S2:

PROBABILITY:

Statistical theory and practice rest largely on the concept of probability.

Probability is a measure of estimation of likelihood of ocurrence of an event.

Ex 1: If a fair coin is tossed, the probability of getting heads is 0.5.

Remark: Probabilities are between 0 and 1. (both included).

Ex 2: If a die is ~~thrown~~ casted, the probability of getting a 3 is $1/6 = 0.1\hat{6}$ ($= 0.16666\ldots$).

Remark: Die (singular) $\rightarrow$ Dice (Plural).

Any probability pertains to the results of a situation which we call an experiment. An <u>experiment</u> is any process by which data are obtained through the observation of uncontrolled events in nature or through controlled procedures in a laboratory.

Any experiment can result in various outcomes. The <u>sample space</u> (Ω) is the set of all possible outcomes that may occur as a result of a particular experiment.

<u>Exs:</u> Coin $\rightarrow$ $\Omega = \{\text{heads}, \text{tail}\} = \{H, T\}$.

Die $\rightarrow$ $\Omega = \{1, 2, 3, 4, 5, 6\}$.

~~A subset of the sample space~~

An <u>event</u> is a subset of the sample space.

<u>Ex:</u> Die: . ~~[illegible]~~ . $\{\text{Even}\}$, $\{\text{Odd}\}$, $\{\geq 3\}$, $\{1\}$.

- Events can be combined. Given two events A and B, then:
  - their union $A \cup B$
  - their intersection $A \cap B$
  - the complementary A^c.

THE PROBABILITY FUNCTION:

The probability function (denoted by P) is the function that quantifies the likelihood of an event. Given an event A, ~~[illegible]~~

Properties:

1. $0 \leq P(A) \leq 1$.
2. $P(\Omega) = 1$.
3. If $A \cap B = \emptyset$ (2 exclusive events), then
   $$P(A \cup B) = P(A) + P(B).$$

3'. If $A_1, \dots, A_n$ are pair wise exclusive events ($A_i \cap A_j = \emptyset$), then
$$P(A_1 \cup \dots \cup A_n) = P(A_1) + P(A_2) + \dots + P(A_n).$$

Notice that $\forall A$, $A^c \cap A = \emptyset$, $A^c \cup A = \Omega$. Hence,
$$P(A^c) + P(A) = 1 \Rightarrow P(A^c) = 1 - P(A).$$

~~Ex~~: In the die problem,

$$P(\{1\})=\frac{1}{6}.$$

$$P(\{1,2\})=\frac{2}{6}=\frac{1}{3}.$$

$$P(\{3,4,5,6\})=\frac{4}{6}=\frac{2}{3}.$$

Notice that the coin and the die satisfy the property of <u>equiprobability</u>: The probability of ~~all~~ result ~~are~~ equal and are $\frac{1}{n}$, where n is the number of possible results. This phenomenon is quite common, but not always true: a biesed coin (or die) will not have this property.

• Another general property of the probability function is that, if two events are independent, then the probability ~~of~~ that both happen is the product of their probabilities.

$$P(A\cap B)=P(A)P(B)$$

Ex: A fair die, we ~~throw~~ cast it twice. What is the probability that the first result is 1 and the second is 3?

$$P(F_1 \cap F_3) = P(F_1)\,P(F_3) = \frac{1}{6^2} \left(= \left(\frac{1}{6} \cdot \frac{1}{6}\right)\right).$$

** $\boxed{\text{ADVANCED}}$ Conditional probability: Given two events A and B, the probability that A occurs given that the event B is certain to occur is denoted by $P(A|B)$, and is calculated by

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Ex: If we know that the result of a die came up even, what is the probability that 2 came up?

$$P(\{2\} | \{2,4,6\}) = \frac{P(\{2\} \cap \{2,4,6\})}{P(\{2,4,6\})} = \frac{\frac{1}{6}}{\frac{3}{6}} = \frac{1}{3}.$$

RANDOM VARIABLES:

We have seen in the previous examples that the results of experiments might be non-numerical (heads, tails) and numerical (1,2,3,4,5,6).

A random variable is ~~the~~ a quantification of the results of an experiment. A random variable is a numerical quantity the value of which is determined by an experiment.

Often we will denote random variables by capital letters X, Y, Z.

Ex: In the ~~die~~ coin example, we can build a random variable $X = \begin{cases} 0 & \text{heads} \\ 1 & \text{tails.} \end{cases}$

Ex: In the ~~d~~ die example, X = result of casting the die.

With the random variables we are still in the game of computing probabilities.

Ex: Die: $P(X<2) = (P(X=1)) = \frac{1}{6}$.

