Applied Mathematics.

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Contents

Some words

These lecture notes are based mainly on the book **Applied Mathematics: Third edition** written by J. David Logan and on the lecture notes written by Professor Lars-Erik Persson, see his web page `http://staff.www.ltu.se/~larserik/`. The main purpose of these notes is to summarize all the topics covered in the course *Tillämpad Matematik* taught at Uppsala University by the author during the Fall 2014. I strongly recommend to go the sources for a better and further exposition on the selected topics.

Notice that in these lecture notes, a lot of exercises appear. I follow the idea that mathematics is learnt through exercises!

Chapter 1

What is Applied Mathematics.

Applied mathematics is a broad subject area dealing with those problems that come *from the real world*. Applied mathematics deals with all the stages for solving these problems, namely:

1. Given a problem, formulate a mathematical model that describes it.

2. By means of analytical or numerical methods, solve the model.

3. Compare the model's results with experimental results. In case that they disagree qualitatively, go back and reformulate the problem.

This previous process is summarized in Figure 1.1.

So, let's rephrase what it means to work in Applied Mathematics: Given a real world problem, we seek for a solution of it. In order to get it, first, we need to propose a model (mathematical model) that describes it. Then, we need to understand this model. This is done by solving it. Once we understand it, we compare the solutions with the output experiments. If they agree, we will say that the model describes the phenomenon. Otherwise, we should rethink the model. Usually, this rethinking process means that, while we constructed the first model, we discarded some things in order of getting a simple model.

Finally, let me say some words about what we will achieve during this course. We will learn how to deal with several of the steps involved in this process. In Chapter 2 we will work on several techniques used in the formulation of the mathematical model, while in the others we will mainly focus on the step of solving it by means of analytical methods.
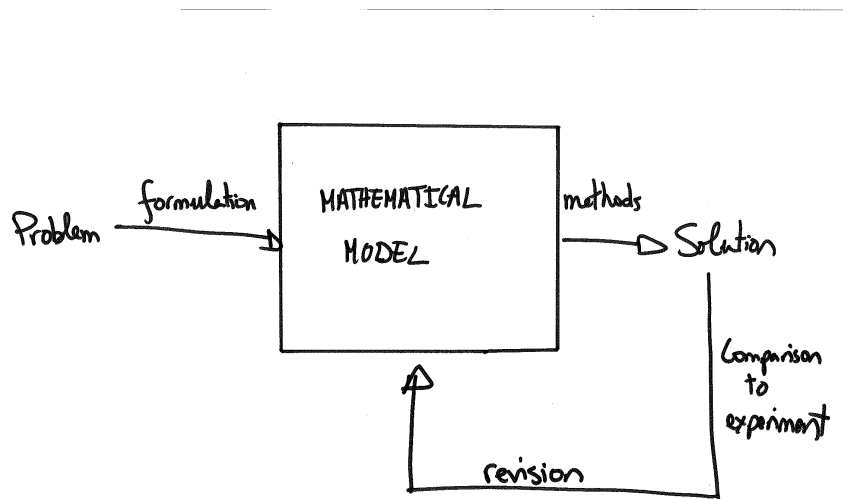
Figure 1.1: Schematic representation of the stages involving the finding for a solution of a real world problem.

Part I

Mathematical modelling

Dimensional analysis and scaling methods deal with the first stage in applied mathematics: finding a mathematical model. With the help of these we can try to construct a mathematical model or, at least, enlight some of the properties of the problem that we have in hand.

Chapter 2

Dimensional analysis and Scaling.

Dimensional analysis is a useful tool for finding mathematical models when the physical law we are studying is *unit free*.

2.1   Dimensions and units.

In dimensional analysis we should distinguish between two different but related concepts: *dimension* and *unit*. A **dimension** is a measure of a variable while a **unit** is a measurement to that.

Example 1. *Time is a dimension, while seconds is a unit.*

Example 2. *Happiness is a dimension, while smiles is a unit.*

A set of **fundamental dimensions (units)** is a set of dimensions (units) from which every dimension (unit) can be generated.

For example, in the SI system, there are seven fundamental units, *kilogram, meter, candela, second, ampere, kelvin and mole* related to seven fundamental dimensions, *mass, length, luminous intensity, time, electric current, temperature and amount of chemical substance.* See Table 2.1 for a detailed presentation of them.

Observation 3. *Usually we use the SI system in physical problems but other sets of fundamental dimensions must be used in other contextes. For example, it could happen that in economics we use dimensions like: population, wealth, happiness...*

A **derived dimension (unit)** is a dimension (unit) that is dimensionless or expressed as product of fundamental dimensions (units).

For example, in the SI system there are plenty of them: velocity, acceleration, frequency, energy, force... See Table 2.2 for other examples.

Fundemantal Unit	Dimension	Symbol
kilogram	Mass	M
meter	Length	L
candela	Luminous intensity	C
second	Time	T
ampere	Electric current	A
kelvin	Temperature	K
mole	Amount of chemical substance	S

Table 2.1: SI fundamental dimensions. Disclaim: the third column contains a non-standard notation for the symbols.

Derived Unit	Dimension	Equivalence to fundamental dimensions
Herz	Frequence	$1/T$
Radian	Angle	1
Newton	Force	ML/T^2
Pascal	Pressure	$M/(LT^2)$
Joule	Energy, work, heat	ML^2/T^2
Watt	Power	ML^2/T^3
Coulomb	Electric charge, quantity of electricity	AT
Volt	Electrical potential difference	$ML^2/(T^3A)$
Farad	Electrical capacitance	$A^2T^4/(ML^2)$
Ohm	Electrical resistance	$ML^2/(T^2A)$
Lux	Illuminance	$C/(M^2)$

Table 2.2: Some SI derived units with respect fundamental dimensions. Dimensionless dimensions are expressed as 1.

2.2 Laws and unit free laws.

Let's define what is a law.

A **law** is defined as the zero set of a function that depends on n variables $q_1, \ldots, q_n$ in $m < n$ fundamental dimensions $L_1, \ldots, L_m$

$$f(q_1, \ldots, q_n) = 0.$$

The dimensions of q_i, denoted by $[q_i]$, are specified explicitly by

$$[q_i] = L_1^{a_{1,i}} \cdots L_m^{a_{m,i}}.$$

With these, we can create **the dimension matrix**. It is the $n \times m$ matrix with integer coefficients

$$\begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ a_{2,1} & \cdots & a_{2,n} \\ \vdots & \ddots & \vdots \\ a_{m,1} & \cdots & a_{m,n} \end{pmatrix}.$$

The definition of law looks a little bit curious, doesn't it? Let's see some examples of laws:

Example 4 (Energy preservation)**.** *Given a system that depends on the position (q), velocity (p) and mass (m), the law of energy preservation in its most classical setting says that the sum of the kinetical and potential energy is constant. That is,*

$$m\frac{p^2}{2} + V(q) = C.$$

Thus, in this example, the function f depends on three variables, p, q, m and it is

$$f(p, q, m) = m\frac{p^2}{2} + V(q) - C.$$

Example 5 (Hooke's law)**.** *The force F needed to extend (or compress) a spring by some distance L is proportional to this distance. That is,*

$$F = kL.$$

Hence, the function f in this case is

$$f(F, L) = F - kL.$$

Notice that Hooke's law implies that the constant k is not dimensionless. This observation should be keeped in mind.

Example 6 (Atomic explosion)**.** *Supose that there is an atomic explosion. In such an explosion a lot of energy E is released instantaneoulsy in a point. A shockwave is then propagated from it. In this process, we assume that the radius r of the shockwave, the air density ρ, the time t and the energy E are the only dimensions that are involved in the law of how the shockwave propagates. Then, we have*

$$f(r, t, \rho, E) = 0.$$

Now that we have seen plenty of examples of laws, and seen that to all laws there is a function associated to it, could you think of a law that has no f related to it? It is hard to imagine it. Once we talk about relations between dimensions/units/quantities, equations appear. And, from each equation, we get a law!

Laws are important because they give as relations between the variables involved. If we know the law, then we know exactly their relation, but just knowing that there is a law tells us that there is some relation.

A **unit free law** is a law that does not depend on the choice of units. More concretely, given a law that depends on n quantities $q_1, \ldots, q_n$ and $m < n$ units $L_1, \ldots, L_m$,

$$f(q_1, \ldots, q_n) = 0,$$

and for any n $\lambda_i > 0$, the law is also true for the new variables $\hat{q}_i$ formed by the new units $\hat{L}_i = \lambda_i L_i$. That is,

$$f(\hat{q}_1, \ldots, \hat{q}_n) = 0,$$

Example 7. *An example of a unit free law is*

$$f(x, g, t) = x - \frac{1}{2}gt^2 = 0, \tag{2.1}$$

where x denotes position (L), g the constant of the gravitational field (L/T^2) and t time (T).

If $\hat{L} = \lambda_1 L$, $\hat{T} = \lambda_2 T$ then, since g has units in L/T^2, we get that

$$f(\hat{x}, \hat{g}, \hat{t}) = 0$$

if and only if Equation (2.1) is also satisfied.

2.3 Pi theorem.

Theorem 8. *Let*

$$f(q_1, \ldots, q_n) = 0$$

be a unit free physical law that relates the dimensioned quantities $q_1, \ldots, q_n$. Let $L_1, \ldots, L_m$ (where $m < n$) be the fundamental dimensions with

$$[q_i] = L_1^{a_{1,i}} \cdots L_m^{a_{m,i}}.$$

and let $r = rank(A)$, where A is the dimension matrix. Then there are $n - r$ independent dimensionless quantities

$$\pi_1, \ldots, \pi_{n-r}$$

that can be formed from $q_1, \ldots, q_n$. That is, for all i,

$$\pi_i = q_1^{\alpha_{1,i}} \cdots q_n^{\alpha_{n,i}}.$$

Moreover, the physical law above is equivalent with an equation

$$F(\pi_1, \ldots, \pi_{n-r}) = 0$$

which is solely expressed in dimensionless quantities.

I will not prove this theorem. If you are interested in seeing a proof, please have a look at Logan's book.

What it is important in this theorem is the information that we can get from it. Let's discuss several examples:

Example 9. *Suppose that a given a unit free law can be reduced to a just one dimensionless variable π_1. Then, Pi Theorem states that this law is equivalent to a law with the form*

$$F(\pi_1) = 0,$$

with $\pi_1 = q_1^{\alpha_{1,1}} \cdots q_n^{\alpha_{n,1}}$. Now, since we can suppose that, generally, zeros of functions of one variable are discrete, then π_1 can only attain discrete values. Hence,

$$\pi_1 = C,$$

with C a constant. This means that we get a relation between the variables q_i of the form

$$q_1^{\alpha_{1,1}} \cdots q_n^{\alpha_{n,1}} = C.$$

This constant C can then be determined by means of experiments.

Example 10. *Suppose now that given a unit free law Pi theorem asserts that there are two dimensionless variables π_1, π_2. Then, this law is equivalent to one with*

$$F(\pi_1, \pi_2) = 0.$$

Now, since the zero set of a function of two variables is, generically, a curve, and using (if possible!) the Implicit Function Theorem we get a relation between π_1 and π_2 of the form

$$\pi_1 = g(\pi_2),$$

with unknown function g. This function g can be deteremined by means of experiments.

