1. Hyperbolic sets and shadowing

1.1. Hyperbolic sets

Let M be a C^1 Riemannian manifold, $U \subset M$ a non-empty open subset, $f : U \mapsto f(U)$ - a C^1 diffeomorphism.

A compact f-invariant subset Λ is *hyperbolic* if $\exists\, \lambda \in (0,1)$ and families of subspaces $E^{\pm}(x) \subset T_xM$, $x \in \Lambda$, s.t. for every $x \in \Lambda$:

- $T_xM = E^+(x) \oplus E^-(x)$;
- $\|D(f^n)(x)|_{E^+(x)}\| \leq C\lambda^n$ and $n \geq 0$;
- $\|D(f^{-n})(x)|_{E^-(x)}\| \leq C\lambda^n$ and $n \geq 0$;
- $Df(x)E^{\pm}(x) = E^{\pm}(f(x))$.

1.2. Horseshoe: an example of a hyperbolic set

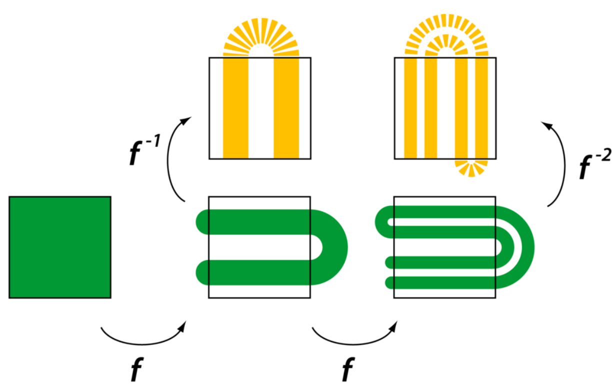

Figure 1. Generating a horseshoe.

A *rectangle* in $\mathbb{R}^{k+l}$ will mean a set of the form $D_1 \times D_2 \subset$ where D_i are disks, $\pi_1 : \mathbb{R}^{k+l} \mapsto \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+l} \mapsto \mathbb{R}^l$ will be two orthogonal projections. $\mathbb{R}^k$ will be called the "horizontal" direction, $\mathbb{R}^l$ - the vertical.

Definition 1. *(Full component) Suppose $\Delta \subset U \subset \mathbb{R}^{k+l}$ is a rectangle and $f : U \mapsto \mathbb{R}^{k+l}$ is a diffeo. A connected component $\Delta_0 = f(\Delta_0')$ of $\Delta \cap f(\Delta)$ is called* <u>*full*</u>*, if*
1) $\pi_2(\Delta_0') = D_2$;
2) *for any $z \in \Delta_0'$, $\pi_1|_{f(\Delta_0' \cap (D_1 \times \pi_2(z)))}$ is a bijection onto D_1.*

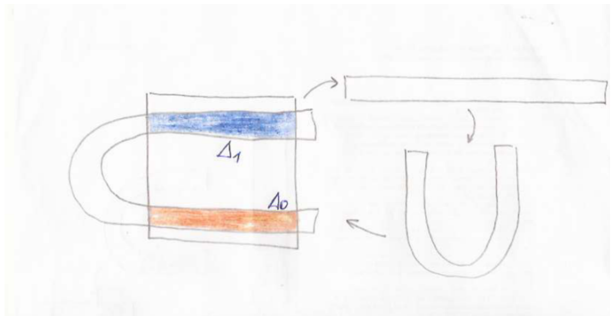

Figure 2. Two horizontal rectangles Δ_0 and Δ_1.

Definition 2. *(Horseshoe) If $U \subset \mathbb{R}^{k+l}$ is open then a rectangle $\Delta = D_1 \times D_2 \subset U \subset \mathbb{R}^k \oplus \mathbb{R}^l$ is called a horseshoe for a diffeo $f : U \mapsto \mathbb{R}^{k+l}$ if $\Delta \cap f(\Delta)$ contains at least two full components Δ_0 and Δ_1 such that for $\Delta' = \Delta_0 \cap \Delta_1$..*
1) $\pi_2(\Delta') \subset \mathrm{int} D_2, \quad \pi_1(f^{-1}(\Delta')) \subset \mathrm{int} D_1$;
2) $D\left(f|_{f^{-1}(\Delta')}\right)$ *preserves and expands a horizontal cone family on* $f^{-1}(\Delta')$;
3) $D\left(f^{-1}|_{\Delta'}\right)$ *preserves and expands a vertical cone family on* $f^{-1}(\Delta')$.

