List of exercises.
Topics in Nonlinear Analysis. Fall 2018.

Instructions: *The basic condition to pass the course is to get at least 100 points out of all the exercises appearing in this list. Choose any exercises you like. Do not hesitate in discussing them with the teachers.*

1. Prove that a linear operator $A : X \to Y$, X and Y being Banach spaces, is continuous if and only if there exists a constant C such that $\|Ax\| \leq C\|x\|$.

   (5 points)

2. Prove existence and uniqueness of solutions to the Cauchy problem

$$\begin{cases} x' = f(x,t) \\ x(t_0) = x_0 \end{cases}$$

   when $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is locally Lipschitz at (t_0, x_0).

   (5 points)

3. Prove existence and uniqueness of solutions to the problem

$$\begin{cases} x'(t) = f(x(t), x(t-1), t) \\ x(s) = g(s) \text{ for } -1 \leq s < 0 \end{cases}$$

   when $f : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is Lipschitz and $g : [-1, 0] \to \mathbb{R}^n$ is continuous.

   (10 points)

4. Consider the holomorphic map $f : \mathbb{C} \to \mathbb{C}$, $f(z) = \frac{1}{2}z + z^2 g(z)$. Prove that there exist a neighborhood $\mathcal{U}$ of 0, and a map $h : \mathcal{U} \to \mathbb{C}$ such that

$$f(h(z)) = h\left(\frac{z}{2}\right),$$

   $h(0) = 0$ and $h'(0) = 1$.

   (10 points)

5. Given a $\mathcal{C}^3$ operator $F : X \to Y$, produce a convergence theorem similar to the Newton-Kantorovich Theorem such that:

   - The nonlinear equation

$$F(x_n) + DF(x_n)(x_{n+1} - x_n) + \frac{1}{2}D^2F(x_n)(x_{n+1} - x_n)^{\otimes 2} = 0$$

     has a unique solution. (x_n is given but x_{n+1} is the unknown).
   - The convergence is cubic:

$$\|F(x_{n+1})\| \sim \|F(x_n)\|^3.$$

   (15 points)

6. Prove the following

   - If $X = C[0,1]$, $Y = C[0,1]$, the map $f : X \to Y$ given by $f(x(t)) = \sin(x(t))$, $0 \leq t \leq 1$, is analytic.
   - If $X = L^2(0,1)$, $Y = L^2(0,1)$, the map $f : X \to Y$ given by $f(x(t)) = \sin(x(t))$, $0 \leq t \leq 1$, is Lipschitz but nowhere differentiable.
   - If $X = H^1(0,1)$, $Y = H^1(0,1)$, the map $f : X \to Y$ given by $f(x(t)) = \sin(x(t))$, $0 \leq t \leq 1$, is analytic.

(5 points)

7. For $u \in C^2[0,1]$, and $v, w \in \mathbb{R}$, define

$$A(u,v,w) = \left(-\frac{d^2u}{dx^2}, 0, 0\right)$$

provided $u_x(0) = v$ and $u_x(1) + u(1) = w$. Extend A to be a closed operator in $L^2(0,1) \times \mathbb{R} \times \mathbb{R}$ and prove that A is sectorial. (7 points)

8. Given $u(\cdot, 0) \in L^2(0,1)$ and $v(0), w(0) \in \mathbb{R}$, prove existence of a solution of

$$u_t = u_{xx}\,, \qquad 0 < x < 1\,, \qquad t > 0\,,$$

with

$$u_x(0,t) = v(t)\,, \qquad u_x(1,t) + u(1,t) = w(t)\,,$$

where

$$\frac{dv}{dt} = \alpha v + \beta w\,, \qquad \frac{dw}{dt} = \gamma v + \delta w + \int_0^1 u(x,t)dx.$$

(8 points)

9. Consider an incompressible fluid with velocity $u(x,t) \in \mathbb{R}^3$ and temperature $T(x,t) \in \mathbb{R}$ in a domain $\Omega \in \mathbb{R}^3$, under the conditions $u = 0$ on $\partial\Omega \times [t_0, \infty)$ and initial value $u(x, t_0) = u_0(x)$ for $x \in \Omega$. The temperature of the fluid is goberned by

$$\rho c_p T_t + \rho c_p (u \cdot \nabla T) = \operatorname{div}(k \nabla T) + \rho q$$

where ρ is the density of the fluid (constant), c_p is the specific heat, k is the conductivity, and q is the rate of production of heat per unit mass. Couple the above equation with the Navier-Stokes equation and prove local existence of strong solutions under suitable assumptions. Consider the cases

- c_p, k, q are constants.
- Now $q = q(T)$ depends on the temperature.
- Now $k = k(T)$ and $q = q(T)$ depend on the temperature.
- Now the viscosity $\mu = \mu(T)$ depends on the temperature.

(15 points)

10. Study the solution set around the point $(0,0)$ of the equation

$$2y^4 - y^3x - xy = 0.$$

(5 points)

11. Study the the solution set around the point $(0,0,0)$ of the system of nonlinear equations

$$\begin{cases} x^2 + y^3 - xy + \sin(z) = 0 \\ 3z + \cos(xy) - 1 = 0 \end{cases}$$

(10 points)

12. *(Hunter-Shu equation)* Consider the equation

$$c\partial_x f + \frac{1}{2}\partial_x \left\{ f^2 Af - fA(f^2) + \frac{1}{3}A(f^3) \right\} + \partial_x Lf = 0,$$

where c is a parameter, $A = \frac{1}{2}|\partial_x|$, $L = \frac{1}{2}|\partial_x|^{-1}$ and f is a 1-periodic even function with zero mean.

Apply Crandall-Rabinowitz theorem to this equation around $f = 0$.

(12 points)