) as

\[ \Phi_n(X) = \text{irrpol}_Q \left( e^{\frac{2\pi i}{n}} \right). \]

Find \( \Phi_n(X) \) for all \( 1 \leq n \leq 8 \).

89. Show that the field \( \mathbb{A} \) of all algebraic numbers has the following properties.

(a) \( [\mathbb{A} : \mathbb{Q}] = \infty \).

(b) \( \mathbb{A} \) is not simple over \( \mathbb{Q} \).

(c) \( \mathbb{A} \) is not finitely generated over \( \mathbb{Q} \).

90. A field \( K \) is called algebraically closed if \( K \) is the only algebraic extension of \( K \). Show that \( \mathbb{C} \) is algebraically closed.

91. Show that \( \mathbb{A} \) is algebraically closed.

92. Show that every field extension \( K \subset E \) of degree 2 is Galois, provided that \( \text{char}(K) \neq 2 \).

93. Prove that \( |\text{Gal}(E/K)| = [E : K] \), whenever \( K \subset E \) is a finite Galois extension.

(Hint. According to the proof of Proposition 56 there exists a primitive element \( \alpha \in E \) such that \( E = K(\alpha) \) is a splitting field for \( q(X) = \text{irrpol}_K(\alpha) \). Let \( R = \{\alpha_1, \ldots, \alpha_n\} \) be the set of all roots of \( q(X) \) in \( E \), with \( \alpha = \alpha_1 \). Show that for each \( \alpha_i \in R \) there is a unique \( \sigma \in \text{Gal}(E/K) \) such that \( \sigma(\alpha) = \alpha_i \).)

Please turn over!
94. Let $E = \mathbb{Q}(\sqrt{2}, \sqrt{3})$.
(a) Determine $\text{Gal}(E/\mathbb{Q})$.
(b) Describe all subgroups of $\text{Gal}(E/\mathbb{Q})$, ordered by inclusion.
(c) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

95. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi i}{13}}$.
(a) Determine $\text{Gal}(E/\mathbb{Q})$.
(b) Describe all subgroups of $\text{Gal}(E/\mathbb{Q})$, ordered by inclusion.
(c) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

96. Let $K \subset E$ be a finite extension of degree $n$. Show that the inequality
$$\deg(\text{irrpol}_K(\alpha)) \leq n$$
holds for all $\alpha \in E$.

97. Show that every irreducible real polynomial has degree 1 or 2.

98. Find the addition table and the multiplication table of a field of order 8.

99. Find the complex roots of the polynomial $f(X) = 12X^3 - 16X^2 + 3X - 4$. 