1. Hyperbolic sets and shadowing

1.1. Hyperbolic sets

Let $M$ be a $C^1$ Riemannian manifold, $U \subset M$ a non-empty open subset, $f : U \mapsto f(U)$ - a $C^1$ diffeomorphism.

A compact $f$-invariant subset $\Lambda$ is hyperbolic if $\exists \lambda \in (0, 1)$ and families of subspaces $E^\pm(x) \subset T_x M$, $x \in \Lambda$, s.t. for every $x \in \Lambda$:

- $T_x M = E^+(x) \oplus E^-(x)$;
- $\|D(f^n)(x)|_{E^+(x)}\| \leq C\lambda^n$ and $n \geq 0$;
- $\|D(f^{-n})(x)|_{E^-(x)}\| \leq C\lambda^n$ and $n \geq 0$;
- $Df(x)E^\pm(x) = E^\pm(f(x))$.

1.2. Horseshoe: an example of a hyperbolic set

A rectangle in $\mathbb{R}^{k+l}$ will mean a set of the form $D_1 \times D_2 \subset$ where $D_i$ are disks, $\pi_1 : \mathbb{R}^{k+l} \mapsto \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+l} \mapsto \mathbb{R}^l$ will be two orthogonal projections. $\mathbb{R}^k$ will be called the “horizontal” direction, $\mathbb{R}^l$ - the vertical.

Definition 1. (Full component) Suppose $\Delta \subset U \subset \mathbb{R}^{k+l}$ is a rectangle and $f : U \mapsto \mathbb{R}^{k+l}$ is a diffeo. A connected component $\Delta_0 = f(\Delta'_0)$ of $\Delta \cap f(\Delta)$ is called full, if

1) $\pi_2(\Delta'_0) = D_2$;
2) for any $z \in \Delta'_0$, $\pi_1|_{f(\Delta'_0 \cap (D_1 \times \pi_2(z)))}$ is a bijection onto $D_1$.

Figure 1. Generating a horseshoe.
Definition 2. (Horseshoe) If $U \subset \mathbb{R}^{k+l}$ is open then a rectangle $\Delta = D_1 \times D_2 \subset U \subset \mathbb{R}^k \oplus \mathbb{R}^l$ is called a horseshoe for a diffeo $f : U \mapsto \mathbb{R}^{k+l}$ if $\Delta \cap f(\Delta)$ contains at least two full components $\Delta_0$ and $\Delta_1$ such that for $\Delta' = \Delta_0 \cap \Delta_1$:

1) $\pi_2(\Delta') \subset \text{int} D_2, \quad \pi_1(f^{-1}(\Delta')) \subset \text{int} D_1$;
2) $D(f\mid_{f^{-1}(\Delta')})$ preserves and expands a horizontal cone family on $f^{-1}(\Delta')$;
3) $D(f^{-1}\mid_{\Delta'})$ preserves and expands a vertical cone family on $f^{-1}(\Delta')$.

- Let us study the maximal invariant subset of $\Delta$. Denote $\Delta_{\omega_1}, \omega_1 = 0, 1$, the two full components of $\Delta \cap f(\Delta)$, and $\Delta^{\omega_1} = f^{-1}(\Delta_{\omega_1}), \omega_1 = 0, 1$.
- The intersection $\Delta \cap f(\Delta) \cap f^2(\Delta)$ consists of four horizontal rectangles:

$$\Delta_{\omega_1\omega_2} = \Delta_{\omega_1} \bigcap f(\Delta_{\omega_2}) = f(\Delta^{\omega_1}) \bigcap f^2(\Delta^{\omega_2}),$$

$\omega_i \in \{0, 1\}$.
- Inductively, the set $\cap_{i=1}^n f^i(\Delta)$ consists of $2^n$ disjoint horizontal rectangles of exponentially decreasing heights.

$$\Delta_{\omega_1...\omega_n} := \bigcap_{i=1}^n f^i(\Delta^{\omega_i}), \omega_i \in \{0, 1\}.$$