• Let us study the maximal invariant subset of Δ. Denote Δ_{ω_1}, $\omega_1 = 0, 1$, the two full components of $\Delta \cap f^1(\Delta)$, and $\Delta^{\omega_1} = f^{-1}(\Delta_{\omega_1})$, $\omega_1 = 0, 1$.
• The intersection $\Delta \cap f(\Delta) \cap f^2(\Delta)$ consists of four horizontal rectangles:

$$\Delta_{\omega_1 \omega 2} = \Delta_{\omega_1} \bigcap f(\Delta_{\omega_2}) = f(\Delta^{\omega_1}) \bigcap f^2(\Delta^{\omega_2}),$$

$\omega_i \in \{0, 1\}$.

• Inductively, the set $\cap_{i=1}^n f^i(\Delta)$ consists of 2^n disjoint horizontal rectangles of exponentially decreasing heights.

$$\Delta_{\omega_1 .. \omega_n} := \bigcap_{i=1}^n f^i(\Delta^{\omega_i}), \ \omega_i \in \{0, 1\}.$$

Each infinite intersection

$$\Delta_\omega := \bigcap_{i=0}^n f^i(\Delta^{\omega_i}), \omega = (\omega_1 \ldots, \omega_n, \ldots) \in \Sigma_2^+,$$

is a horizontal fiber (a curve connecting the left and the right sides of Δ, such that the projection π_1 on the disk D_1 is a bijection).

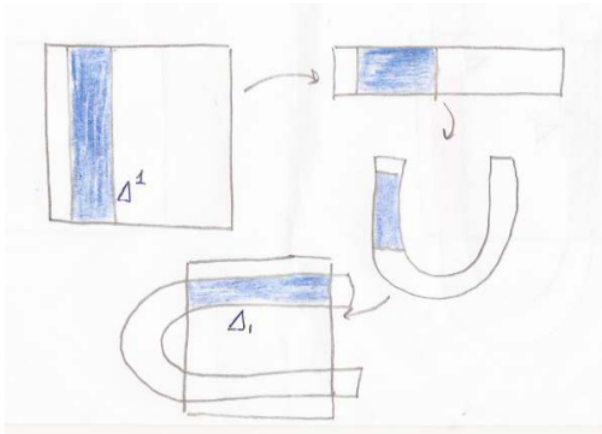

Figure 3. A vertical rectangle is a preimage of the horizontal $\Delta^1 = f^{-1}(\Delta_1)$. Therefore, it gets mapped into the horizontal Δ_1 by f..

• Similarly, the sets

$$\Delta^{\omega_{-n}\dots\omega_0} := \bigcap_{i=0}^{n} f^{-i}(\Delta^{\omega_{-i}}),\ \omega_{-i} \in \{0,1\},$$

are vertical rectangles, the sets

$$\Delta^{\omega} := \bigcap_{i=0}^{n} f^{-i}(\Delta^{\omega_{-i}}),\ \omega = (\dots,\omega_{-n},\dots,\omega_{-1},\omega_0) \in \Sigma_2^+,$$

are vertical fibers.

• The intersection of any vertical fiber with the set of horizontal fibers projects to a Cantor set Λ_2 in D_2, while the intersection of any horizontal fiber with the vertical ones projects to a Cantor set Λ_1 in D_1:

$$\Lambda_2 := \Delta^{\dots\omega_{-n}\dots\omega_{-1},\omega_0} \bigcap \left(\bigcap_{i=1}^{\infty} f^{i}(\Delta) \right),$$

$$\Lambda_1 := \Delta_{\omega_1\dots\omega_n\dots} \bigcap \left(\bigcap_{i=0}^{\infty} f^{-i}(\Delta) \right).$$

