UPPSALA UNIVERSITY
DEPARTMENT OF MATHEMATICS
ERNST DIETERICH

ALGEBRAIC NUMBER THEORY
SPRING TERM 2012
MARCH 22

# Home assignments

SECOND SET

5. Show that if $p$ is a prime number, then $p$ divides $\binom{p}{i}$ for all $1 \leq i \leq p-1$.

6. Prove the following

**Proposition 3.** *If $a$ and $b$ are elements in a commutative ring $R$ and $p$ is a prime number, then*

$$(a+b)^p \equiv a^p + b^p \pmod{p}.$$

*If moreover* $\mathrm{char}(R) = p$, *then*

$$(a+b)^p = a^p + b^p.$$

7. Let $p$ be a prime number and $\zeta = e^{\frac{2\pi}{p}i}$. Show that there exists a cyclotomic unit $\varepsilon \in \mathbb{Z}[\zeta]^\iota$ for which the equation $\xi^p = \varepsilon$ has no solution $\xi \in \mathbb{Z}[\zeta]$.

8. Let $R$ be a commutative ring.

(a) What does it mean that an element $r \in R$ is prime? (Reproduce the definition!)

(b) What does it mean that an ideal $I \subset R$ is prime? (Reproduce the definition!)

(c) What does it mean that a commutative ring $S$ is an integral domain? (Reproduce the definition!)

Prove that the following statements (d) and (e) hold true for every element $r \in R$, and for every ideal $I \subset R$.

(d) $r$ is a prime element if and only if $(r)$ is a prime ideal.

(e) $I$ is a prime ideal if and only if $R/I$ is an integral domain.

*Every exercise gives at most 5 points. Your assignments should be handed in to me or my mailbox not later than Thursday, 29 March, 10 a.m. Delayed exercises will in general be ignored. Exceptions are possible, but they require your explanation and my approval in advance.*