The previous examples use the following principle [1] that can be deduced using the Implicit Function Theorem:

Principle 11. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Then, the zero set $\{z \in \mathbb{R}^n : f(z) = 0\}$ is, generically, a $n - 1$ dimensional hypersurface. Furthermore, if $z = (z_1, \ldots, z_n)$, for most of the z_i there exists a function $g_i : \mathbb{R}^{n-1} \to \mathbb{R}$ such that the zero set is locally equivalent to the solutions of the equation*

$$z_i = g_i(z_1, \ldots, z_{i-1}, z_{i+1}, \ldots, z_n).$$

2.3.1   Example 1: Atomic bomb.

As we saw in Example 6, there is a unit free law

$$f(r, t, \rho, E) = 0.$$

that depends on the radius r, the air density ρ, the time t and the energy E. Now, $[t] = T$, $[r] = L$, $[E] = ML^2/T^2$ and $[\rho] = M/L^3$. The dimension matrix is

$$\begin{pmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Notice that there are $n = 4$ dimensioned quantities and the rank of the dimension matrix is 3. Hence, Pi Theorem asserts that there is just 1 dimensionless quantity π_1 that can be formed from these 4 quantities. Also, the law is equivalent to

$$F(\pi_1) = 0.$$

With a little bit of algebra we get that

$$\pi_1 = \frac{r^5 \rho}{t^2 E}$$

so, we deduce from Principle 11 that

$$\frac{r^5 \rho}{t^2 E} = C,$$

where C is (an unknown) constant.

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[1]I distinguish between principles and theorems. The former are vague versions of the latter.

2.3.2 Example 2: Heat transfer problem.

At time $t = 0$ an amount of heat energy e, concentrated at a point, is released into a region at temperature 0. We want to determine the temperature u as a function of r and t. Other quantities that play a role are the heat capacity c of the region and the thermal diffusivity k.

As in the previous example we have that the dimensions are $[t] = T$, $[r] = L$, $[u] = K$, $[e] = E$, $[c] = EK^{-1}L^{-3}$ and $[k] = L^2/T$. [2]

The dimension matrix is

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & -3 & 2 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

and, since the number of dimensioned quantities is $n = 6$ and the rank of the dimension matrix is 4, the Pi Theorem asserts that there are two independent dimensionless quantities. Doing some algebra we get

$$\pi_1 = \frac{r}{\sqrt{kt}}$$

and

$$\pi_2 = \frac{uc}{e}(kt)^{\frac{3}{2}}.$$

Finally, using the Principle 11 we have that there exists a smooth (but unknown) function g such that

$$\frac{uc}{e}(kt)^{\frac{3}{2}} = g\left(\frac{r}{\sqrt{kt}}\right).$$

So, the temperature u behaves like

$$u = \frac{e}{c}(kt)^{-\frac{3}{2}}g\left(\frac{r}{\sqrt{kt}}\right).$$

2.4 Scaling.

Scaling is another procedure useful in formulating mathematical models. Scaling is about scaling the variables in their correct magnitude. A lot of systems evolve in time but not all of them are *well measured* if we use seconds. For example, it is not the same measuring time when we study galaxies or when we study atomic reactions. Another example could be measuring distances: Galaxies and atomic reactions are not measured using the same scale.

Every problem has its own scales (for each of the dimensions). And this scale, called the **characteristic scale** is the one that should be used.

Once the characteristic scale is identified, a new dimensionless variable is formed by dividing the former with the latter. For example, in the case of time in galaxies, the

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[2]Notice that in this problem the heat energy is a fundamental dimension, since it can not be deduced from the others.

characteristic scale could be something around $t_c = 10^6$ years, and the dimensionless time will be

$$\bar{t} = \frac{t}{t_c}.$$

After scaling all the variables of the model in hand, we get a dimensionless form of the problem. This process is called **non-dimensionalization**.

Part II

Analytical methods.

Chapter 3

Perturbation methods.

Perturbation methods are used for studying problems that are close to a known problem.

Example 12. *When considering the motion of planets, it is well known that the 2 body problem (e.g. Sun-Earth system) is a problem with a well known solution: the bodies orbit around their center of mass along elliptical orbits. In this setting, if we consider the problem of 3 masses with one of them much smaller with respect the other two (e.g. Sun-Earth-satellite system) then we have a perturbed system.*

The idea behind the perturbation methods is computing approximate solutions of the system in terms of Taylor (or other) expansions.

Example 13. *Consider the equation*

$$x^2 - 1 + \varepsilon x = 0.$$

For $\varepsilon = 0$ the equation has solutions $x(0) = \pm 1$. Without loss of generality, set $x(0) = 1$. It is natural to expect that for values of ε small enough, solution to the equation will be close to $x(0) = 1$. If we do the ansatz that the solutions can be written in Taylor form

$$x(\varepsilon) = x_0 + x_1 \varepsilon + x_2 \varepsilon^2 + \cdots$$

then we have that $x_0 = x(0) = 1$.

Question 14. *Could you compute the terms x_1 and x_2 that appear on the previous example?*

Generally, perturbation methods deal with equations of the form

$$F(t, y, y', y'', \ldots, y^{n)}, \varepsilon) = 0,$$

satisfying that for $\varepsilon = 0$ a solution to it is known.

Observation 15. *Perturbation methods do not only deal with ODEs, but also with PDEs, integral equations... In general, pertubation methods deal with all type of equations that depend on a small parameter ε for which a solution is known when $\varepsilon = 0$.*

3.1   Regular perturbations.

The basic idea behind regular perturbations is the one behind Example 13: We do not need to perform any change in the equation and the Taylor expansion works fine.

Example 16. *Consider the initial value problem*

$$\begin{cases} mv' &= -av + bv^2 \\ v(0) &= V_0 \end{cases},$$

with $b \ll a$.

First, we introduce dimensionless variables

$$y = \frac{v}{V_0}, \quad \tau = \frac{at}{m},$$

obtaining the scaled initial value problem

$$\begin{cases} \dot{y} &= -y + \varepsilon y^2 \\ y(0) &= 1 \end{cases}, \tag{3.1}$$

where $\varepsilon = \dfrac{bV_0}{a} \ll 1$.

After this change of variables, the solution to Equation (3.1) when $\varepsilon = 0$ is

$$y_0(t) = e^{-t}.$$

Now, performing the ansatz that solutions to Equation (3.1) are of the form

$$y(t) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + \cdots$$

and substituting it into Equation (3.1) we obtain

$$y_0'(t) + \varepsilon y_1'(t) + \varepsilon^2 y_2'(t) + h.o.t. = -y_0(t) - \varepsilon y_1(t) - \varepsilon^2 y_2(t) + \varepsilon \left(y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) \right)^2 + h.o.t.$$

which is equivalent to,

$$y_0'(t) + \varepsilon y_1'(t) + \varepsilon^2 y_2'(t) + h.o.t. = -y_0(t) - \varepsilon y_1(t) - \varepsilon^2 y_2(t) + \varepsilon y_0(t)^2 + \varepsilon^2 2 y_0(t) y_1(t) + +h.o.t.$$

From this last equality we get

$$\begin{aligned} y_0(t) &= e^{-t}, \\ y_1(t) &= e^{-t} - e^{-2t}, \\ y_2(t) &= e^{-t} - 2e^{-2t} + e^{-3t}. \end{aligned}$$

Anders Lindstedt. Swedish mathematician. 1854-1939.

Henri Poincaré. French mathematician. 1854-1912.

3.1.1 Poincaré-Lindstedt method.

The Poincaré-Lindstedt method is used for uniformly approximate periodic solutions in perturbed systems, when the period of the perturbed periodic solution changes with respect ε.

Example 17. *Consider the Duffing equation*

$$x'' + x + \varepsilon x^3 = 0$$

with initial conditions

$$x(0) = 1, x'(0) = 0.$$

This equation has solution $x_0(t) = \cos(t)$ for $\varepsilon = 0$.

If we perform the ansatz that the perturbed solution is of the form

$$x(t) = x_0(t) + \varepsilon x_1(t) + h.o.t. \tag{3.2}$$

then we get that

$$x_1(t) = \frac{1}{32}(\cos(3t) - \cos(t)) - \frac{3}{8}t\sin(t) \ .$$

*Notice that x_1 contains the term $t\sin(t)$ (called **the secular term**). Since the Taylor expansion in Equation (3.2) should approximate a periodic orbit, there is a problem with this secular term. It does not approximate any periodic solution!*

The way we overcome this is by also letting power series in the time variable. Let

$$\tau = \omega t, \ \text{where } \omega = \omega_0 + \varepsilon\omega_1 + h.o.t.$$

Notice that $\omega_0 = 1$. Using the change of coordinates $\tau = \omega t$ we get the initial value problem

$$\begin{cases} \omega^2 \ddot{x}(\tau) + x(\tau) + \varepsilon x(\tau)^3 = 0, \\ x(0) = 1, \\ \dot{x}(0) = 0 \end{cases} \ .$$

Then, expanding in power series

$$x(\tau) = x_0(\tau) + \varepsilon x_1(\tau) + h.o.t.$$

and using the series expansion of τ we obtain

$$\ddot{x}_0(\tau) + x_0(\tau) = 0, \quad x_0(0) = 1, \quad \dot{x}_0(0) = 0$$
$$\ddot{x}_1(\tau) + x_1(\tau) = -2\omega_1 \ddot{x}_0(\tau) - x_0(\tau)^3, \quad x_1(0) = \dot{x}_1(0) = 0.$$

Their solutions are

$$x_0(\tau) = \cos(\tau),$$
$$x_1(\tau) = \frac{1}{32}\left(\cos(3\tau) - \cos(\tau)\right) + \left(\omega_1 - \frac{3}{8}\right)\tau\sin(\tau).$$

Notice that choosing $\omega_1 = \dfrac{3}{8}$ we avoid the secular term.

Exercise 18. *Prove that the Duffing equation has periodic orbits with initial conditions $x(0) = 1, x'(0) = 0$ for all $|\varepsilon| \ll 1$.*

Exercise 19. *Consider the ODE*

$$x'' + x + \varepsilon x^5 = 0$$

with initial conditions

$$x(0) = 1, x'(0) = 0.$$

Use the Poincaré-Lindstedt method for computing the periodic solutions with that initial conditions.

3.1.2   Big O and little o notation.

We write $f(s) = O(g(s))$, as $s \to A$ if there exists a constant $C > 0$ such that for all $s \approx A$

$$|f(s)| < C|g(s)|.$$

This is the so-called **big O notation**.
    We write $f(s) = o(g(s))$, as $s \to A$ if

$$\lim_{s \to A} \frac{f(s)}{g(s)} = 0.$$

This is the so-called **little o notation**.

Exercise 20. *Prove the following assertions:*

   1. $x^2 = o(x)$ *as $x \to 0$.*

2. $\sin(x) = O(x)$ *as* $x \to 0$.

3. $\ln(1 + x) = O(x)$ *as* $x \to 0$.

4. *For all* $\varepsilon > 0$, $\ln(x) = o(x^{\varepsilon})$ *as* $x \to +\infty$.

This notation is very useful because it help us to specify how, among other things, the approximate solutions computed using perturbation methods approximate the true solutions.