Each infinite intersection

$$\Delta_{\omega} := \bigcap_{i=0}^n f^i(\Delta^{\omega_i}), \omega = (\omega_1, \ldots, \omega_n, \ldots) \in \Sigma_2^+,$$

is a horizontal fiber (a curve connecting the left and the right sides of $\Delta$, such that the projection $\pi_1$ on the disk $D_1$ is a bijection).
Figure 3. A vertical rectangle is a preimage of the horizontal $\Delta^1 = f^{-1}(\Delta_1)$. Therefore, it gets mapped into the horizontal $\Delta_1$ by $f$.

• Similarly, the sets

$$\Delta^{\omega_n \ldots \omega_0} := \bigcap_{i=0}^{n} f^{-i}(\Delta^{\omega_i}), \ \omega_i \in \{0, 1\},$$

are vertical rectangles, the sets

$$\Delta^{\omega} := \bigcap_{i=0}^{n} f^{-i}(\Delta^{\omega_i}), \ \omega = (\ldots, \omega_n, \ldots, \omega_1, \omega_0) \in \Sigma^+_2,$$

are vertical fibers.

• The intersection of any vertical fiber with the set of horizontal fibers projects to a Cantor set $\Lambda_2$ in $D_2$, while the intersection of any horizontal fiber with the vertical ones projects to a Cantor set $\Lambda_1$ in $D_1$:

$$\Lambda_2 := \Delta^{\omega_n \ldots \omega_1, \omega_0} \bigcap \left( \bigcap_{i=0}^{\infty} f^{-i}(\Delta) \right),$$

$$\Lambda_1 := \Delta^{\omega_1 \ldots \omega_n} \bigcap \left( \bigcap_{i=1}^{\infty} f^{i}(\Delta) \right).$$

• Finally, the set

$$\Lambda := \bigcap_{i=-\infty}^{\infty} f^{-i}(\Delta)$$
Figure 4. An approximation of the invariant hyperbolic set.

is an invariant set, equal to the product of two Cantor sets $\Lambda_1$ and $\Lambda_2$, hence a Cantor set itself. The map $h : \Sigma_2 \mapsto \Lambda$, given by

$$h(\omega) = \bigcap_{i=-\infty}^{\infty} f^{-i}(\Delta^{\omega_i})$$

is the homeomorphism conjugating the shift $\sigma|\Sigma_2$ to $f|\Lambda$.

**Corollary 3.** The horseshoe is a hyperbolic set. $f|\Lambda$ is topologically conjugate to $\sigma|\Sigma_2$.

**Proof.** Hyperbolicity follows from the invariance of the cone families and stretching of the vectors inside the cones. \qed

**Corollary 4.** $f|\Lambda$ is topologically mixing. Periodic points of $f$ are dense in $\Lambda$, and the number of periodic points of period $p$ is $2^p$.

For stable/unstable manifolds, horseshoe, the attractor, etc for the Hénon family check [this applet](#).

### 1.3. Homoclinic and heteroclinic intersections

**Definition 5.** (Homoclinic points) Let $p$ be a hyperbolic periodic point of a diffeo $f : U \mapsto M$. A point $q$ is homoclinic to $p$ if $q \neq p$ and $q \in W^s(p) \cap W^u(p)$. It is transverse homoclinic if, additionally, $W^s(p)$ and $W^u(p)$ intersect transversely at $q$.

**Definition 6.** (Heteroclinic points) Suppose $p_1, \ldots, p_k$ be periodic points (of possibly different periods) of $f : U \mapsto M$. Suppose $W^u(p_i)$ intersects $W^s(p_{i+1})$ at $q_i$, $i = 1, \ldots, k$ ($p_{k+1} = p_1$. $q_i$ are called heteroclinic points.
Figure 5. Horseshoe for a Hénon map, taken from [this applet].