• Finally, the set

$$\Lambda := \bigcap_{i=-\infty}^{\infty} f^{-i}(\Delta)$$

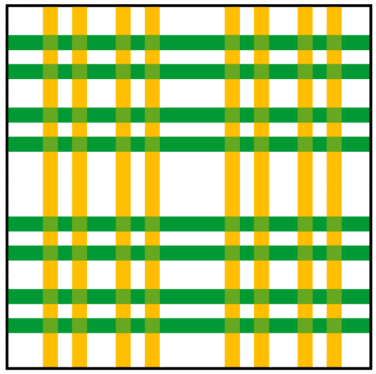

Figure 4. An approximation of the invariant hyperbolic set.

is an invariant set, equal to the product of two Cantor sets Λ_1 and Λ_2, hence a Cantor set itself. The map $h : \Sigma_2 \mapsto \Lambda$, given by

$$h(\omega) = \bigcap_{i=-\infty}^{\infty} f^{-i}(\Delta^{\omega_i})$$

is the homeomorphism conjugating the shift $\sigma|_{\Sigma_2}$ to $f|_\Lambda$.

Corollary 3. *The horseshoe is a hyperbolic set.* $f|_\Lambda$ *is topologically conjugate to* $\sigma|_{\Sigma_2}$.

Proof. Hyperbolicity follows from the invariance of the cone families and stretching of the vectors inside the cones. □

Corollary 4. $f|_\Lambda$ *is topologically mixing. Periodic points of* f *are dense in* Λ*, and the number of periodic points of period* p *is* 2^p.

For stable/unstable manifolds, horseshoe, the attractor, etc for the Hénon family check this applet.

1.3. Homoclinic and heteroclinic intersections

Definition 5. *(Homoclinic points) Let* p *be a hyperbolic periodic point of a diffeo* $f : U \mapsto M$. *A point* q *is homoclinic to* p *if* $q \neq p$ *and* $q \in W^s(p) \cap W^u(p)$. *It is transverse homoclinic if, additionally,* $W^s(p)$ *and* $W^u(p)$ *intersect transversely at* q.

Definition 6. *(Heteroclinic points) Suppose* $p_1, \ldots, p_k$ *be periodic points (of possibly different periods) of* $f : U \mapsto M$. *Suppose* $W^u(p_i)$ *intersects* $W^s(p_{i+1}$ *at* q_i, $i = 1, \ldots, k$ *(*$p_{k+1} = p_1$. q_i *are called heteroclinic points.*

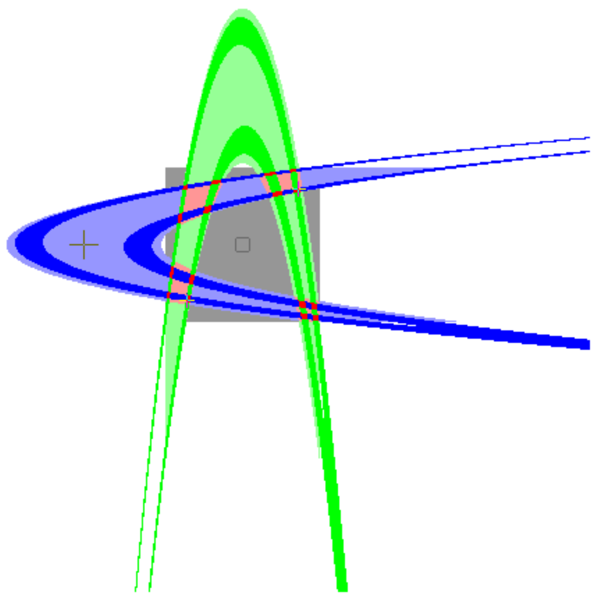

Figure 5. Horseshoe for a Hénon map, taken from this applet.

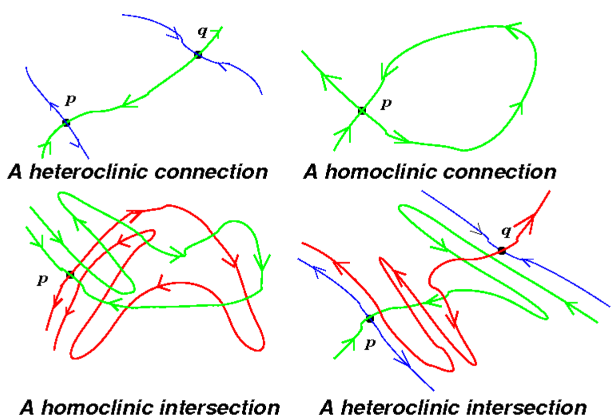

Figure 6. Some possible configurations of homoclinic/heteroclinic intersections

Theorem 7. *Let p be a hyperbolic periodic point of a diffeo $f : U \mapsto M$ and let q be a transverse homoclinic point to p. Then for every $\epsilon > 0$ the union of ϵ-neighborhoods of the orbits of p and q contains a horseshoe of f.*