Example 21. *In Example 17 it could be proved that the true solution* $x(\tau)$ *and the approximate solution* $x_0(\tau) + \varepsilon x_1(\tau)$ *satisfy*

$$x(\tau) - (x_0(\tau) + \varepsilon x_1(\tau)) = O(\varepsilon^2).$$

3.2 Singular perturbations.

Singular could perturbations appear when some of the following happen:

1. The small parameter multiplies the highest derivative in an ODE. For example,

$$\varepsilon y'' + y' + y^3 = 0.$$

2. The small parameter multiplies the term with highest degree in an algebraic equation. For example,
$$\varepsilon x^4 + x + 1 = 0.$$

3. The problem occurs in infinite domains.

4. When singular points occur in the domain of interest.

5. When the equations have multiple scales. For example,

$$\begin{cases} \varepsilon x' &= f(x, y) \\ y' &= g(x, y) \end{cases}.$$

Let's give an explicit example.

Example 22. *Consider the algebraic equation*

$$\varepsilon x^5 + x - 1 = 0. \tag{3.3}$$

Notice that for $\varepsilon = 0$ *Equation (3.3) has just one solution,* $x = 1$, *but for* $\varepsilon \neq 0$ *it has 5 solutions. Hence, there is a family of solutions given by the Taylor expansion*

$$x(\varepsilon) = 1 + \varepsilon x_1 + \varepsilon^2 x_2 + h.o.t.,$$

but the four other solutions are missing. What is wrong with Equation (3.3)? It is wrong that the degree of the polynomial changes with respect the value of ε.

Let's try to find a way of computing them. Perform a change of variables

$$x = \frac{y}{f(\varepsilon)},$$

with unknown function $f(\varepsilon)$. Applying this change of variables to Equation (3.3) we obtain the equation

$$f(\varepsilon)^{-4}\varepsilon y^5 + y - f(\varepsilon) = 0. \tag{3.4}$$

Now, choosing $f(\varepsilon) = \varepsilon^{\frac{1}{4}}$ the leading term of Equation (3.4) is 1 and we obtain

$$y^5 + y - \varepsilon^{\frac{1}{4}} = 0.$$

Now, for this last equation we can perform a regular perturbation analysis and obtain that there are five solutions when $\varepsilon = 0$:

$$0,\ e^{2\pi i\frac{1}{4}},\ e^{2\pi i\frac{1}{2}},\ e^{2\pi i\frac{3}{4}},\ 1$$

We discard $y = 0$ because it is not a solution to Equation (3.3). Consequently, we get four solutions for x:

$$\varepsilon^{-\frac{1}{4}}e^{2\pi i\frac{1}{4}},\ \varepsilon^{-\frac{1}{4}}e^{2\pi i\frac{1}{2}},\ \varepsilon^{-\frac{1}{4}}e^{2\pi i\frac{3}{4}},\ \varepsilon^{-\frac{1}{4}}.$$

Now, if we want a Taylor-like expansion of these four roots, we proceed as in the regular case for the y variable and obtain, for each of them, an expansion of the form

$$y(\varepsilon) = y_0 + \varepsilon^{\frac{1}{4}}y_1 + (\varepsilon^{\frac{1}{4}})^2 y_2 + h.o.t.$$

using Equation (3.4).

Exercise 23. *Give a second order approximation of the following algebraic equations:*

1. $\varepsilon x^3 + x + 1 = 0$.

2. $\varepsilon x^3 + 1 = 0$.

3. $\varepsilon x^6 + x^2 - 2x - 2 = 0$.

For the case of how to solve ODEs with singular perturbations, see Sections 3.3 and 3.4.

3.3   Boundary layers.

We introduce the boundary layers method via an example.

Example 24. *Consider the boundary value problem*

$$\begin{cases} \varepsilon y'' + (1 + \varepsilon)y' + y = 0, \\ y(0) = 0, \\ y(1) = 1. \end{cases} \tag{3.5}$$

Notice that this example has an explicit solution,

$$y(x) = \frac{e^{-x} - e^{-\frac{x}{\varepsilon}}}{e^{-1} - e^{-\frac{1}{\varepsilon}}},$$

but we will only use it for checking if our approximation method succeeds. See Figure 3.1 for a visual representation of it. Observe in this figure how the solution behaves near the origin.
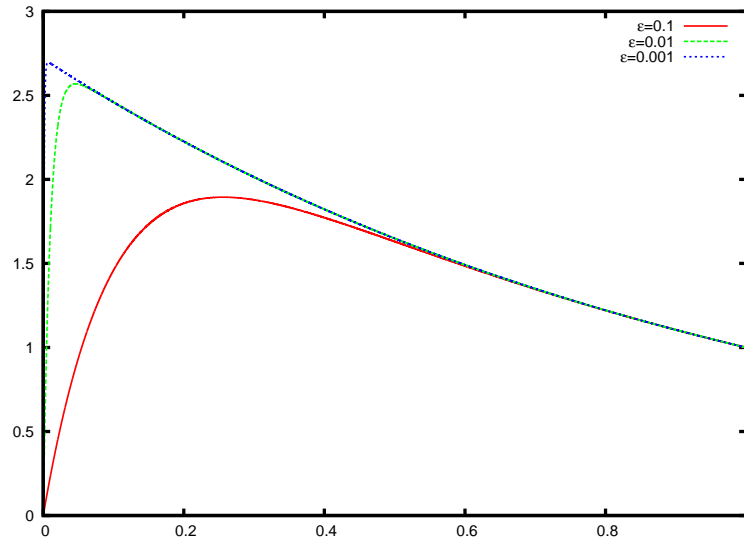
Figure 3.1: Graph of the function $y(x)$ for different values of the parameter ε.

The boundary value problem (3.5) satisfies that solving it as in the regular case, via Taylor expansions of the form

$$y(x) = y_0 + \varepsilon y_1 + \cdots ,$$

then $y_0 = Ce^{-x}$, where C is a constant, does not satisfy both boundary values.

This is overcome by approximating the solution to (3.5) by inner and outer layers.

Outer layer:

The outer layer is the one away $x = 0$. Since there $\varepsilon y''$ and $\varepsilon y'$ are small, it is computed using

$$\begin{cases} y_o' + y_o = 0 \\ y_o(1) = 1 \end{cases},$$

which has as solution $y_o(x) = e^{-x}$.

Inner layer:
We scale the boundary problem (3.5) with

$$\tau = \frac{x}{f(\varepsilon)}$$

obtaining

$$\frac{\varepsilon}{f(\varepsilon)^2}\ddot{y} + \frac{1+\varepsilon}{f(\varepsilon)}\dot{y} + y = 0. \tag{3.6}$$

This last ODE has coefficients

$$\frac{\varepsilon}{f(\varepsilon)^2}, \frac{\varepsilon}{f(\varepsilon)}, \frac{1}{f(\varepsilon)}, 1.$$

We will choose $f(\varepsilon)$ so that the leading coefficient $\frac{\varepsilon}{f(\varepsilon)^2}$ has the same order as another and the other two are small in comparison. This leads to the choice $f(\varepsilon) = \varepsilon$.
   Then, an approximate solution of Equation (3.6) of order $O(\varepsilon)$ is $y_i(x) = a(1 - e^{-\frac{x}{\varepsilon}})$.

   Now, the problem is finding the value of constant a in the inner approximation so the solutions match. This matching should be done outside the inner ($x = O(\varepsilon)$) and outer ($x = O(1)$) regions. For example, when $x = O(\sqrt{\varepsilon})$. We perform the change of variables $\nu = \frac{x}{\sqrt{\varepsilon}}$ and impose the condition

$$\lim_{\varepsilon \to 0} y_i(\sqrt{\varepsilon}\nu) = \lim_{\varepsilon \to 0} y_o(\sqrt{\varepsilon}\nu).$$

With this, we obtain that $a = e$.

Exercise 25. *Perform the same analysis as in Example 24 in the initial value problem*

$$\begin{cases} \varepsilon y' + y = e^{-x}, \\ y(0) = 0. \end{cases}$$

3.4   The WKB approximation.

The WKB method (Wentzel-Kramer-Brillouin) is a perturbation method that applies to problems of the form:

$$\varepsilon^2 y'' + q(x)y = 0, \, 0 < \varepsilon \ll 1 \tag{3.7}$$
$$y'' + (\lambda^2 p(x) - q(x))y = 0, \, \lambda \gg 1 \tag{3.8}$$
$$y'' + q(\varepsilon x)^2 y = 0. \tag{3.9}$$

Example 26. *The time-independent Schrödinger equation*

$$-\frac{\hbar^2}{2m}y'' + (V(x) - E)y = 0,$$

is an example where the WKB method can be applied.

Let's consider Equation (3.7). The method consists on doing the ansatz that the solution is of the form

$$y(x) = e^{\frac{u(x)}{\varepsilon}}.$$

Thus, we obtain equation

$$\varepsilon f' + f^2 + q(x) = 0,$$

where $f = u'$. Finally, using a regular perturbation of f

$$f = f_0 + \varepsilon f_1 + h.o.t.$$

we obtain that

$$f_0 = \pm\sqrt{q(x)},$$
$$f_1 = -\frac{q'(x)}{4q(x)}.$$

Hence, we obtain an approximation of the form

$$f(x) = \pm\sqrt{q(x)} - \varepsilon\frac{q'(x)}{4q(x)} + O(\varepsilon^2).$$

In terms of y it is

$$y(x) = \frac{1}{\sqrt[4]{q(x)}}e^{\pm\frac{1}{\varepsilon}\int_a^x \sqrt{q(x)}dx}(1 + O(\varepsilon)).$$

Exercise 27. *Apply the WKB method in the following equations:*

1. *$\varepsilon y'' + xy = 0$, $0 < \varepsilon \ll 1$.*

2. *$y'' + \lambda\cos(x)y = 0$, $1 \ll \lambda$.*

Chapter 4

Calculus of variations.

The calculus of variations deals with the study of minimizing/maximizing functionals. These are functions that map functions to the reals. Examples are: minimize the length of a curve, maximize the area given a fixed length, minimize the area...

4.1 Variational problems.

Recall that given a function

$$f : \mathbb{R}^n \to \mathbb{R}$$

a **local minimum** is defined as a point $x_0 \in \mathbb{R}^n$ such that in a neighbourhood of it

$$f(x_0) \leq f(x).$$

A global minimum will be a local minimum for all neighbourhoods. Similarly, we define local and global maxima.

If the function f is differentiable, a necessary condition of being a local minimum is that

$$\nabla f(x_0) = 0.$$

But, this is not a sufficient condition.

Exercise 28. *Prove the previous statement.*

Question 29. *Give an example of a function with a point x_0 satisfying that*

$$\nabla f(x_0) = 0$$

but not being a minimum or a maximum.

Definition 30. *Let $\mathcal{X}$ be a (normed) vector space formed by functions. A functional is a map*

$$J : \mathcal{X} \to \mathbb{R}.$$

Some vector spaces of functions are:

Notation 31.   • $\mathcal{C}^0(\mathcal{U}, \mathbb{R}^m)$, where $\mathcal{U} \subset \mathbb{R}^n$, is the space of continuous maps

$$f : \mathcal{U} \to \mathbb{R}^m$$

with norm

$$\|f\|_0 := \sup_{x \in \mathcal{U}} \|f(x)\|.$$

• $\mathcal{C}^r(\mathcal{U}, \mathbb{R}^m)$, where $\mathcal{U} \subset \mathbb{R}^n$, is the space of r-times differentiable maps

$$f : \mathcal{U} \to \mathbb{R}^m$$

with norm

$$\|f\|_r := \sum_{k=0}^{r} \|D^k f\|_0.$$

• $\mathcal{A}(\mathcal{U}, \mathbb{C})$, where $\mathcal{U} \subset \mathbb{C}$, is the space of analytic maps

$$f : \mathcal{U} \to \mathbb{C}$$

with norm

$$\|f\|_{\mathcal{U}} := \sup_{x \in \mathcal{U}} |f(x)|.$$

The problem that we are interested in is, given a functional, find its (local) minima or maxima.