Figure 6. Some possible configurations of homoclinic/heteroclinic intersections

Theorem 7. Let $p$ be a hyperbolic periodic point of a diffeo $f : U \to M$ and let $q$ be a transverse homoclinic point to $p$. Then for every $\epsilon > 0$ the union of $\epsilon$-neighborhoods of the orbits of $p$ and $q$ contains a horseshoe of $f$. 
1.4. Shadowing

An $\epsilon$-orbit (a pseudo-orbit) if $f : U \mapsto M$ is a finite or infinite set $\{x_n\}$ s.t. $\text{dist}(f(x_n), x_{n+1}) < \epsilon$ for all $n$.

**Question:** When are orbits of a perturbed dynamical system are $\epsilon$-orbits of the original one? This might give us a way to conjugate the perturbed and the original systems.

The following theorem answers this question.

**Theorem 8.** Shadowing Theorem Let $\Lambda \subseteq M$ be a hyperbolic set for a $C^1$-diffeo $f : M \mapsto M$ of a smooth manifold $M$. Then there exists a nbhd. $V$ of $\Lambda$ and a neighborhood $W$ of $f$ in $C^1(M, M)$ such that for all $\delta > 0$ there exists $\epsilon > 0$ s.t. for all topological spaces $X$, homeos $g : X \mapsto X$ and continuous maps $h_0 : X \mapsto X$ the following holds.

If $\tilde{f} \in W$ is such that $d_{C^0}(h_0 \circ g, \tilde{f} \circ h_0) < \epsilon$ then

1) **(existence of a conjugacy)** there is a continuous $h_1 : X \mapsto V$ s.t.

$$h_1 \circ g = \tilde{f} \circ h_1, \text{ and } d_{C^0}(h_0, h_1) < \delta;$$

2) **(uniqueness of the conjugacy)** $\exists \delta_0 = \delta_0(\Lambda, f) > 0$, s.t. if $h'_1 : X \mapsto V$ is a cont. map satisfying $h'_1 \circ g = \tilde{f} \circ h'_1$ and $d_{C^0}(h'_1, h_1) < \delta_0$ then $h'_1 = h_1$;

3) **(continuity of the conjugacy)** $h_1$ depends continuously on $\tilde{f}$

**Proof.** The proof will be based on the Contraction Mapping Principle.

1) Set

$$\Gamma(X, h_0^*TM) = \{\xi \in C^0(X, TM) : \xi(x) \in T_{h_0(x)}M\}.$$
the space of continuous vector fields field “along” $h_0$, endowed with the sup. norm. Now, let $V_1$ be any relatively compact nbhd. of $\Lambda$.

There is $\theta = \theta(V_1, M) > 0$ such that $\mathcal{A} : B_\theta(h_0) \mapsto \Gamma(X, h_0^*TM) \subset C^0(X, V_1)$

\[\Box\]

**Definition 9.** Let $(X, f)$ be a dyn. sys. on a metric space $X$. An $\epsilon$-pseudo-orbit $\{x_k\}_{k \in \mathbb{Z}}$ is $\delta$-shadowed by an orbit of $x \in X$ under $f$ if $d_X(x_k, f^k(x)) < \delta$ for all $k \in \mathbb{Z}$.

**Orbits of a hyperbolic dynamical system shadow pseudo-orbits:**

**Corollary 10.** *(Shadowing Lemma)* Let $\Lambda$ be a hyperbolic set for $f : U \mapsto M$. Then $\exists$ an open nbhd $V \supset \Lambda$ s.t. for every $\delta > 0$ there is $\epsilon > 0$ so that every $\epsilon$-pseudo-orbit in $V$ is $\delta$-shadowed by an orbit of $f$.

Furthermore, there is $\delta_0$ s. t. if $\delta < \delta_0$ then the orbit of $f$ shadowing the given pseudo-orbit is unique.

