1. (a) Define the dihedral group $D_n$ for any natural number $n \geq 2$.
(b) Show that every non-trivial morphism $\varphi : D_2 \to D_3$ has the property $|\ker \varphi| = 2 = |\text{im} \varphi|$.
(c) Use (b) to find the number of morphisms from $D_2$ to $D_3$.

2. (a) The centre of a group $G$ is a subset $Z(G) \subset G$. Which one? Reproduce its definition!
(b) Show that $Z(G)$ always is a normal subgroup of $G$.
(c) What can you say about $Z(G)$ in case $G$ has prime squared order? Motivate your answer!

3. Classify all groups of order 529.

4. (a) What is a subring of a ring $R$? Reproduce the definition!
(b) Show that $H = \left\{ \begin{pmatrix} w & z \\ -z & w \end{pmatrix} \middle| w, z \in \mathbb{C} \right\}$ is a subring of the ring $\mathbb{C}^{2 \times 2}$ of all complex $2 \times 2$-matrices.

5. (a) Reproduce the definition of a unit (also called invertible element) of a ring $R$, and show that the set $R^*$ of all units in $R$ is a multiplicative group.
(b) Determine the unit group $R^*$ for each of the following rings $R$, and motivate your description: (i) $R = \mathbb{Z}$, (ii) $R = \mathbb{Q}[X]$, (iii) $R = \mathbb{R}^{3 \times 3}$, (iv) $R = H$, the ring defined in problem 4(b).

6. (a) Reproduce the cubic formula, expressing the roots of a complex cubic $f(X) = X^3 + qX + r$ in terms of its coefficients $q$ and $r$.
(b) Express the roots of the cubic $f(X) = X^3 + 3X + 2$ in terms of its coefficients 3 and 2.

Please turn over!
7. Let $E = \mathbb{Q}(\zeta)$, where $\zeta = e^{\frac{2\pi i}{19}}$.

(a) When is a field extension called *separable*, when is it called *normal*, and when is it called *Galois*? Reproduce the definitions!

(b) Explain why $\mathbb{Q} \subset E$ is a finite Galois extension.

(c) Determine $\text{Gal}(E/\mathbb{Q})$, up to isomorphism.

(d) Describe all subgroups of $\text{Gal}(E/\mathbb{Q})$, ordered by inclusion.

(e) Describe all intermediate fields $\mathbb{Q} \subset F \subset E$, ordered by inclusion.

8. Let $f(X) = X^5 - 4X + 2 \in \mathbb{Q}[X]$. Let $R = \{\alpha_1, \ldots, \alpha_5\}$ be the set of all complex roots of $f(X)$, set $E = \mathbb{Q}(\alpha_1, \ldots, \alpha_5)$, and denote for every $\sigma \in \text{Gal}(E/\mathbb{Q})$ by $\sigma_R$ the permutation of $R$ induced by $\sigma$. Give reasons for each of the following statements.

(a) The polynomial $f(X)$ is irreducible in $\mathbb{Q}[X]$.

(b) All the roots $\alpha_i$ are simple.

(c) There is a $\tau \in \text{Gal}(E/\mathbb{Q})$ such that $\tau_R$ is a transposition.

(d) There is a $\gamma \in \text{Gal}(E/\mathbb{Q})$ such that $\gamma_R$ is a 5-cycle.

(e) The equation $f(X) = 0$ is not solvable by radicals.

*Good luck!*