Let's see some examples of functionals.

Example 32.   • Let $x_0 \in \mathbb{R}$, then $J : \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ with

$$J(f) := f(x_0)$$

is a functional.

• Let $a, b \in \mathbb{R}$, then $J : \mathcal{C}^0(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ with

$$J(f) := \int_a^b f(x)dx$$

is a functional.

• Let $a, b \in \mathbb{R}$, then $J : \mathcal{C}^2(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ with

$$J(f) := \int_a^b f(x)(f''(x)^2 + \cos(f(x)))dx$$

is a functional.

- (Arclength) Let $a, b \in \mathbb{R}$, then $J : \mathcal{C}^1(\mathbb{R}, \mathbb{R}) \to \mathbb{R}$ with

$$J(f) := \int_a^b \sqrt{1 + f'(x)^2} dx$$

   is a functional.

Exercise 33. *(Brachistochrone problem) Let $p = (0, b)$ and $q = (a, 0)$ be two points lying on a vertical plane under the force of the gravity g (vertical). Let a wire joining p and q be given by the graph of a function $y(x)$.*

Prove that the time that a bead takes to travel across the wire, starting at p and finishing at q, is

$$T = \int_0^a \frac{\sqrt{1 + y'(x)}}{\sqrt{2g(b - y(x))}} dx.$$

In a more general setting, in classical calculus of variations the types of functionals are of the form

$$J(y) = \int_a^b L(x, y, y') dx,$$

where L is a given function. This function L is called the **Lagrangian**.

4.2 Necessary conditions for extrema.

4.2.1 Normed linear spaces.

A **normed linear space** V is a vector space equipped with a norm.

A **norm** is a map $\| \cdot \| : V \to \mathbb{R}$ that satisfies

1. $\|y\| = 0$ if and only if $y = 0$.

2. $\|y\| \geq 0$ for all $y \in V$.

3. $\|ay\| = |a| \|y\|$ for all $a \in \mathbb{C}$ and all $y \in V$.

4. $\|y_1 + y_2\| \leq \|y_1\| + \|y_2\|$ for all $y_1, y_2 \in V$.

Exercise 34. *Prove that the vector spaces in Notation 31 are normed vector spaces with the norms specified there.*

4.2.2 Derivatives of functionals.

Given a functional $J : A \subset \mathcal{X} \to \mathbb{R}$, its **directional derivative** with direction v at the point y_0 is (if it exists)

$$\delta J(y_0, v) := \frac{d}{d\varepsilon} J(y_0 + \varepsilon v)_{|\varepsilon=0}.$$

Notice that in the definition of directional derivative we are using an auxiliary construction: a function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\varepsilon \to J(y_0 + \varepsilon v).$$

Exercise 35. *Compute the directional derivatives of the following functionals at the specified point y_0 and with direction v:*

1.
$$J(y) = \int_0^1 y^2 dx, \ y_0 = \cos(x), \ v = \sin(x).$$

2.
$$J(y) = \int_0^1 y'^2 dx, \ y_0 = \cos(x), \ v = \sin(x).$$

3.
$$J(y) = \int_0^1 \cos(y) dx, \ y_0 = x, \ v = x^2.$$

Now, with the help of directional derivatives we can give necessary conditions for the existence of minima/maxima of functionals.

Theorem 36. *Let $J : A \subset \mathcal{X} \to \mathbb{R}$ be a functional defined on an open subset of a normed vector space $\mathcal{X}$. If $y_0 \in A$ is a minimum (maximum) of J, then*

$$\delta J(y_0, v) = 0$$

for all v where the directional derivative exists.

Exercise 37. *Consider the functional $J : \mathcal{C}^0([2, 4], \mathbb{R}) \to \mathbb{R}$,*

$$J(y) = \int_2^4 y(x)^2 dx.$$

Prove that $y_0(x) = 0$ is a minimum and check that

$$\delta J(0, v) = 0$$

for all $v \in \mathcal{C}^0([2, 4])$.

4.3   The simplest problem.

The *simplest problem* in calculus of variations is to consider the functional

$$J(y) = \int_a^b L(x, y, y') dx \tag{4.1}$$

defined for functions $y \in \mathcal{C}^2[a, b]$ with the extra condition $y(a) = A$, $y(b) = B$. The function L should satisfy that it is twice differentiable in $[a, b] \times \mathbb{R}^2$.

When computing direrctional derivatives it is required that $v(a) = v(b) = 0$, so $J(y + \varepsilon v)$ is well-defined.

Exercise 38. *Prove that in the simplest problem*

$$\delta J(y_0, v) = \int_a^b \partial_y L(x, y, y')v + \partial_{y'} L(x, y, y')v' dx.$$

From Exercise 38 we deduce that a necessary condition for y_0 being a minimum is that

$$\int_a^b \partial_y L(x, y_0, y_0')v + \partial_{y'} L(x, y_0, y_0')v' dx = 0 \tag{4.2}$$

for all v with $v(a) = v(b) = 0$.

Using integration by parts we deduce that Equation (4.2) is equivalent to

$$\int_a^b \left(\partial_y L(x, y_0, y_0') - \frac{d}{dx} \partial_{y'} L(x, y_0, y_0') \right) v dx = 0.$$

From this last equation we get the following result.

Theorem 39. *Given a functional J of the form (4.1) and defined for functions $y \in C^2[a, b]$ with the extra condition $y(a) = A$, $y(b) = B$, a necessary condition for y_0 being a minimum (maximum) is that*

$$\partial_y L(x, y_0, y_0') - \frac{d}{dx} \partial_{y'} L(x, y_0, y_0') = 0. \tag{4.3}$$

Observation 40. *This theorem is based on the following fact: Let f be a continuous function defined on $[a, b]$. If*

$$\int_a^b f(x)g(x)dx = 0$$

for all twice differentiable functions g with $g(a) = g(b) = 0$, then f is identically the zero function.

Equation (4.3) is called **Euler-Lagrange equation**. Their solutions are extremals.

Exercise 41. *Prove that extremals of functionals of the form*

$$\int_a^b L(x, y, y')dx,$$

with $\partial_y L = 0$ satisfy

$$\partial_{y'} L = C,$$

with C being a constant.
  Also, if $\partial_x L = 0$, then

$$L - y' \partial_{y'} L = C,$$

with C being a constant.
  Finally, if $\partial_{y'} L = 0$, then

$$\partial_y L = 0.$$

Exercise 42. *Find the extremals of the following functionals:*

1.
$$J(y) = \int_0^1 ((y')^2 + 3y + 2x)dx,$$

with $y(0) = 0$, $y(1) = 1$.

2.
$$J(y) = \int_0^1 \sqrt{1 + (y')^2}dx,$$

with $y(0) = a$, $y(1) = b$.

3.
$$J(y) = \int_0^a \frac{\sqrt{1 + (y')^2}}{\sqrt{2g(b - y)}}dx,$$

with $y(0) = b$, $y(a) = 0$.

4.4   Generalizations.

There are different ways of generalizing the Euler-Lagrange equations.

4.4.1   Higher derivatives.

One way of generalizing the Euler-Lagrange equations are by increasing the degree of derivatives involve. For example, the second-order problem

$$\int_a^b L(x, y, y', y'')dx,$$

where $y \in \mathcal{C}^4[a, b]$ satisfying the boundary conditions $y(a) = A_1, y(b) = B_1, y'(a) = A_2, y'(b) = B_2$. In this case, and proceeding as before, we obtain that the (generalized) Euler-Lagrange equations are

$$\partial_y L - \frac{d}{dx}\partial_{y'}L + \frac{d^2}{dx^2}\partial_{y''}L = 0.$$

More generally, in the case of having the Lagrange function L depending on the derivatives of y up to the n-th order, then $y \in \mathcal{C}^{2n}[a, b]$, y should satisfy boundary conditions up the $(n-1)$-th derivatives, and the Euler-Lagrange equations are

$$\partial_y L - \frac{d}{dx}\partial_{y'}L + \frac{d^2}{dx^2}\partial_{y''}L + \cdots + (-1)^n \frac{d^n}{dx^n}\partial_{y^{n-1)}}L = 0.$$

Exercise 43. *Find the extremals of the functional*

$$\int_0^2 \sqrt{1 + (y'')^2}dx,$$

with $y(0) = 0, y'(0) = 1, y(2) = 1, y'(2) = 1$.

4.4.2 Several functions.

Another way is by allowing several functions involved. For example, if two are involved, we get the functional

$$J(y) = \int_a^b L(x, y_1, y_1', y_2, y_2')dx,$$

with boundary conditions $y_1(a) = A_1, y_2(a) = A_2, y_1(b) = B_1, y_2(b) = B_2$. In this case, we get the system of equations

$$\begin{cases} \partial_{y_1} L - \dfrac{d}{dx}\partial_{y_1'} L = 0, \\ \partial_{y_2} L - \dfrac{d}{dx}\partial_{y_2'} L = 0. \end{cases}$$

4.4.3 Natural boundary conditions.

Another way of generalizing the Euler-Langrange equations is by allowing one of the boundaries free. For example, consider the functional

$$\int_a^b L(x, y, y')dx,$$

with boundary conditions $y(a) = A$ and $y(b)$ free. In this case, we get the system of equations

$$\begin{cases} \partial_y L - \dfrac{d}{dx}\partial_{y'} L = 0, \\ \partial_{y'} L(b, y(b), y'(b)) = 0. \end{cases}$$

4.5 More problems.

Exercise 44. *Find the extremal paths connecting two points lying on a sphere.*

Exercise 45. *Find the extremal paths connecting two points lying on a cylinder.*

Exercise 46. *Find the extremals of*

1.

$$J(y) = \int_0^1 (y^2 + y'^2 - 2y\sin(x))dx,$$

where $y(0) = 1$ and $y(1) = 2$.

2.

$$J(y) = \int_1^2 \frac{y'^2}{x^3}dx,$$

where $y(1) = 1$ and $y(2) = 0$.

3.

$$J(y) = \int_0^2 (y^2 + y'^2 + 2ye^x)dx,$$

where $y(0) = 0$ and $y(2) = 1$.

Exercise 47. *Find the Euler-Lagrange equation of the functional*

$$\int_a^b f(x)\sqrt{1 + y'^2}dx,$$

and solve it for $y(a) = A$, $y(b) = B$.

Exercise 48. *Find an extremal for*

$$J(y) = \int_1^2 \frac{\sqrt{1 + y'^2}}{x}dx,$$

where $y(1) = 0$ and $y(2) = 1$.