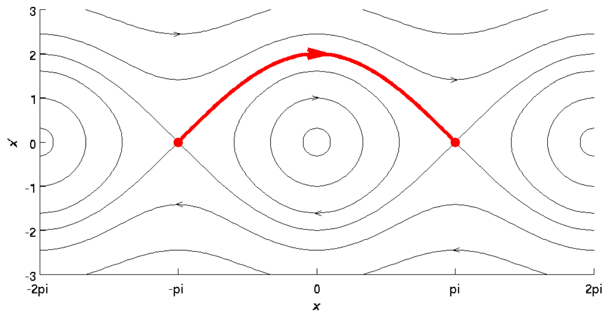

Figure 7. A heteroclinic connection in a pendulum

1.4. Shadowing

An ϵ-orbit (a pseudo-orbit) if $f : U \mapsto M$ is a finite or infinite set $\{x_n\}$ s.t. s to $\mathrm{dist}(f(x_n), x_{n+1}) < \epsilon$ for all n.

Question: *When are orbits of a perturbed dynamical system are ϵ-orbits of the original one? This might give a us a way to conjugate the perturbed and the original systems.*

The following theorem answers this question.

Theorem 8. *Shadowing Theorem Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1-diffeo $f : M \mapsto M$ of a smooth manifold M. Then there exists a nbhd. V of Λ and a neighborhood W of f in $C^1(M, M)$ such that for all $\delta > 0$ there exists $\epsilon > 0$ s. t. for all topological spaces X, homeos $g : X \mapsto X$ and continuous maps $h_0 : X \mapsto X$ the following holds.*

If $\tilde{f} \in W$ is such that $d_{C^0}(h_0 \circ g, \tilde{f} \circ h_0) < \epsilon$ then

1) *(existence of a conjugacy) there is a continuous $h_1 : X \mapsto V$ s.t.*
$$h_1 \circ g = \tilde{f} \circ h_1, \text{ and } d_{C^0}(h_0, h_1) < \delta;$$
2) *(uniqueness of the conjugacy) $\exists\ \delta_0 = \delta_0(\Lambda, f) > 0$, s.t. if $h_1' : X \mapsto V$ is a cont. map satisfying $h_1' \circ g = \tilde{f} \circ h_1'$ and $d_{C^0}(h_1', h_1) < \delta_0$ then $h_1' = h_1$;*
3) *(continuity of the conjugacy) h_1 depends continuously on $\tilde{f}$*

Proof. The proof will be based on the Contraction Mapping Principle.

1) Set
$$\Gamma(X, h_0^* TM) = \left\{\xi \in C^0(X, TM) : \xi(x) \in T_{h_0(x)}M\right\},$$

the space of continous vector fields field "along" h_0, endowed with the sup. norm. Now, let V_1 be any relatively compact nbhd. of Λ.

There is $\theta = \theta(V_1, M) > 0$ such that $\mathcal{A} : B_\theta(h_0) \mapsto \Gamma(X, h_0^* TM) \subset C^0(X, V_1)$ $\square$

Definition 9. *Let (X, f) be a dyn. sys. on a metric space X. An ϵ-psedu-orbit $\{x_k\}_{k\in\mathbb{Z}}$ is δ-shadowed by an orbit of of $x \in X$ under f if $d_X(x_k, f^k(x)) < \delta$ for all $k \in \mathbb{Z}$.*

Orbits of a hyperbolic dynamical system shadow pseudo-orbits:

Corollary 10. *(Shadowing Lemma) Let Λ be a hyperbolic set for $f : U \mapsto M$. Then $\exists$ an open nbhd $V \supset \Lambda$ s.t. for every $\delta > 0$ there is $\epsilon > 0$ so that every ϵ-pseudo-orbit in V is δ-shadowed by an orbit of f.*

Furthermore, there is δ_0 s. t. if $\delta < \delta_0$ then the orbit of f shadowing the given pseudo-orbit is unique.

