FINAL EXAMINATION

1MA208 Ordinary Differential Equations II

Due March 20, 2017

Name: ____
45% to 62% of the maximum point total - 3
62% to 80% of the maximum point total - 4
$\geq$ 80% of the maximum point total - 5

Problem 1. (Picard-Lindelöf Theorem). 10 points

a) Consider the equation

$$y'(x) = \frac{-xy(x)}{\ln(y(x))}.$$

- Where in $\mathbb{R}$ is the vector field a Lipschitz continuous function?
- For which initial conditions and for which x-intervals the IVP has a unique solution?
- Solve the IVP $y(0) = e^2$. Does it have a unique solution, for which x?

b) Derive several first terms in the Taylor series for $\sin 2t$ by applying Picard's iterations to the first-order system corresponding to the second-order initial value problem

$$x'' = -4x, \quad x(0) = 0, \quad x'(0) = 2.$$

Problem 2. (Dependence on the initial conditions), 9 points

a) Let $F(x,t)$ be a continuous non-autonomous vector field on $\mathbb{R}^n \times \mathbb{R}$ that satisfies

$$\|F(x,t) - F(y,t)\| \leq L(t)\|x - y\|.$$

Show that the solution $\phi_t(x_0)$ of

$$x' = F, \ x(0) = x_0$$

satisfies

$$\|\phi_t(x_0) - \phi_t(y_0)\| \leq \|x_0 - y_0\| e^{\left|\int_0^t L(s)ds\right|}.$$

b) Suppose that $F(x,t)$ is a continuous non-autonomous vector field on $\mathbb{R} \times \mathbb{R}$ which is continuously differentiable in x. Show that we have

$$\frac{\partial \phi_t(x)}{\partial x} = \exp\left(\int_0^t F_1(\phi_s(x), s)ds\right),$$

where $F_1(x,t) := \frac{\partial F(x,t)}{\partial x}$,

Remark: This expression shows how quickly the solution for a smooth vector field in the 1D case (n=1) changes as the initial condition is changed.

Problem 3. (Linearization. Bifurcations). 10 points

Consider the system

$$\begin{aligned} x'(t) &= x(t)^2 + y(t), \\ y'(t) &= x(t) - y(t) + a, \end{aligned}$$

where a is a real parameter.

a) Find all equilibrium points and compute the linearized equation at each.

b) Describe the behaviour of the linearized system at each equilibrium point.

c) Describe any bifurcation that occur.

Problem 4. (Lyapunov function). 10 points

Consider the system

$$\begin{aligned} x'(t) &= (\epsilon x(t) + 2y(t))(z(t) + 1), \\ y'(t) &= (-x(t) + \epsilon y(t))(z(t) + 1), \\ z'(t) &= -z^3(t). \end{aligned}$$

a) Show that the origin is not asymptotically stable when $\epsilon = 0$.

b) Construct a Lyapunov function and show that when $\epsilon < 0$, the basin of attraction of the origin contains the region $z > -1$.

Problem 5. (Poincare-Bendixson Theorem), problem 8 page 223 in HSD, 10 points

Let A be an annular region in $\mathbb{R}^2$ (Fig. 10.12). Let F be a planar vector field in $\mathbb{R}^2$ that points inward along the two boundary curves in A. Suppose that F has no equilibria.

a) Prove that A contains a closed orbit.

b) If there are exactly seven closed orbits in A, show that one of them has orbits spiraling toward it from both sides.

Problem 6. (Limit cycles), 10 points

Consider the system

$$\begin{aligned} r'(t) &= \mu r(t) + ar(t)^3, \\ \theta'(t) &= \omega + br(t)^2. \end{aligned}$$

a) For which values of parameters μ, a and b there is a periodic orbit?

b) Suppose $a < 0$ is fixed. For which values of μ there is a periodic orbit? For which there is none? Is the periodic orbit an ω- or an α-limit cycles? What happens to the equilibrium at the origin at the critical value of μ? (*This is the so called supercritical Poincaré-Andronov-Hopf bifurcation*)

c) Suppose $a > 0$ is fixed. Answer the same questions. (*This is the so called subrcritical Poincaré-Andronov-Hopf bifurcation*)

Problem 7. (Lorenz attractor), problems 14.1 and 14.8 in HSD, pages 325-326, 15 points

a) Consider the non-zero equilibria $Q_\pm$ of the Lorenz flow. Linearize the flow at those points, and consider the linear stability of $Q\pm$. For which values of parameters are they stable, for which are they unstable?

b) Consider the system

$$\begin{aligned} x'(t) &= -3x(t), \\ y'(t) &= 2y(t), \\ z'(t) &= -z(t), \end{aligned}$$

Let R_1 to be a piece of of the upper face of the unit box in $\mathbb{R}^3$, that is R_1 is given by $|x| \leq 1$, $0 < y \leq 1$ and $z = 1$.

Let R_2 to be a piece of of the right face of the unit box in $\mathbb{R}^3$, that is R_2 is given by $|x| \leq 1$, $0 < z \leq 1$ and $y = 1$.

Consider the "flow map" h that takes a point from R_1 to R_2. Show that it is given by

$$(\tilde{x}, \tilde{z}) = h(x, y) = (xy^{\frac{3}{2}}, y^{\frac{1}{2}}).$$

c) Consider a map Φ on a rectangle R as shown in Fig. 14.13, where Φ has similar properties to the model Lorenz map. How many periodic points of period n does Φ have?

(Hint: use the *Schauder fixed-point theorem*: Let C be a nonempty closed convex subset of a Banach space V, if $f : C \mapsto C$ is continuous with a compact image, then f has a fixed point.)