Exercise 49. *Show that the area of a surface given by the graph of a function* $z = f(x, y)$ *defined on a domain* D *is given by the double integral*

$$\iint_D \sqrt{1 + (\partial_x f)^2 + (\partial_y f)^2}dxdy.$$

It can be proved that a minimal surface satisfies the PDE

$$(1 + (\partial_x f)^2)\partial_{yy}f - 2\partial_x f\partial_y f\partial_{xy}f + (1 + (\partial_y f)^2)\partial_{xx}f = 0. \tag{4.4}$$

Prove that the surface given by $z = \arctan yx$ *satisfies Equation (4.4).*
Could you give an idea of the proof of Equation (4.4)?

Exercise 50. *Find the extremals of*

1.

$$J(y) = \int_0^1 (y'^2 + y^2)dx,$$

where $y(0) = 1$ and $y(1)$ free.

2.

$$J(y) = \int_1^e \left(\frac{1}{2}x^2y'^2 - \frac{1}{8}y^2\right)dx,$$

where $y(1) = 1$ and $y(e)$ free.

Chapter 5

Dynamical systems.

A **dynamical system** is a rule for time evolution on a state space. This means that given a space describing all possible statuses, a dynamical system is a set of rules that describe how a given initial status evolves in time.

Example 51. *Consider the phase space $\mathbb{R}$ describing the height of a particle. Then an example of a dynamical system on this phase space is*

$$x_{n+1} = x_n - 1.$$

This dynamical system describes how the height of the particle evolves in (discrete) time.

Example 52. *As before, consider the phase space $\mathbb{R}$ describing the height of a particle. Another example of a dynamical system on this phase space is*

$$\dot{x} = -1.$$

This dynamical system describes how the height of the particle evolves in (continuous) time.

As shown in the previous two examples, there are different types of dynamical systems. A general classification of all possible dynamical systems is out of the scope of this lecture notes but, we could say that there is a dichotomy depending on the nature of time: discrete or continuous.

In this notes we will concentrate in two types of dynamical systems. When the time evolution is discrete, we will consider dynamical systems described by the iteration of a map

$$F : X \to X,$$

where X will represent the phase space. When the time evolution is continuous, we will consider dynamical systems described by ODEs

$$\dot{x} = F(x, t),$$

where $x \in X$.

The goal of studying a dynamical system is to describe, if possible, the behaviour of the particles when time evolves. Questions that are usually asked are: do all particles converge to a point? Do all particles converge to a set? Are there stationary particles? How is the evolution of the particles near a stationary one?

5.1   Discrete dynamical systems.

As said before, a discrete dynamical system is given by the evolution of a system of the form

$$x_{n+1} = f(x_n, n).$$

Exercise 53. *Consider the dynamical system that models the evolution of your savings in your bank account. This system is given by*

$$x_{n+1} = x_n + rx_n,$$

with $r \in \mathbb{R}$ being a real parameter. Describe the evolution of all initial states x_0 under this system.

Exercise 54. *Find a closed formula of the forward iterations of the (linear) dynamical system*

$$x_{n+1} = Ax_n$$

in terms of the eigenvalues and eigenvectors of the matrix A. Consider the case that all eigenvalues have multiplicity one.
   Apply this to the system with

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}.$$

In Exercises 53 and 54 we could find an explicit formula for the evolution of the system. Usually this is not the case.

5.1.1   Equilibria and stability.

Definition 55. *Given a discrete dynamical system $x_{n+1} = f(x_n)$, a **fixed point** (or **equilibrium solution**) is a point x_* such that its forward evolution is stationary. That is,*

$$f(x_*) = x_*.$$

Exercise 56. *Find all fixed points of the system in Exercise 53.*

Exercise 57. *Find all fixed points of the system*

$$x_{n+1} = x_n^2 + a.$$

Definition 58. *Given a discrete dynamical system $x_{n+1} = f(x_n)$, a **periodic orbit** of period k is a point x_* such that its forward evolution is k-periodic. That is,*

$$f^k(x_*) = x_*.$$

Exercise 59. *Find all period 2 orbits of the system*

$$x_{n+1} = -x_n.$$

Exercise 60. *Find all fixed points and period 2 orbits of the system*

$$\begin{cases} x_{n+1} &=& 1 - ax_n^2 + by_n \\ y_{n+1} &=& x_n \end{cases}.$$

Fixed points and periodic orbits consitute the simplest orbits a dynamical system has. Once they are computed, the next question that one should ask is: how is the behaviour of nearby orbits? Do they converge to the fixed point? Are they repelled? In order of answering these questions, we should introduce linear stability.

Definition 61. *Given a dynamical system $x_{n+1} = f(x_n)$, we will say that a fixed (or periodic) point x_* is **asymptotically stable** if all points y in a neighbourhood of it converge satisfy*

$$\lim_{n \to +\infty} d(f^n(y), f^n(x_*)) = 0.$$

*On the other hand, it is **asymptotically unstable** if it is asymptotically stable when iterating backwards.*

Exercise 62. *Prove that 0 in Exercise 53 is asymptotically stable if $|1 + r| < 1$. Similarly, it is asymptotically stable if $|1 + r| > 1$.*

Usually, checking the asymptotic stability of a fixed point (or periodic orbit) is done applying the following theorem.

Theorem 63. *If the spectrum of the linearization of the fixed point*

$$Df(x_*)$$

is contained in the unit circle, then it is asymptotically stable.
  If the spectrum of the lineariztion contains an eigenvalue with modulus larger than 1, then it is asymptotically unstable.
  In the case of k periodic orbits, replace f with f^k in the definition.

The procedure in Theorem 63 is sometimes called the study of the linear behaviour. This is because for points x near a fixed point,

$$f(x) = x_* + Df(x_*)(x - x_*) + O(|x - x_*|^2),$$

so $f(x) \simeq x_* + Df(x_*)(x - x_*)$.

Exercise 64. *Compute the fixed points and their linear stability of the dynamical system*

$$x_{n+1} = 2 + x_n - x_n^2.$$

Exercise 65. *Compute the fixed points and their linear stability of the dynamical system*

$$\begin{cases} x_{n+1} &=& 1 - ax_n^2 + by_n \\ y_{n+1} &=& x_n \end{cases},$$

when $a = 1.4$ and $b = 0.3$.

5.2   Continuous dynamical systems.

As we saw, ODEs define continuous dynamical systems. This is so because, given an initial condition $x(0) = x_0$, there exists a unique solution satisfying it (when the ODE satisfies some mild conditions such as differentiability).

Some ODEs are easy to solve analytically, while others do not have known analytic solution. For example, linear ODEs have a closed form of solutions.

Exercise 66. *Consider the (linear) ODE*

$$\dot{x} = Ax,$$

where A is a matrix. Write down the solution of it in terms of the eigenvalues/eigenvectors of it. Consider just the case that all eigenvalues have multiplicity 1.

For the ODEs that no known explicit solution is known, other methods for studying them are needed. Let's see some of them in the next subsections.

For the sake of simplicity, from now on we will only consider autonomous ODEs: $\partial_t F(x,t) = 0$.

5.2.1   Vector fields and phase space portraits.

An ODE

$$\dot{x} = F(x)$$

defines a **vector field**. For each point x of the phase space it is associated a vector. Vector fields are very useful for the understanding of the dynamics.

Examples of vector fields are seen every day in the news. Have you ever seen the weather forecast? The wind field is a vector field! A dust particle will flow under the wind field following the vector directions.

If a particle sits at a point $p(0)$, it should follow the vector sitting in it:

$$p(h) \simeq p(0) + hF(p(0), 0),$$

where h is a small time advance. Of course, this last equation is just an approximation of how solutions of the vector fields behave.

The **phase portrait** of an ODE is the portrait of all its solutions.

Exercise 67. *Plot the vector fields and phase portraits of the following vector fields:*

   1.

$$\dot{x} = 1.$$

   2.

$$\dot{x} = x.$$

3.

$$\begin{cases} \dot{x} = 2 \\ \dot{y} = -1 \end{cases}$$

4.

$$\begin{cases} \dot{x} = x + y \\ \dot{y} = y - x \end{cases}$$

5.

$$\begin{cases} \dot{x} = x^2 + y^2 \\ \dot{y} = \cos(x) \end{cases}$$

Exercise 68. *Consider the following vector fields in polar coordinates. Plot them and their phase portraits:*

1.

$$\begin{cases} \dot{r} = 1 \\ \dot{\theta} = 1 \end{cases}$$

2.

$$\begin{cases} \dot{r} = r^3 - r \\ \dot{\theta} = 1 \end{cases}$$

3.

$$\begin{cases} \dot{r} = 0 \\ \dot{\theta} = r \end{cases}$$

5.2.2   Stationary orbits and stability.

A **stationary orbit** of an ODE is an orbit that does not evolve in time. This is equivalent to $\dot{x} = 0$. Hence, stationary orbits of ODEs satisfy that they vanish the vector field,

$$F(x) = 0.$$

As in the discrete case, the stability of a stationary orbit is dictated by its linearization around it.

Theorem 69. *Consider an ODE*

$$\dot{x} = F(x),$$

with a stationary orbit x_0. If the spectrum of its linearization,

$$DF(x_0)$$

is strictly contained in the left side of the y axis, then the stationary orbit is asymptotically stable. Similarly, if part of the spectrum is on the right hand side, it is asymptotically unstable.

Exercise 70. *Find the stationary orbits of the following ODEs and study their stability:*

1.

$$\dot{x} = x^2 + x.$$

2. *The dumped pendulum*

$$\begin{cases} \dot{x} = y \\ \dot{y} = \cos(x) + \frac{1}{2}y \end{cases}$$

3.

$$\begin{cases} \dot{x} = x^2 + y^2 - 1 \\ \dot{y} = x - y \end{cases}$$

5.2.3   Periodic orbits.

A **periodic orbit** in a continuous dynamical system is a solution that repeats in time with the same frequency. Finding periodic orbits is not an easy task and requires, in general, advance techniques out of the scope of this course. Nevertheless, in some examples it is possible to localize them.

Exercise 71. *Find the periodic orbits of the following systems:*

1. *Polar coordinates:*

$$\begin{cases} \dot{r} = r^3 - r \\ \dot{\theta} = 1 \end{cases}$$

2. *Cartesian coordinates:*

$$\begin{cases} \dot{x} = x^3 - x + y^2 x - y \\ \dot{y} = y^3 - y + x^2 y + x \end{cases}$$

5.3   Chaotic systems.

The definition of a chaotic system is a little bit involved, but roughtly speaking, a dynamical system is **chaotic** in a region $\mathcal{U}$ if, for every trajectory starting at $\mathcal{U}$ diverges from its nearby points. This means that if $x_0 \in \mathcal{U}$ then, for all y_0 close to x_0

$$d(x_n, y_n)$$

diverge. This condition ensures that the system is sensitive under initial conditions.

Observation 72. *The definition of a chaotic system is more technical than the idea expressed above. It involves two other conditions: that the system has a dense set of periodic orbits and that it is topologically mixing: all neighborhoods in $\mathcal{U}$ mix under the action of the system.*

Exercise 73. *Convince yourself that the discrete dynamical system defined on the unit interval* $[0, 1]$

$$x_{n+1} = f(x_n),$$

where

$$f(x) = \begin{cases} 2x, & x < \frac{1}{2} \\ 2 - 2x, & x \geq \frac{1}{2} \end{cases}$$

is chaotic. That is, prove that it has a dense set of periodic orbits and its sensitive to initial conditions.