$P(X>3) = (P(X=3 \text{ or } X=4 \text{ or } X=5 \text{ or } X=6)) = \frac{4}{6} = \frac{2}{3}$.

Ex: Height of students.

$P(X<1.80) = ?$

$P(160 < X < 200) = ?$

DISCRETE RANDOM VARIABLES:

Notice that the possible results of casting a die or the height of students differ substantially. The height of students is quite special: what is the probability that $P(X=180)$? Discuss it.

Let $p(x)=P(X=x)$. We will say that X is a discrete random variable if

- $p(x)\geq 0 \ \forall x$.
- $p(x)=0$ for all ~~x~~ out of a finite or numerable set.
- $\left(\sum' p(x)=1.\right)$
- $\left(P(X\in A)=\sum'_{x\in A} p(x)\right.$.

Ex: . Die $\to$ discrete.

- Height $\to$ non-discrete.
- Coin $\to$ discrete.
- $X=\{$the day of my next son's birth$\}$. (Consider me as inmortal! :-). $\to$ That's a discrete (but not finite) random veriable.

Distribution function:

Given X, it's <u>distribution function</u> $F(x)$ is defined by $F(x) = P(X \le x)$. [The probability that $X \le x$].

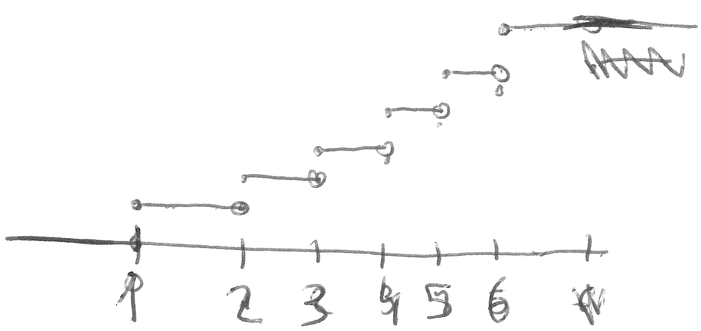
Sometimes distribution functions are called <u>cumulative distribution functions</u>.

Properties:

- $0 \le F(x) \le 1$.
- $\lim_{x \to -\infty} F(x) = 0$, $\lim_{x \to +\infty} F(x) = 1$.

<u>Example:</u> Die:

$$F(x) = \begin{cases} 0 & x < 1 \\ 1/6 & x < 2 \\ 2/6 & x < 3 \\ 3/6 & x < 4 \\ 4/6 & x < 5 \\ 5/6 & x < 6 \\ 6/6 & x \ge 6. \end{cases}$$



Independent random variables:

X and Y are <u>independent random variables</u> if

$$P(a \le X \le b,\ c \le Y \le d) = P(a \le X \le b) \cdot P(c \le Y \le d).$$

Expected value:

The expected value of a (discrete) random variable X is the "expected result" of it. It is the weighted mean of all its possible values by their probabilities.

$$E(X) = \sum_i x_i \, p(x_i) = \sum_i x_i \, P(X = x_i)$$

Ex: Die: $E(X) = \frac{1}{6} \cdot 1 + \frac{1}{6} \cdot 2 + \frac{1}{6} \cdot 3 + \dots + \frac{1}{6} \cdot 6 = 3.5$.

Properties:

- $E(f(X)) = \sum f(x) \, p(x)$
- $E(aX+b) = aE(X)+b$
- $E(X+Y) = E(X)+E(Y)$
- If $X_1, \dots, X_n$ form $\overline{X} = \frac{1}{n}(X_1 + \dots + X_n)$ and $E(X_i) = \mu$ then $E(\overline{X}) = \mu$.

The random variable $\overline{X}$ will be called the mean random variable.

Variance and standard deviation

Variance: $\sigma^2(X) = E((X - E(X))^2) = \sum (x_i - \mu)^2 p(x)$

Standard deviation: $\sigma(X) = \sqrt{\sigma^2(X)} = \sqrt{\sum (x_i - \mu)^2 p(x)}$

Properties:

- $\sigma^2(X) = E(X^2) - E(X)^2$
- $\sigma^2(aX+b) = a^2\sigma^2(X)$

Before finishing, let's point out several properties of the expected value and the variance of independent random variables:

- $E(XY) = E(X)E(Y)$
- $\sigma^2(X+Y) = \sigma^2(X) + \sigma^2(Y)$
- $X_1, \ldots, X_n \sim \mu, \sigma^2 \Rightarrow E(\overline{X}) = \mu$ and $\sigma^2(\overline{X}) = \frac{\sigma^2}{n}$.