Proof. Take $X = \mathbb{Z}$ (with discrete topology); $g : X \mapsto X$ given by $g(k) = k + 1$; $h_0 : X \mapsto V$ given by $h_0(k) = x_k$; and $\tilde{f} = f$. By the Shadowing Theorem $\exists\, h_1 : X \mapsto V$ such that $h_1 \circ g = f \circ h_1$ and $d_{C^0}(h_0, h_1) < \delta$, i.e.

$$h_1(k+1) = f(h_1(k)), \text{ for all } k \in \mathbb{Z} \text{ or } h_1(k) = f^k(x),$$

where $x = h_1(0)$, and $d(x_k f^k(x)) < \delta$ for all $k \in \mathbb{Z}$ as requested. $\square$

Periodic orbits of a hyperbolic dynamical system shadow pseudo-orbits "uniformly":

Corollary 11. *(Anosov Closing Lemma) Let Λ be a hyperbolic set for $f : U \mapsto M$. Then $\exists$ an open nbhd $V \supset \Lambda$ and $C, \epsilon_0 > 0$, s.t. for every $\epsilon < \epsilon_0$ and any periodic ϵ-orbit $(x_0, x_1, \ldots, x_m) \subset V$, there is a point $y \in U$ s. t. $f^m(y) = y$ and $\mathrm{dist}(f^k(y), x_k) < C\epsilon$ for $k = 0, 1, \ldots, m-1$.*

Proof. Choose $X = \mathbb{Z}_m$, $g(k) = k + 1 mod m$, $h_0(k) = x_k$ and $\tilde{f} = f$ in the Shadowing Theorem. $\square$

Remark 12. *In particular, consider an almost periodic orbit, i.e. an orbit segment s. t.* $\operatorname{dist}(f^m(x_0), x_0) < \epsilon$ *(this is a pseudo-orbit). Thus Anosov Closing Lemma implies that close to any orbit in a hyperbolic set* Λ *that "almost" returns to itself, there is a true periodic orbit (but not necessarily in* Λ*).*

Finally, the Shadowing Theorem leads to the *structural stability* of hyperbolic sets:

Theorem 13. *(Persistence of hyperbolic sets) Let* $\Lambda \subseteq M$ *be a hyperbolic set for a* C^1*-diffeo* $f : M \mapsto M$ *Then there exists an open nbhd.* $V \supset \Lambda$ *s.t. for any* C^1 *diffeo* $g : M \mapsto M$ *sufficiently* C^1*-close to* f*, the completely invariant set*

$$\Lambda_V^g = \bigcap_{m \in \mathbb{Z}} g^m(\bar{V})$$

is hyperbolic for g*, if not empty. In particular,* $\Lambda_V^f \supseteq \Lambda$ *is hyperbolic.*

Proof. 1) Extend the invariant splitting $T_x M = E_x^+ \oplus E_x^-$ defined for $x \in \Lambda$ to a continuous (but not nec. invariant splitting) on an open $V_1 \supset \Lambda$. Given $\gamma > 0$, let

$$H_x^\gamma := \left\{ u + v \in T_x M I u \in E_x^+, v \in E_x^-, \|v\| \le \gamma \| \|u\| \right\}$$

be the corresponding horizontal cone in $T_x M$, and let V_x^g be the complimentary vertical cone.

2) $\exists$ (λ, μ)-splitting on $\Lambda \implies$

$$\begin{aligned} Df[x]\left(H_x^\gamma\right) \quad & \subseteq H_{f(x)}^{\gamma\lambda/\mu} \subset \operatorname{int} H_{f(x)}^\gamma \cap \{0\}, \\ (Df[x])^{-1}\left(V_{f(x)}^\gamma\right) & \subseteq V_x^{\gamma\lambda/\mu} \subset \operatorname{int} V_x^\gamma \cap \{0\}, \end{aligned}$$

and

$$u + v \in H_x^\gamma \implies \|Df[x](u+v)\| \ge \frac{\mu - \lambda\gamma}{1+\gamma} \|u+v\|, \tag{1.1}$$

$$u + v \in (Df[x])^{-1}\left(V_{f(x)}^\gamma\right) \implies \|Df[x](u+v)\| \le (1+\gamma)\lambda\|u+v\|. \tag{1.2}$$

Now, by continuity, for any $\delta > 0$ we can find a rel. compact nbhd $V \subseteq V_1$ of Λ and a nbhd f in C^1-topology s.t. (7.14) and (7.15) remain valid with μ substituted by $\mu - \delta$ and λ by $\lambda + \delta$ for all $x \in V$ and $g \in W$.