Exercise 74. *Convince yourself that the discrete dynamical system defined on the circle* $[0, 1]/\mathbb{Z}$

$$x_{n+1} = f(x_n) \ (mod \ 1),$$

where

$$f(x) = 2x \ (mod \ 1)$$

is chaotic. That is, prove that it has a dense set of periodic orbits and its sensitive to initial conditions.

Where could we find this dynamical system?

Chapter 6

Introduction to partial differential equations.

Partial differential equations occur in a lot of problems in applied mathematics. They model chemical reactions, physical laws, reaction-diffusion systems in biology and chemistry, gas dynamics, fluid dynamics... The list is endless.

The goal of this chapter is to give an introduction to this topic.

6.1 Some examples.

Some examples of PDEs are:

Example 75. *(Heat equation.)*
    The heat equation

$$u_t = \alpha \Delta u$$

describes the distribution of heat in a given region over time.
    The operator Δ, the laplacian, is defined as $\Delta u = \sum_{k=0}^{n} u_{x_k x_k}$.

Example 76. *(Wave equation.)*
    The wave equation

$$u_{tt} = \alpha \Delta u$$

describes the evolution in time of waves, such as sound waves, light waves and water waves.

Example 77. *(Reaction-diffusion equation.)*
    The reaction-diffusion equation

$$u_t = \Delta u + f(u)$$

describes how the concentration of one or more substances distributed in space changes under the influence of two processes: diffussion which causes the substances to spread out, and reactions which causes the substances to transform themselves.

6.2   Basic concepts

We will concentrate our efforts in second-order PDEs. These are of the form

$$G(t, x, u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}) = 0,$$

and their solutions are twice differentiable functions.

If the PDE has a variable that represents time, then the PDE studies the evolution of the solutions with respect time. If all the variables are spatial, then the PDE studies steady-state problems. (These are just conventions).

In general, a PDE has an infinite number of solutions. Just as the general solution of an ODE depends on some constants, the solution of a PDE depends on some arbitrary function.

Let's solve some easy PDEs.

Exercise 78. *Solve the following PDEs:*

1.
$$u_t = \cos(x).$$

2.
$$u_{tx} = x.$$

3.
$$u_{xx} = y.$$

Exercise 79. *Solve the following PDEs:*

1.
$$uu_x = \cos(x) + \sin(y).$$

2.
$$u_t + u_{xt} = 1.$$

3.
$$u_t + cu_x = f(x, t).$$

Use the change of coordinates $z = x - ct$, $t = t$.

Exercise 80. *Find all solutions of the heat equation $u_t = ku_{xx}$ of the form $u(x, t) = U(z)$, with $z = \frac{x}{\sqrt{kt}}$.*

Most of the times, solving analytically a PDE is impossible. We will study some examples where analytic solutions are known.

To give a classification of all PDEs is out of the scope of this text. Nevertheless, we will see that some PDEs can be classified into three categories: **elliptic** (those that govern equilibrium phenomena), **hyperbolic** (those that govern wave propagation) and **parabolic** (those that govern diffusion processes).

Example 81. • *Elliptic:*
$$u_{xx} + u_{yy} = 0.$$

• *Hyperbolic:*
$$u_{tt} = u_{xx}.$$

• *Parabolic:*
$$u_t = u_{xx}.$$

6.3 Linearity and superposition.

PDEs can be of two types: linear and nonlinear. In this section we will concentrate on the linear ones.

A **linear PDE** (just with two variables x, t) is of the form:

$$L(\partial_x, \partial_t, x, t)u(x, t) = f(x, t),$$

where f is a known function and u is the unknown. The linear operator L is of the form

$$L = \sum a(x, t)\partial_x^i \partial_t^j,$$

with a known functions.

Exercise 82. *Tell if the following PDEs are linear or not:*

1.
$$u_{xx} + \cos(x)u_x = 0.$$

2.
$$u_{xx} + u_x^2 = 0.$$

Another way of seeing that a PDE is linear if it satisfies the following two properties:

1. $L(u + w) = Lu + Lw$.

2. $L(cu) = cLu$, for $c \in \mathbb{R}$.

Linear PDEs satisfy the following interesting property: **superposition**. This property is that if u_1 and u_2 are solutions of the PDE

$$Lu = 0,$$

then any linear combination $c_1 u_1 + c_2 u_2$, with $c_i \mathbb{R}$, is also a solution of it.

Superposition is very helpful because the general solution of the PDE

$$Lu = f, \tag{6.1}$$

is of the form

$$u_p + c_1 u_1 + \cdots + c_n u_n,$$

where u_p is a particular solution of Equation (6.1), and $c_i \in \mathbb{R}$ and u_i are solutions of equation $Lu = 0$.

Another type of superposition is that, if $u(x, t; \alpha)$ are all solutions of the equation $Lu = 0$, then

$$\int_{-\infty}^{\infty} g(\alpha) u(x, t; \alpha) d\alpha$$

is a solution also for every function g.

Exercise 83. *Prove that*

$$u(x, t; \alpha) = \frac{1}{\sqrt{4\pi k t}} e^{\frac{-(x-\alpha)^2}{4kt}}, t > 0$$

is a solution of the heat equation

$$u_t = k u_{xx}.$$

Use the superposition principle for showing that

$$u(x, t) = \int_{-\infty}^{\infty} c(\alpha) u(x, t; \alpha) d\alpha$$

is also a solution for any function c.

6.4   Laplace's equation.

Laplace's equation is the prototype of elliptic equations. It is the PDE

$$\Delta u = u_{xx} + u_{yy} = 0.$$

Its solutions model equilibrium problems because, for example, these are the time independent solutions of the heat and the wave equations.

There are different ways of stating the Laplace's equation, depending on which type of conditions we impose on the boundaries.

- (Dirichlet condition.) Given a region $\Omega \in \mathbb{R}^2$,

$$\begin{cases} \Delta u = f \\ u_{|\partial\Omega} = g(x). \end{cases}$$

- (Neuman condition.) Given a region $\Omega \in \mathbb{R}^2$,

$$\begin{cases} \Delta u = f \\ \frac{du}{dn}_{|\partial\Omega} = g(x). \end{cases}$$

In the case of the Dirichlet problem there is unicity of solutions.

Theorem 84. *If the boundary of the region Ω is smooth and the function f is continuous on $\bar{\Omega}$, then, if the Dirichlet problem*

$$\begin{cases} \Delta u = f \\ u_{|\partial\Omega} = g(x). \end{cases}$$

has a solution $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\bar{\Omega})$, then it is unique.

Solutions to the Laplace's equation $\Delta u = 0$ on Ω satisfies that they have a maximum principle: Its maxima and minima are attained on $\partial\Omega$.

Exercise 85. *Consider the Laplace's equation in $\mathbb{R}^2$:*

$$u_{xx} + u_{yy} = 0.$$

Write it down in polar coordinates.

Exercise 86. *Consider the Laplace's equation in $\mathbb{R}^3$:*

$$u_{xx} + u_{yy} + u_{zz} = 0.$$

Write it down in spherical and cylindrical coordinates.

Exercise 87. *Find all solutions of the Laplace's equation in $\mathbb{R}^3$ such that they are radially symmetric. That is, $u(x, y, z) = \varphi(x^2 + y^2 + z^2)$.*

6.5 Evolution problems.

Consider the prototype of parabolic equation, the heat equation, defined on a bounded domain Ω:

$$\begin{cases} u_t = k\Delta u \\ u_{|\partial\Omega} = f(x). \end{cases}$$

An initial value problem for it is defined once we fix $u(x, 0 = u_0(x)$.

Its steady state solutions are solutions of the Laplace's equation

$$\begin{cases} \Delta u = 0, \\ u_{|\partial\Omega} = f(x). \end{cases}$$

Exercise 88. *Consider the one dimensional heat equation defined on the interval $\Omega = [0, l]$ with boundary condition $u_{|\partial\Omega} = 0$. Let's discretize it by considering just N values of u, u_k, equidistant along the $[0, l]$ interval. Prove that the discretized version of the heat equation, where the Laplacian is substituted by the finite difference*

$$\Delta u(x, t) \approx \frac{u(x+h, t) - 2u(x, t) + u(x-h, t)}{h^2},$$

with $h = \frac{l}{N}$, has as globally attracting fixed point $u(x, t) = 0$.

Observation 89. *In general, and under mild conditions, all initial conditions evolve, under the heat equation flow, to the solution of the Laplace's equation. This is the same as saying the they are globally attracting solutions.*

Exercise 90. *Consider the two dimensional heat equation on the unit square $\Omega = [0,1]^2$ with boundary condition $u_{|\partial\Omega} = 0$. If we assume that all functions on it with these boundary conditions are of the form*

$$\sum_{n,m \in \mathbb{Z}^2} a_{n,m} \sin(2\pi nx)\sin(2\pi my),$$

convince yourself that the zero solution is a globally attracting solution.

Exercise 91. *Consider the reaction-diffusion equation of the form*

$$u_t = ku_{xx} + F(u),$$

with Neuman conditions $u_x(t,0) = u_x(t,L) = 0$.
   Prove that the zeros of F are steady-states.

Exercise 92. *Consider the reaction-diffusion equation*

$$u_t = u_x x + u(1-u),$$

with $x \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ and Dirichlet conditions $u(\pm\frac{\pi}{2}, t) = 3$.
   Show that $u(x,t) = \frac{3}{1+\cos(x)}$ is a steady-state solutions.

6.6   Eigenfunction expansions.

In this section we will see how to solve linear PDEs using eigenfunction expansions.
   Consider the linear PDE

$$\begin{cases} Lu = f, \ u \in \Omega \\ B(u) = 0, \ u \in \partial\Omega \end{cases} \tag{6.2}$$

where $B(u) = 0$ are the boundary conditions (Dirichlet or Neumann). The idea for solving (6.2) is, if possible, finding a basis of eigenfunctions v_k, with associated eigenvalues λ_k, satisfying the eigenvalue problem

$$\begin{cases} Lv_k = \lambda_k v_k, \ v_k \in \Omega \\ B(v_k) = 0, \ v_k \in \partial\Omega \end{cases}.$$

If this is the case then, decomposing f as

$$f(x) = \sum f_k u_k(x)$$

we obtain that a solution u of Equation (6.2) is

$$u(x) = \sum \frac{f_k}{\lambda_k} v_k(x).$$

Notice that we should require that the eigenvalues λ_k are not zero and that the quotients $\frac{f_k}{\lambda_k}$ do not blow up.

Example 93. *Consider the PDE*

$$u_{xx} + u_{yy} = f$$

on $\Omega = [0, \pi]^2$ with Dirichlet condition $u_{|\partial\Omega} = 0$. Then, a set of eigenvectors is

$$v_{n,k}(x, y) = \sin(nx)\sin(my),$$

with associated eigenvalues $\lambda_{n,k} = n^2 + m^2$.

Exercise 94. *Use the eigenfunction expansion method to solve the heat equation*

$$u_t = ku_{xx}$$

on $\Omega = [0, \pi]$ with $u(0, t) = u(\pi, t) = 0$ and $u(x, 0) = f(x)$.

Exercise 95. *Use the eigenfunction expansion method to solve the PDE*

$$u_t = ku_{xx} + \sin(\pi x)$$

on $\Omega = [0, \pi]$ with $u(0, t) = u(\pi, t) = 0$ and $u(x, 0) = f(x)$.

