Uppsala University
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Algebraic number theory
Spring term 2012
April 26

Home assignments

Fifth set

In all exercises, p is a prime number and $\zeta = e^{\frac{2\pi}{p}i}$.

17. Every non-zero ideal $I < \mathbb{Z}[\zeta]$ has a unique factorization $I = \prod_{i=1}^{\ell} P_i^{m_i}$ into distinct prime ideals $P_1, \ldots, P_\ell$. The exponent $m_i = \mathrm{mult}_I(P_i)$ is called the *multiplicity* of P_i in I. Show that the following statements hold true for all non-zero ideals $I < \mathbb{Z}[\zeta]$ and all $m \in \mathbb{N}$.

(a) If $(1-\zeta)^m \supset I$, then $m \leq \mathrm{mult}_I(1-\zeta)$.

(b) If $(1-\zeta)^m \supset I$ and $(1-\zeta)^{m+1} \not\supset I$, then $m = \mathrm{mult}_I(1-\zeta)$.

18. Let $m \geq 2$ be a natural number. Let $x, y, z \in \mathbb{Z}[\zeta] \setminus (1-\zeta)$ be cyclotomic integers satisfying the identity of principal ideals

$$\prod_{i=0}^{p-1}(x + \zeta^i y) = (1-\zeta)^{pm}(z)^p.$$

Show that if $\mathrm{mult}_{(x+y)}(1-\zeta) = p(m-1)+1$ and $\mathrm{mult}_{(x+\zeta^i y)}(1-\zeta) = 1$ for all $1 \leq i \leq p-1$, then there exist ideals $C_0, \ldots, C_{p-1} < \mathbb{Z}[\zeta]$ such that

(a) $(x+y) = (1-\zeta)^{p(m-1)+1}(x,y)C_0$,

(b) $(x+\zeta^i y) = (1-\zeta)(x,y)C_i$ for all $1 \leq i \leq p-1$, and

(c) $(1-\zeta) \not\supset C_i$ for all $0 \leq i \leq p-1$.

19. Let N, Z and $C_0, \ldots, C_{p-1}$ be non-zero ideals in $\mathbb{Z}[\zeta]$ such that

$$N^p \prod_{i=0}^{p-1} C_i = Z^p.$$

Assume moreover that $C_0, \ldots, C_{p-1}$ are pairwise relatively prime. Prove that there exist ideals $D_0, \ldots, D_{p-1}$ in $\mathbb{Z}[\zeta]$ such that $C_i = D_i^p$ for all $0 \leq i \leq p-1$.

Please turn over!

20. Let $C_0, \ldots, C_{p-1}$ and $D_0, \ldots, D_{p-1}$ be as in exercise 19. Assume in addition that p is regular, $(1-\zeta) \not\supset C_i$ for all $0 \leq i \leq p-1$, and that the fractional ideals $(D_i D_0^{-1})^p$ are principal fractional for all $1 \leq i \leq p-1$. Prove that for each $1 \leq i \leq p-1$ there exist cyclotomic integers $\alpha_i, \beta_i \in \mathbb{Z}[\zeta] \setminus (1-\zeta)$ such that
$$D_i D_0^{-1} = \mathbb{Z}[\zeta] \frac{\alpha_i}{\beta_i}.$$

Every exercise gives at most 5 points. Your assignments should be handed in to me or my mailbox not later than Thursday, May 3, 10 a.m. Delayed exercises will in general be ignored. Exceptions are possible, but they require your explanation and my approval in advance.