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Prov i matematik
Algebraiska strukturer
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*Skrivtid: 8.00-13.00. Inga hjälpmedel förutom skrivdon. Lösningarna skall åtföljas av förklarande text! Varje uppgift ger maximalt 5 poäng.*

1. (a) Define Klein's four-group $\mathrm{V}_4$ and the symmetric group $\mathrm{S}_3$ respectively.

(b) Show that the groups $\mathrm{Aut}(\mathrm{V}_4)$ and $\mathrm{S}_3$ are isomorphic.

2. Classify all abelian groups of order 1080.

3. (a) Reproduce the statements of the three Sylow theorems.

(b) Show that every group of order 56 has a nontrivial proper normal subgroup.

4. A ring is called *simple* if it has precisely two two-sided ideals. Decide for each of the following rings $R$ whether they are simple or not, and prove your statements.

(a) $R = \{0\}$ ;

(b) $R = \mathbb{Z}$ ;

(c) $R = K^{n \times n}$, where $K$ is a field and $n \in \mathbb{N} \setminus \{0\}$.

5. Let $R$ be a commutative ring.

(a) Show that for each $r \in R$, the substitution map

$$\sigma_{X+r} : R[X] \to R[X],\ \sigma_{X+r}(a(X)) = a(X+r)$$

is an automorphism of the polynomial ring $R[X]$.

(b) Exhibit a subgroup $H < \mathrm{Aut}(R[X])$, together with a group isomorphism $\gamma : (R,+) \tilde{\to} H$.

Please turn over!

6. (a) Reproduce the definition of the notion "domain".

(b) Reproduce the definition of the notion "field of fractions of a domain".

(c) Show that the ring of Gaussian integers $\mathbb{Z}[i] = \{a+bi \mid a, b \in \mathbb{Z}\}$ is a domain.

(d) Show that the field of fractions of $\mathbb{Z}[i]$ is isomorphic to the ring of Gaussian numbers $\mathbb{Q}[i] = \{u + vi \mid u, v \in \mathbb{Q}\}$.

7. Find the addition table and the multiplication table of a field of order 9.

8. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi}{17}i}$.

(a) Explain why $\mathbb{Q} \subset E$ is a finite Galois extension.

(b) Determine $\mathrm{Gal}(E/\mathbb{Q})$, up to isomorphism.

(c) Describe all subgroups of $\mathrm{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(d) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

LYCKA TILL!