Exercises

Exercise 1. Let $\mathcal{A}_\triangle$ be the set of all graphs which contain $\triangle$ as a subgraph. Fix a constant $0 < p \leq 1$, and show that $\mathbb{P}(G(n,p) \in \mathcal{A}_\triangle) \to 1$.

Exercise 2. Prove the following (for the second part it may help to use Chebyshev's inequality: for X be a random variable and $t > 0$; $\mathbb{P}(|X - \mathbb{E}[X]| \geq t) \leq \mathbb{V}[X]/t^2$):

Let $X_1, X_2, \ldots$ be a sequence of random variables each taking non-negative integer values. If $\mathbb{E}[X_n] \to 0$ then

$$\mathbb{P}(X_n = 0) \to 1,$$

and if $\mathbb{E}[X_n] > 0$ for each n, and $\mathbb{V}[X_n]/(\mathbb{E}[X_n])^2 \to 0$ then

$$\mathbb{P}(X_n = 0) \to 0.$$

Exercise 3. Show whp $np \to \infty$ implies whp G_n contains $\triangle$ i.e. a 3-cycle.
Let Y_n count the number of $\triangle$ in G_n and for any 3-subset of vertices $S \subset V(G)$ let A_S be the event that G_n restricted to the vertices S is a $\triangle$.

(a) Show by linearity of expectation that:

$$\mathbb{V}(Y_n) = \sum_{S,T \in \binom{[n]}{3}} \Big(\mathbb{P}(A_S \ \& \ A_T) - \mathbb{P}(A_S)\mathbb{P}(A_T) \Big)$$

where $\binom{[n]}{3}$ denotes the set of sets of three vertices in the graph.

(b) After some case analysis and from (a) show: $\mathbb{V}(Y_n) \leq n^4 p^5 + n^3 p^3$.

(c) From (b) conclude that whp $Y_n > 0$.

Exercise 4. Show that the function $p^*(n) = \frac{1}{n^{2/3}}$ is a threshold for $G(n,p)$ containing $\boxtimes$ as a subgraph.

Exercise 5. Given $k \in \mathbb{N}$, let $\mathcal{P}_k$ be the set of graphs which have a path on k vertices as a subgraph.

(a) Find the threshold function for $\mathcal{P}_3$ (notice $\mathcal{P}_3$ is the set of graphs containing the path $\wedge$ as a subgraph).

(b) Find the threshold for $\mathcal{P}_4$.

(c) Let $k \in \mathbb{N}$ be a constant. Find the threshold for $\mathcal{P}_k$ in terms of k and n.

Exercise 6. We can define an interated majority function for $n = 3^k$. The base case is $\text{Imaj}_1(x_1, x_2, x_3) = \text{Maj}_3(x_1, x_2, x_3)$ and

$$\text{Imaj}_k(x) = \text{Maj}_3(\text{Imaj}_{k-1}(x_1, \ldots, x_{3^{k-1}}), \text{Imaj}_{k-1}(x_{3^{k-1}+1}, \ldots, x_{2.3^{k-1}}), \text{IMaj}_{k-1}(x_{2.3^{k-1}+1}, \ldots, x_{3^k})).$$

(a) Calculate the influence of the i-th bit $I_i^p(\text{Imaj}_2)$ and total influence $I^p(\text{Imaj}_2)$.

(b) For $p = 1/2$ calculate $I_i^p(\text{Imaj}_k)$ and $I^p(\text{Imaj}_k)$.

Exercise 7. Suppose $\mathcal{A}$ is non-trivial monotone and let $p_c(n)$ be such that

$$\mathbb{P}(G(n, p_c(n)) \in \mathcal{A}_n) = \frac{1}{2}$$

and then show that for $p_b(n) = 1 - (1 - p_c(n))^k$ we have

$$\mathbb{P}(G(n, p_b(n)) \in \mathcal{A}_n) = 1 - \frac{1}{2^k}.$$