The sequence of differentials $Dg(g^m(x))$ admits a (λ', μ') splitting with

$$\begin{aligned} \lambda' &= (1+\gamma)(\lambda+\delta), \\ \mu' &= \frac{\mu - \lambda\gamma - (1+\gamma)\delta}{1+\gamma}, \end{aligned}$$

and if δ and γ are small, we still have $\lambda' < 1 < \mu'$, the set Λ_V^g is hyperbolic for g. $\square$

Theorem 14. *(Structural stability of hyperbolic sets) Let $\Lambda \subseteq M$ be a hyperbolic set for C^1 diffeomorphism $f : M \mapsto M$ of a smooth manifold M. Then for every open nbhd. V of Λ and every $\eta > 0$ there exists a nbhd. W of f in $C^1(M, M)$ such that for all diffeomorphisms $\tilde{f} \in W$ there is a hyperbolic set $\tilde{\Lambda} \subset V$, and a homeomorphism $H : \Lambda \mapsto \tilde{\Lambda}$ with*

$$h \circ f = \tilde{f} \circ h$$

on Λ and $d_{C^0}(\mathrm{id}, h) + d_{C^0}(\mathrm{id}, h^{-1}) < \eta$. Furthermore, h is unique if δ is small enough.

Proof.
i) Apply the Shadowing Theorem taking $\delta < \min\{\delta_0, \eta/2\}$, $X = \Lambda$, $h_0 = \mathrm{id}_\Lambda$ and $g = f$. Get a nbhd $V_1 \subset V$ of Λ, and a nbhd W_1 of f, such that $d_{C^0}(\tilde{f}, f) < \epsilon$ for all $\tilde{f} \in W_1$, and a unique $h_1 : \Lambda \mapsto V_1$ such that $h_1 \circ f = \tilde{f} \circ h_1$ and $d_{C^0}(\mathrm{id}_\Lambda, h_1) < \delta$.

In particular, $\tilde{\Lambda} = h_1(\Lambda)$ is completely $\tilde{f}$-invariant and hyperbolic by Theorem 48 (after, possibly, a shrinking of W_1).

ii) To prove that h_1 is injective, we apply the Shadowing Theorem again taking δ as before, $X = \tilde{\Lambda}$ and $h_0 := \mathrm{id}_{\tilde{\Lambda}}$ and $g = \tilde{f}$, we get the same nbhd W_1 as soon as ϵ is small. Then we have a unique $h_2 : \tilde{\Lambda} \mapsto V$ s.t. $h_2 \circ \tilde{f} = f \circ h_2$ and $d_{C^0}(\mathrm{id}_{\tilde{\Lambda}}, h_2) < \delta$.

iii) To end the proof, it is sufficient to show that $h_2 \circ h_1 = \mathrm{id}_\Lambda$. We apply again the Shadowing Theorem with $X = \Lambda$, $h_0 = \mathrm{id}_{\tilde{\Lambda}}$ and $g = \tilde{f} = f$. Since

$$d_{C^0}(\mathrm{id}_\Lambda, h_2 \circ h_1) \leq d_{C^0}(\mathrm{id}_\Lambda, h_1) + d_{C^0}(h_1, h_2 \circ h_1) = d_{C^0}(\mathrm{id}_\Lambda, h_1) + d_{C^0}(\mathrm{id}_{\tilde{\Lambda}}, h_2) < 2\delta < \delta_0,$$

we can apply the uniqueness statement in the Shadowing Theorem to get

$$h_2 \circ h_1 = \mathrm{id}_\Lambda,$$

because they both commute with f and are close to h_1. $\square$