UPPSALA UNIVERSITET

**Written examination**
Applied mathematics 1MA060
2015–01–17

*Allowed aids: writing material. For the grades 3, 4 and 5 one should obtain at least 18, 25 and 32 points, respectively. Solutions should be accompained with explanatory text.*

1. *(6p)* Consider the initial value problem
$$\begin{cases} y' + (\cos(\varepsilon) - 1)y = 0 \\ y(0) = 1 \end{cases}$$
Compute the first three terms of the Taylor expansion
$$y(t;\varepsilon) = y_0(t) + \varepsilon y_1(t) + \varepsilon^2 y_2(t) + h.o.t.,$$
centered at $\varepsilon = 0$.

2. *(6p)* Consider the equation
$$\varepsilon x^5 + x^2 - 1 = 0.$$
When $\varepsilon = 0$, $x_0 = 1$ is a solution. Compute the first three terms of the Taylor expansion
$$x(\varepsilon) = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + h.o.t.,$$
centered at $\varepsilon = 0$, with $x(0) = 1$.

3. *(6p)* Compute all stationary solutions and sketch the phase portrait of the continuous dynamical system
$$x' = x^2 - 1.$$

4. *(6p)* Determine all fixed points of the discrete dynamical system
$$x_{n+1} = e^{x_n} - 2.$$
Determine if they are asymptotically stable.

5. *(6p)* Find all the global minima of the functional
$$\mathcal{J}(y) = \int_0^1 (y(x) + y'(x))^2 + 5y(x)^4 dx,$$
where $y$ is twice-differentiable and subject to the boundary conditions $y(0) = 0$ and $y(1) = 0$. *(A global minimum $y_*$ is a function such that for all other functions $y$, $\mathcal{J}(y) \geq \mathcal{J}(y_*)$).*

6. *(5p)* Express the solution of the Volterra equation
$$u(y) = y^4 + 5\int_0^y u(s)ds, \ 0 \leq y \leq 1$$
in the form
$$u(y) = \sum_{k=0}^{\infty} u_k y^k.$$

7. *(5p)* Consider the equation
$$u(y) = 1 + \varepsilon y + \varepsilon^2 \int_0^2 u(s)ds, \ 0 \leq y \leq 2.$$
When $\varepsilon = 0$, a solution is $u_0(y) = 1$. Find the first three terms of the Taylor expansion
$$u(y;\varepsilon) = u_0(y) + \varepsilon u_1(y) + \varepsilon^2 u_2(y) + h.o.t.$$
of the solution.

**May the Math be with you.**