Chapter 7

Sturm-Liouville problems.

An ODE of the form
$$(p(x)y')' + q(x)y' = \lambda y, \tag{7.1}$$
defined on an interval $[a, b]$ with boundary conditions
$$\begin{cases} a_1 y(a) + a_2 y'(a) = 0 \\ b_1 y(b) + b_2 y'(b) = 0 \end{cases}$$
defined a **Sturm-Liouville problem**, (SLP). If both functions p and q are continuous and p does not change sign in $[a, b]$ then the SLP problem is called **regular**.

Notice that solutions of Equation (7.1) depend on the parameter λ, and that $y(x) = 0$ is always a solution. A value λ for which Equation (7.1) has a non trival solution is called an **eigenvalue**, and the corresponding solution is called an **eigenvector**.

Regular SLP problems satisfy that they have infinite eigenvalues with associated eigenvectors of finite multiplicity. These eigenvectors form a complete, orthogonal set. Moreover, all eigenvalues are real.

Exercise 96. *Find all eigenvalues and eigenvectors of the regular SLP problems:*

1.
$$y'' = \lambda y,$$
with boundary conditions $y(0) = y(\pi) = 0$.

2.
$$y'' = \lambda y,$$
with boundary conditions $y(0) = y(l) = 0$.

3.
$$y'' + y' = \lambda y,$$
with boundary conditions $y'(0) = y(1) = 0$.

Chapter 8

Theory of transforms.

The idea behind the theory of transforms is to transform our problem into an easier problem. The applied transformations are usually of integral type: Laplace transform and Fourier transform. The former is applied on functions defined on the positive real line, while the latter is applied on functions defined on the entire line. As we will see, the theory behind these two transforms is parallel to each other.

In general, transforms are applied in problems in order that we change an undesired property of it to an easy-handling one. For example, Fourier and Laplace transforms are useful for replacing derivatives by algebraic equations.

Example 97. *Let's illustrate the idea behind transform methods.*
   Consider the nonlinear system of equations

$$\begin{cases} x^2 y^3 z &= 8 \\ xy &= 7 \\ \frac{x^3 y^5}{z} &= 1 \end{cases}$$

This system is, in principle, intractable. But, if we perform the logarithm transform, we get the system

$$\begin{cases} 2X + 3Y + Z &= \log(8) \\ X + Y &= \log(7) \\ 3X + 5Y - Z &= 0 \end{cases},$$

where $X = \log(X)$, $Y = \log(Y)$, $Z = \log(Z)$. This last system is easily solvable. Once we have its solutions, we pull them back with the help of the inverse of the logarithm transform, the exponential.

8.1   Laplace transform.

The **Laplace transform** is defined as

$$\mathcal{L}(y)(s) = \int_0^\infty y(t) e^{-st} dt.$$

We usually denote by capital letters the transformed function: $\mathcal{L}(y) = Y$.

Exercise 98. *Compute the Laplace transforms of the following functions:*

1. 1.

2. t.

3. t^n.

4. e^{at}.

The inverse of the Laplace transfrom is defined as

$$\mathcal{L}^{-1}(Y)(t) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} Y(s)e^{st}dt,$$

where the integration path is a vertical line on the complex plane from bottom to top and a is chosen in a way that all singularities of the function Y lie on the left side of the vertical line with real part a.

The Laplace transform satisfies very nice properties.

Theorem 99. *The Laplace transform satisfies that*

$$\mathcal{L}(y^{(n)})(t) = s^n Y(s) - \sum_{k=0}^{n-1} s^{n-k-1} y^{(k)}(0).$$

Theorem 100. *Let's define the convolution of two functions $y_1, y_2 : [0, \infty) \to \mathbb{R}$ as*

$$(y_1 * y_2)(t) = \int_0^t y_1(t-s)y_2(s)ds.$$

Then, the Laplace transform satisfies that

$$\mathcal{L}(y_1 * y_2)(s) = Y_1(s)Y_2(s).$$

Furthermore,

$$\mathcal{L}^{-1}(Y_1 Y_2)(t) = (y_1 * y_2)(t).$$

Exercise 101. *Prove Theorems 99 and 100.*

Exercise 102. *Prove that the Laplace transform defines a linear map. That is, $\mathcal{L}(af+bg) = a\mathcal{L}(f) + b\mathcal{L}(g)$, where a and b are constants.*

Have a look at Table 8.1, where some of the most common Laplace transforms appear.

Exercise 103. *With the help of Laplace transforms, solve the following problems:*

$y(t)$	$Y(s)$
1	$\frac{1}{s}$, $s > 0$
t^n with $n > 0$ integer	$\frac{n!}{s^{n+1}}$, $s > 0$
e^{at}	$\frac{1}{s-a}$, $s > a$
$\sin(at)$	$\frac{a}{s^2+a^2}$, $s > 0$
$\cos(at)$	$\frac{s}{s^2+a^2}$, $s > 0$
$\sinh(at)$	$\frac{a}{s^2-a^2}$, $s > |a|$
$\cosh(at)$	$\frac{s}{s^2-a^2}$, $s > |a|$
$e^{bt}\sin(at)$	$\frac{a}{(s-b)^2+a^2}$, $s > b$
$e^{bt}\cos(at)$	$\frac{s-b}{(s-b)^2+a^2}$, $s > b$
$t^n e^{at}$	$\frac{n!}{(s-a)^{n+1}}$, $s > a$
$H(t-a)$	$\frac{e^{-as}}{s}$, $s > 0$
$\delta(t-a)$	e^{-as}
$H(t-a)f(t-a)$	$F(s)e^{-as}$
$f(t)e^{-at}$	$F(s+a)$
$\operatorname{erf}(\sqrt{t})$	$\frac{1}{s\sqrt{s+1}}$, $s > 0$
$\frac{1}{\sqrt{t}}e^{-\frac{a^2}{4t}}$	$\sqrt{\frac{\pi}{s}}e^{-a\sqrt{s}}$, $s > 0$
$1 - \operatorname{erf}(\frac{a}{2\sqrt{t}})$	$\frac{1}{s}e^{-a\sqrt{s}}$, $s > 0$
$\frac{a}{2t^{\frac{3}{2}}}e^{-\frac{a^2}{4t}}$	$\sqrt{\pi}e^{-a\sqrt{s}}$, $s > 0$

Table 8.1: Laplace transforms. $H(t)$ is the Heaviside function, while $\delta(t)$ is the Dirac delta.

1. *The ODE*

$$u'' + u' + u = \sin(t).$$

2. *The ODE*

$$\sum_{k=0}^{N} a_k u^{(k)} = \sum_{k=0}^{M} b_k x^k,$$

   where a_k and b_k are real values.

3. *The PDE*

$$\begin{cases} u_t + u_x = x, \ x > 0, \ t > 0. \\ u(x,0) = 0, \ x > 0, \\ u(0,t) = 0, \ t > 0. \end{cases}$$

4. *The PDE*

$$\begin{cases} u_t - u_{xx} = 0, \ x > 0, \ t > 0, \\ u(x,0) = 0, \ x > 0, \\ u(0,t) = 1, \ t > 0, \\ u(x,t) \ \text{is bounded.} \end{cases}$$

5. *The PDE*

$$\begin{cases} u_t - k u_{xx} = 0, \ x > 0, \ t > 0, \\ u(x,0) = 0, \ x > 0, \\ u(0,t) = g(t), \ t > 0. \end{cases}$$

Write the solution in the form $u(x,t) = \int_0^t K(x,t-\tau)g(\tau)d\tau$.

8.2   Fourier transform.

The **Fourier transform** is defined as

$$\mathcal{F}(y)(s) = \int_{-\infty}^{\infty} y(t)e^{ist}dt.$$

We usually denote by adding a hat on the transformed function: $\mathcal{F}(y) = \hat{y}$.

The inverse of the Fourier transform is

$$\mathcal{F}^{-1}(\hat{y})(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{y}(t)e^{-ist}dt.$$

Observation 104. *The definition of the Fourier transform presented in these notes differs from others given in other textbooks. Although I personally do not like it, I follow the notation on Logan's book in order of not creating any confusing situation.*

The Fourier transform satisfies similar properties to the ones satisfied by the Laplace transform.

Exercise 105. *Prove that the Fourier transform defines a linear map. That is, $\mathcal{F}(af+bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$, where a and b are constants.*

Exercise 106. *Compute the Fourier transform of n-th derivative $y^{(n)}$ and of the convolution $y_1 * y_2$.*

In Table 8.2 there are some Fourier transforms.

$y(t)$	$\hat{y}(s)$
$\delta(t-a)$	e^{ias}
e^{-at^2}	$\sqrt{\frac{\pi}{a}} e^{-\frac{s^2}{4a}}$
$H(t)$	$\pi\delta(s) - \frac{i}{s}$
$e^{-a\lvert t\rvert}$	$\frac{2a}{a^2+s^2}$
$y^{(n)}(t)$	$(-is)^n \hat{y}(s)$
$(y_1 * y_2)(t)$	$\hat{y}_1(s)\hat{y}_2(s)$

Table 8.2: Fourier transforms. $H(t)$ is the Heaviside function, while $\delta(t)$ is the Dirac delta.

Exercise 107. *With the help of Fourier transforms, solve the following problems:*

1. *The ODE*
$$u'' - u = f(x),\ x \in \mathbb{R}.$$

2. *The PDE (Heat equation)*
$$u_t - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,\ u(x,0) = f(x).$$

3. *The PDE (Wave equation)*
$$u_{tt} - c^2 u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,\ u(x,0) = f(x),\ u_t(x,0) = g(x).$$

4. *The PDE (Laplace equation on the half-plane)*
$$u_{xx} + u_{yy} = 0,\ x \in \mathbb{R},\ y > 0,\ u(x,0) = f(x).$$

5. *The PDE (Laplace equation on a strip)*
$$u_{xx} + u_{yy} = 0,\ x \in \mathbb{R},\ b > y > 0,\ u(x,0) = f(x),\ u(x,b) = g(x).$$

6. *The PDE (Advection-diffusion equation)*
$$u_t - cu_x - u_{xx} = 0,\ x \in \mathbb{R},\ t > 0,\ u(x,0) = f(x).$$

7. *The PDE (Non-homogeneous heat equation)*
$$u_t = u_{xx} + F(x,t),\ x \in \mathbb{R},\ t > 0,\ u(x,0) = 0.$$

8.3   Other transforms.

There are other type of transforms.

For example, when dealing with periodic functions $y : \mathbb{R} \to \mathbb{R}$, $y(x) = y(x+1)$, there is the (periodic) Fourier transform

$$\mathcal{F}(y)(s) = \int_0^1 y(t)e^{-2\pi i s t}dt.$$

This transform is defined only for integer values s.

Example 108. *The periodic Fourier transform is useful for solving PDEs where solutions are periodic in one of the variables. For example, the heat equation on the circle. It is the PDE*

$$u_t = u_{xx},$$

with $u(t, x) = u(t, x+1)$, $u(0, x) = f(x)$.

Another example is when dealing with sequences. Then we can use the Z transform, defined as

$$\mathcal{Z}(x)(z) = \sum_{k=0}^{\infty} x_n z^{-n}.$$

This transform is useful in finite differences equations.

Chapter 9

Integral equations.

