UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
ERNST DIETERICH

SPRING TERM 2012
MASTER PROGRAMME IN MATHEMATICS
15 MARCH 2012

Algebraic number theory, 5 credits

General information

Course literature. [1] M. Jonsson, *Fermat's last theorem for regular prime exponents*, M.Sc. thesis, Uppsala University, 1999.

[2] A. Fröhlich, M.J. Taylor, *Algebraic number theory*, Cambridge University Press, second edition, 2010.

[3] I. Stewart, D. Tall, *Algebraic number theory and Fermat's last theorem*, third edition, AK Peters, 2002.

Course homepage. At *Studentportalen* and on my homepage

http://www2.math.uu.se/staff/pages/?uname=ernstdie

you will find all current information concerning the course.

Requirements. 120 credits, including Algebraic structures.

Instruction. 15 lectures and home assignments.

Assessment. Written examination at the end of the course, scheduled for Friday, 1 June. *Please don't forget to register for the examination online at www.math.uu.se, not later than May 18!*

Home assignments will be distributed during the course, once every week. They are compulsory in the sense that a total score of at least 50% is a necessary requirement for passing the course. Moreover, a total score of at least 75% gives you 2 credit points for the written examination.

Grading System. U = failed, 3 = passed, 4 = passed with credit, 5 = passed with distinction.

Learning outcomes

At the end of the course the student should be able to

- define the concepts algebraic number, algebraic integer, Dedekind ring, fractional ideal, ideal class, ideal class group, regular prime, irregular prime, norm, trace, ramification index, residue class degree;
- give an account of the fundamental theorem of Dedekind rings;
- in elementary cases calculate the norm and trace of an algebraic number;
- explain how the ideal class number reflects the deviation of a ring of algebraic integers from being factorial;
- outline Kummer's lemma;
- outline Kummer's proof of Fermat's great theorem for regular prime exponents;
- outline the link between the ramification indices and the residue class degrees of a prime ideal, with respect to an algebraic field extension.

Contents

Integral elements in commutative ring extensions. Dedekind rings, their ideal theory and ideal class group. Quadratic number fields. Cyclotomic fields and cyclotomic integers. Regular and irregular primes. Kummer's lemma. Kummer's proof of Fermat's great theorem for regular prime exponents. Norm and trace of an algebraic number. Ramification index and residue class degree of a prime ideal with respect to an algebraic field extension.

Course plan in outline

The course is built up from three parts.

Part 1 presents Kummer's proof of Fermat's great theorem for regular prime exponents. The presentation will be detailed and kept on an elementary level, up to a few facts about Dedekind rings which at this stage will be stated without proof but applied, thus both explaining their origin, proving their strength, and motivating their importance.

Part 2 develops the ideal theory of Dedekind domains in the ring theoretic axiomatic setting, leading up to the fundamental theorem of Dedekind domains and the notions of the ideal class group and the ideal class number of a Dedekind domain.

Part 3 explains how Dedekind domains arise naturally and in abundance in algebraic number theory, and that, for these Dedekind domains, the ideal class number is always finite. Since these rings include the rings of p-th cyclotomic integers for all prime numbers p, part 3 renders part 2 applicable to part 1, and thereby it closes the ring theoretic and ideal class group theoretic gaps that were left open in part 1.

Subdivision of lectures and literature. Part 1 (the first seven lectures) is covered by [1], which we will read fully. Part 2 (the subsequent four lectures) and part 3 (the last four lectures) are covered by selected portions of [2] and [3], according to the tentative plan below.

Plan of lectures 8-15, covering part 2 and part 3

Lecture	Date	Section in book	Content
L8	23/4	[2] I.2 (2.1)–(2.5)	Integrality
L9	26/4	[2] II.1 (1.1)–(1.6)	The notion of a Dedekind domain
L10	3/5	[2] II.1 (1.7)–(1.13)	Weakly invertible ideals
L11	7/5	[2] II.1 Theorem 2 and Theorem 3	Fundamental theorem of Dedekind domains
L12	10/5	[2] II.1 Theorem 5 with Corollary	Dedekind domains of algebraic integers
L13	14/5	[3] (3.4)–(3.10) and Theorem 3.5	$\mathbb{Z}[\zeta]$ is a Dedekind domain
L14	16/5	[2] IV.1 Theorem 31	Finiteness of the ideal class number
L15	21/5	[2] II.1 (1.31)–(1.33) and III.1 Theorem 20	Quadratic number fields, ramification