**Proof.** Take $X = \mathbb{Z}$ (with discrete topology); $g : X \mapsto X$ given by $g(k) = k + 1$; $h_0 : X \mapsto V$ given by $h_0(k) = x_k$; and $\tilde{f} = f$. By the Shadowing Theorem $\exists h_1 : X \mapsto V$ such that $h_1 \circ g = f \circ h_1$ and $d_{C^0}(h_0, h_1) < \delta$, i.e.

$$h_1(k + 1) = f(h_1(k)),$$

for all $k \in \mathbb{Z}$ or $h_1(k) = f^k(x)$,

where $x = h_1(0)$, and $d(x_kf^k(x)) < \delta$ for all $k \in \mathbb{Z}$ as requested.

**Periodic orbits of a hyperbolic dynamical system shadow pseudo-orbits “uniformly”:**

**Corollary 11.** *(Anosov Closing Lemma)* Let $\Lambda$ be a hyperbolic set for $f : U \mapsto M$. Then $\exists$ an open nbhd $V \supset \Lambda$ and $C, \epsilon_0 > 0$, s.t. for every $\epsilon < \epsilon_0$ and any periodic $\epsilon$-orbit $(x_0, x_1, \ldots, x_m) \subset V$, there is a point $y \in U$ s. t. $f^m(y) = y$ and $\text{dist}(f^k(y), x_k) < C\epsilon$ for $k = 0, 1, \ldots, m - 1$.

**Proof.** Choose $X = \mathbb{Z}_m$, $g(k) = k + 1 \mod m$, $h_0(k) = x_k$ and $\tilde{f} = f$ in the Shadowing Theorem.
Remark 12. In particular, consider an almost periodic orbit, i.e. an orbit segment s.t. dist(\(f^m(x_0), x_0\)) < \(\epsilon\) (this is a pseudo-orbit). Thus Anosov Closing Lemma implies that close to any orbit in a hyperbolic set \(\Lambda\) that “almost” returns to itself, there is a true periodic orbit (but not necessarily in \(\Lambda\)).

Finally, the Shadowing Theorem leads to the structural stability of hyperbolic sets:

**Theorem 13. (Persistence of hyperbolic sets)** Let \(\Lambda \subseteq M\) be a hyperbolic set for a \(C^1\)-diffeo \(f : M \mapsto M\) Then there exists an open nbhd. \(V \supset \Lambda\) s.t. for any \(C^1\) diffeo \(g : M \mapsto M\) sufficiently \(C^1\)-close to \(f\), the completely invariant set

\[
\Lambda^g_V = \bigcap_{m \in \mathbb{Z}} g^m(V)
\]

is hyperbolic for \(g\), if not empty. In particular, \(\Lambda^g_V \supset \Lambda\) is hyperbolic.

**Proof.** 1) Extend the invariant splitting \(T_xM = E^+_x \oplus E^-_x\) defined for \(x \in \Lambda\) to a continuous (but not nec. invariant splitting ) on an open \(V_1 \supset \Lambda\). Given \(\gamma > 0\), let

\[
H^\gamma_x := \{u + v \in T_xM | u \in E^+_x, v \in E^-_x, \|v\| \leq \gamma \|u\|\}
\]

be the corresponding horizontal cone in \(T_xM\), and let \(V^\gamma_x\) be the complimentary vertical cone.

2) \(\exists (\lambda, \mu)\)-splitting on \(\Lambda \implies \)

\[
Df[x](H^\gamma_x) \subseteq H^\gamma_{f(x)} \subset \text{int}H^\gamma_{f(x)} \cap \{0\},
\]

\[
(Df[x])^{-1}(V^\gamma_{f(x)}) \subseteq V^\gamma_{f(x)} \subset \text{int}V^\gamma_{f(x)} \cap \{0\},
\]

and

\[
u + v \in H^\gamma_x \implies \|Df[x](u + v)\| \geq \frac{\mu - \lambda \gamma}{1 + \gamma} \|u + v\|, (1.1)
\]

\[
u + v \in (Df[x])^{-1}(V^\gamma_{f(x)}) \implies \|Df[x](u + v)\| \leq (1 + \gamma)\lambda \|u + v\|. (1.2)
\]

Now, by continuity, for any \(\delta > 0\) we can find a rel. compact nbhd \(V \subseteq V_1\) of \(\Lambda\) and a nbhd \(f\) in \(C^1\)-topology s.t. (7.14) and (7.15) remain valid with \(\mu\) substituted by \(\mu - \delta\) and \(\lambda\) by \(\lambda + \delta\) for all \(x \in V\) and \(g \in W\).