An integral equation is an equation where the unknown is a function and integrals are involved.

Example 109.
$$\int_0^1 f(x)dx = f(2).$$

Example 110.
$$\int_0^x f(t)dt = f(0) + f(x).$$

These equations appear in a lot of problems: reformulation of ODEs, modelling...
For example, given an ODE
$$\dot{x}(t) = f(x(t), t), \tag{9.1}$$
it is equivalent to the integral equation

$$x(t) = x(0) + \int_0^t f(x(s), s)ds. \tag{9.2}$$

Exercise 111. *Prove that a solution of Equation (9.2) is a solution of the ODE (9.1).*

Two classical examples of linear integral equations are the **Volterra** and **Fredholm** equations. The former is of the form

$$\int_a^x k(x, y)u(y)dy - \lambda u(x) = f(x),\ a \leq x \leq b$$

while the latter is
$$\int_a^b k(x, y)u(y)dy - \lambda u(x) = f(x),\ a \leq x \leq b.$$

In both examples, the unknown function is u, while k and f are known. The function k is usually called the **kernel**.

Observation 112. *Notice that both problems look similar. They only differ on the fact that for the Volterra equations the limits of integration depend on x, while for the Fredholm are fixed. As we will see, this small detail changes dramatically the way each problem is addressed.*

Let's discuss in more detail these equations. Notice that both equations can be written in the form

$$(K - \lambda Id)u = f, \tag{9.3}$$

where K denotes the linear integral operator. Hence, the equations will have a solution u if the function f is on the range of the linear operator $K - \lambda Id$. For example. if it is invertible:

$$u = (K - \lambda Id)^{-1}f.$$

Observation 113. *If the operator $K - \lambda Id$ fails to be invertible, it is still possible that for some (but not all) f Equation (9.3) has solutions.*

To study the invertibility of $K - \lambda Id$ it is important to understand for which λs the eigenvalue equation

$$Ku = \lambda u$$

is satisfied. For these, invertibility will fail.

The following exercise shows why studying the spectrum of a linear operator A is useful for solving linear systems.

Exercise 114. *Consider the real symmetric $n \times n$ matrix A. Give a solution of the nonhomogeneous system*

$$Av = \lambda v + f$$

in terms of the eigenvalues and eigenvectors of the matrix A. Use the fact that there exists an orthogonal basis of eigenvectors, and that the eigenvalues are all real.

9.1   Volterra equations.

As said before, Volterra equations are of the form

$$\int_a^x k(x,s)u(s)ds = \lambda u(x) + f(x), \ a \le x \le b. \tag{9.4}$$

There are special cases where the Volterra equation has an easy solution. Let's see some of these.

Exercise 115. *Suppose that the kernel k does not depend on the first variable x ($k(x,t) = g(t)$). Prove that a solution of Equation (9.4) satisfies the ODE*

$$u'(x) = \frac{1}{\lambda}(g(x)u(x) - f'(x)).$$

Exercise 116. *Solve the following Volterra equations:*

1.

$$\int_0^x u(t)dt = u(x) + x.$$

2.

$$\int_0^x tu(t)dt = 2u(x) + \cos(x).$$

Exercise 117. *Suppose that the kernel k in Equation (9.4) is of the form $k(x,t) = g(x-t)$. Prove that the solution of Equation (9.4) can be solved by means of the Laplace transform.*

(Hint: Remember that the Laplace transform of the convolution is the product of the Laplace transforms.)

Exercise 118. *Solve the Volterra equations*

1.

$$u(x) + \int_0^x (x - t)u(t)dt = t.$$

2.

$$u(x) = \int_0^x e^{x-t}u(t)dt.$$

In general, Volterra equations are solved by means of the Picard's method. If we write down the Volterra equation as

$$u = (K - \lambda Id)^{-1}f = (Id - \hat{K})^{-1}\hat{f},$$

with $\hat{K} = \frac{1}{\lambda}K$ and $\hat{f} = \frac{-1}{\lambda}f$. The solution is of the form

$$u = \sum_{n=0}^{\infty} \hat{K}^n f, \tag{9.5}$$

where $\hat{K}^n$ denotes the n-th composition of the operator $\hat{K}$. This series is called the **Neumann series**.

There is a theorem that assures that this procedures works.

Theorem 119. *If f, k are continuous, then the solution to the Volterra equation is given by (9.5).*

Observation 120. *Since solution (9.5) involves an infinite series, approximate solutions are required. It can be proven that*

$$|\hat{K}^n f| \le \max|f| \frac{(b-a)^n \max|\frac{k}{\lambda}|^n}{n!}.$$

Hence, an approximate solution is given by the truncated series

$$u = \sum_{n=0}^{N} \hat{K}^n f,$$

and an upper bound of the error of it is given by

$$\max |f| \sum_{n=N+1}^{\infty} \frac{(b-a)^n \max |\frac{k}{\lambda}|^n}{n!}. \tag{9.6}$$

Exercise 121. *Prove that an upper bound of* (9.6) *is*

$$\max |f| e^{\frac{(b-a)^{N+1} \max |\frac{k}{\lambda}|^{N+1}}{(N+1)!}}. \tag{9.7}$$

Exercise 122. *Find approximate solutions to the following Volterra equations using Neumann series:*

1.

$$u(x) + \lambda \int_0^x u(s)ds = x.$$

2.

$$\lambda u(x) + \int_0^x (x-s)u(s)ds = x.$$

9.2   Fredholm equations.

As said before, Fredholm equations are of the form

$$\int_a^b k(x,y)u(y)dy - \lambda u(x) = f(x),\ a \le x \le b. \tag{9.8}$$

In the case of Volterra equations we saw that all the linear equations have a solution, given by the Neumann series. In the case of Fredholm equations, this is no longer true. However, as we will see, there are cases that we can treat.

9.2.1   Fredholm equations with degenerate kernel.

A Fredholm equation with degenerate kernel is one that its kernel $k(x,y)$ can be expressed in the form

$$\sum_{i=0}^{n} \alpha_i(x)\beta_i(x).$$

In this special case, the solution to the Fredholm equation (9.8) can be reduced to a finite dimensional linear algebra problem. Notice that it is equivalent to

$$\sum_{i=0}^{n} \alpha_i(x) \int_a^b \beta_i(y)u(y)dy - \lambda u(x) = f(x). \tag{9.9}$$

Let's denote by (f, g) the integrals

$$\int_a^b f(y)g(y)dy.$$

Multiplying Equation (9.9) by $\beta_j(x)$ and integrating with respect x we obtain the n linear equations of the form

$$\sum_{i=0}^{n} (\alpha_i, \beta_j)(\beta_i, u) - \lambda(\beta_j, u) = (\beta_j, f).$$

This system is of the form

$$Aw - \lambda w = b, \tag{9.10}$$

where A is the matrix with (i, j) entry (α_i, β_j), and w and f are vectors with entries (β_i, u) and (β_j, f_j).

If the linear system (9.10) has a solution w, then a solution to the Fredholm equation with degenerate kernel will be

$$u(x) = \frac{1}{\lambda}\left(-f(x) + \sum_{i=0}^{n} \alpha_i(x)w_i\right).$$

Observation 123. *Notice that the linear system (9.10) has a solution for all f if and only if λ is not an eigenvalue of the matrix A.*

It is easily proven in this case the following theorem, sometimes called the *Fredholm alternative.*

Theorem 124. *Consider the Fredholm equation (9.8) with degenerate kernel. Then, if λ is not an eigenvalue of the matrix A, the problem has a unique solution. If, on the contrary, it is an eigenvalue, either the problem has none or infinite number of solutions.*

Exercise 125. *Solve the Fredholm equation*

$$\int_0^1 xtu(t)dt + u(x) = \cos(2\pi x).$$

Exercise 126. *Solve the Fredholm equation*

$$\int_0^1 (xt + x^2t^2)u(t)dt + u(x) = \cos(2\pi x).$$

9.2.2   Symmetric kernels.

A symmetric kernel $k(x, y)$ is one that satisfies $k(x, y) = k(y, x)$. With these kind of kernels, the eigenvalue problem

$$Ku = \lambda u$$

satisfies that if an eigenvalue exists, it is real, and all the eigenvectors corresponding to distinct eigenvalues are orthogonal.

The existence of eigenvalues is a very difficult problems, and out of the scope of this course. Nevertheless, we can give some conditions for their existence.

Theorem 127. *If the Fredholm equation satisfies that its kernel is symmetric, continuous and non-degenerate, then the eigenvalue problem*

$$Ku = \lambda u$$

has infinite eigenvalues λ_i, each with finite multiplicity, such that then can be ordered

$$0 < \cdots < |\lambda_2| < |\lambda_1|$$

with $\lim_{n \to \infty} = 0$. Moreover, there exists an orthonormal basis formed by eigenfunctions ϕ_i such that all square integrable function f can be expressed uniquely as

$$f(x) = \sum_{k=1}^{\infty} a_k \phi_k(x).$$

The coefficients a_k are equal to $\int_a^b f(x)\phi_k(x)dx$.

Notice that in the case of the previous theorem, solving the linear equation

$$Ku - \lambda u = f$$

is easy once we know all the eigenvalues and eigenfunctions of the operator K. See Exercise 114 for an analogue solution.

Exercise 128. *Find the eigenvalues and eigenvectors of the operator*

$$Ku(x) = \int_{-1}^{1} (1 - |x - y|)u(y)dy.$$

9.3   Perturbation methods.

Perturbation methods can be applied in Volterra and Fredholm equations. These methods could be very helpful for solving nonlinear integral equations or, more generally, integro-differential equations. Let's see some examples through some exercises.

Exercise 129. *Find approximate solutions of the following equations by means of perturbation series around $\varepsilon = 0$:*

1.
$$u(x) + \varepsilon \int_0^x u(x)^2 dx = 1, \ 0 \le x \le 1.$$

2.
$$u(x) + \int_0^x u(x)dx + \varepsilon(u(x)^3 - u(x)) = x, \ 0 \le x \le 1.$$

3.
$$u(x) + \int_0^1 u(x)dx + \varepsilon u'(x) = 1, \ 0 \le x \le 1.$$

Appendices

Appendix A

Solving some ODEs.

A.1 First order linear ODEs.

First order linear ODEs are of the form

$$y' + p(x)y = q(x).$$ (A.1)

First, we multiply Equation (A.1) by a function $f(x)$, obtaining

$$f(x)y' + f(x)p(x)y = f(x)q(x).$$

We will choose $f(x)$ such that

$$f'(x) = f(x)p(x).$$ (A.2)

Observation 130. *The solution to Equation* (A.2) *is*

$$f(x) = Ke^{\int p(x)dx}.$$

Thus, we get that

$$(f(x)y)' = f(x)p(x),$$

so

$$y(x) = \frac{1}{f(x)} \int f(x)p(x)dx.$$

A.2 Second order linear ODEs.

These are ODEs of the form

$$y'' + p(x)y' + q(x)y = r(x).$$ (A.3)

First, we find solutions to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

These are of the form

$$y_H(x) = Ay_1(x) + By_2(x).$$

Then, a general solution of Equation (A.3) is found by finding a particular solution of the form

$$y_P(x) = A(x)y_1(x) + B(x)y_2(x),$$

with the extra condition

$$A'(x)y_1(x) + B'(x)y_2(x) = 0.$$