The sequence of differentials \(Dg(g^m(x))\) admits a \((\lambda', \mu')\) splitting with

\[
\lambda' = (1 + \gamma)(\lambda + \delta),
\]

\[
\mu' = \frac{\mu - \lambda \gamma - (1 + \gamma)\delta}{1 + \gamma},
\]

and if \(\delta\) and \(\gamma\) are small, we still have \(\lambda' < 1 < \mu'\), the set \(\Lambda^g_V\) is hyperbolic for \(g\).
Theorem 14. (Structural stability of hyperbolic sets) Let $\Lambda \subseteq M$ be a hyperbolic set for $C^1$ diffeomorphism $f : M \mapsto M$ of a smooth manifold $M$. Then for every open nbhd. $V$ of $\Lambda$ and every $\eta > 0$ there exists a nbhd. $W$ of $f$ in $C^1(M,M)$ such that for all diffeomorphisms $\tilde{f} \in W$ there is a hyperbolic set $\tilde{\Lambda} \subset V$, and a homeomorphism $H : \Lambda \mapsto \tilde{\Lambda}$ with

$$h \circ f = \tilde{f} \circ h$$

on $\Lambda$ and $d_{C^0}(\text{id}, h) + d_{C^0}(\text{id}, h^{-1}) < \eta$. Furthermore, $h$ is unique if $\delta$ is small enough.

Proof.

i) Apply the Shadowing Theorem taking $\delta < \min\{\delta_0, \eta/2\}$, $X = \Lambda$, $h_0 = \text{id}_\Lambda$ and $g = f$. Get a nbhd $V_1 \subset V$ of $\Lambda$, and a nbhd $W_1$ of $f$, such that $d_{C^0}(\tilde{f}, f) < \epsilon$ for all $\tilde{f} \in W_1$, and a unique $h_1 : \Lambda \mapsto V_1$ such that $h_1 \circ f = \tilde{f} \circ h_1$ and $d_{C^0}(\text{id}_\Lambda, h_1) < \delta$.

In particular, $\tilde{\Lambda} = h_1(\Lambda)$ is completely $\tilde{f}$-invariant and hyperbolic by Theorem 48 (after, possibly, a shrinking of $W_1$).

ii) To prove that $h_1$ is injective, we apply the Shadowing Theorem again taking $\delta$ as before, $X = \tilde{\Lambda}$ and $h_0 := \text{id}_{\tilde{\Lambda}}$ and $g = \tilde{f}$, we get the same nbhd $W_1$ as soon as $\epsilon$ is small. Then we have a unique $h_2 : \tilde{\Lambda} \mapsto V$ s.t. $h_2 \circ \tilde{f} = f \circ h_2$ and $d_{C^0}(\text{id}_{\tilde{\Lambda}}, h_2) < \delta$.

iii) To end the proof, it is sufficient to show that $h_2 \circ h_1 = \text{id}_\Lambda$. We apply again the Shadowing Theorem with $X = \Lambda$, $h_0 = \text{id}_\Lambda$ and $g = \tilde{f} = f$. Since

$$d_{C^0}(\text{id}_\Lambda, h_2 \circ h_1) \leq d_{C^0}(\text{id}_\Lambda, h_1) + d_{C^0}(h_1, h_2 \circ h_1) = d_{C^0}(\text{id}_\Lambda, h_1) + d_{C^0}(\text{id}_{\tilde{\Lambda}}, h_2) < 2\delta < \delta_0,$$

we can apply the uniqueness statement in the Shadowing Theorem to get

$$h_2 \circ h_1 = \text{id}_\Lambda,$$

because they both commute with $f$ and are close to $h_1$. \qed