Problem 4

2) Prove of $\lambda_1 > 0$ was done in class, also see

To prove existence of solutions, consider the lin. functional on $H_0^{1,2}(\Omega)$

$$F[v] = -\int_\Omega fv = -\int(\Delta u + cu)v = -\int(-\nabla u \nabla v + cuv)$$
$$= \int \nabla u \nabla v - cuv \equiv B(u,v)$$

We will prove that B is bounded and positive:

- Positivity:

$$B(u,u) = \int|\nabla u|^2 - c\int|u|^2 \geq \int|\nabla u|^2 \cdot \left(1 - c\frac{\int|u|^2}{\int|\nabla u|^2}\right) \geq \int|\nabla u|^2 \cdot \left(1 - \sup c\frac{\int|u|^2}{\int|\nabla u|^2}\right)$$

$$\geq \int|\nabla u|^2 \left(1 - \frac{c}{\inf \frac{\int|\nabla u|^2}{\int|u|^2}}\right) = \int|\nabla u|^2\left(1 - \frac{c}{\lambda_1}\right)$$

If $c < \lambda_1$ then $\left(1 - \frac{c}{\lambda_1}\right) = A > 0$, i.e.

$$B(u,u) = A\int|\nabla u|^2$$

By Poincare inequality $|u|_{1,2} \equiv \int|\nabla u|^2$ is comparable with $\|u\|_{1,2}$, i.e. $\exists$ a constant C, s.t.

$B(u,u) \geq C\|u\|_{1,2}$ – positivity

- Boundedness:

$$B(u,v) = \int \nabla u \nabla v - cuv \leq$$

$$\leq \int |\nabla u \cdot \nabla v| + |c| \int |u||v| \leq \text{Cauchy-Schw} \leq$$

$$\leq \left[\int |\nabla u|^2\right]^{\frac{1}{2}} \left[\int |\nabla v|^2\right]^{\frac{1}{2}} + |c| \left[\int |u|^2\right]^{\frac{1}{2}} =$$

$$= |u|_{1,2} \cdot |v|_{1,2} + |c| \, \|u\|_2 \, \|v\|_2 \leq$$

$$\leq |u|_{1,2} \cdot |v|_{1,2} + |c| \, \|u\|_{1,2} \, \|v\|_{1,2}. \quad < \text{P. ineq.}$$

$$\leq \text{const} \, \|u\|_{1,2} \|v\|_{1,2} + |c| \, \|u\|_{1,2} \|v\|_{1,2} \leq$$

$$\leq \text{const} \, \|u\|_{1,2} \, \|v\|_{1,2} \quad \text{- bounded}$$

By Lax-Milgram, the weak solution in $H_0^{1,2}(\Omega)$ exists and is unique.

Problem 5

First, we show that $F(u)$ is C^1:

$$\frac{1}{\varepsilon}\{F(u+\varepsilon y) - F(u)\} = \frac{1}{\varepsilon}\left[\int_\Omega L(\nabla u + \varepsilon \nabla y, x) - L(\nabla u, x)\right]$$

Since L is $C^{1,1}(\mathbb{R}^n \times \Omega)$ we get

$$\frac{1}{\varepsilon}\left(L(\nabla u + \varepsilon \nabla y, x) - L(\nabla u, x)\right) \xrightarrow[\varepsilon \to 0]{} \frac{\partial}{\partial p} L(\nabla u, x) \nabla y$$

$\frac{\partial}{\partial p}$ here is the gradient ∇_p

Since Ω is bounded, $\int_\Omega \frac{\partial}{\partial p} L(\nabla u, x)\, dx$ is also bounded, and

$$F'(u)y = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\left[F(u+\varepsilon y) - F(u)\right] =$$

$$= \int_\Omega \nabla_p L(\nabla u, x)\, \nabla y \quad \text{exists.}$$

Convexity; use Taylor series "up to second order". To be more precise, since $L \in C^{2,2}(\mathbb{R}^n \times \Omega)$, $\exists$ a point p^*, s.t.

$$L(\nabla u + \nabla y, x) = L(\nabla u, x) + \nabla_p L(\nabla u, x) \cdot \nabla y + \frac{1}{2}\sum_{i,j} L_{p_i p_j}(p^*, x) \frac{\partial}{\partial x_i} y \frac{\partial}{\partial x_j} y \text{, then}$$

$$F(u+y) = F(u) + \int_\Omega \nabla_p L(\nabla u, x) \cdot \nabla y + \frac{1}{2}\sum L_{p_i p_j}(p^*, x)\, y_{x_i} y_{x_j} \geq F(u) + \int_\Omega \nabla_p L(\nabla u, x) \cdot \nabla y + \varepsilon \int_\Omega |\nabla y|^2$$

Since $y \in H^{1,2}(\Omega)$, the last integral is bounded and

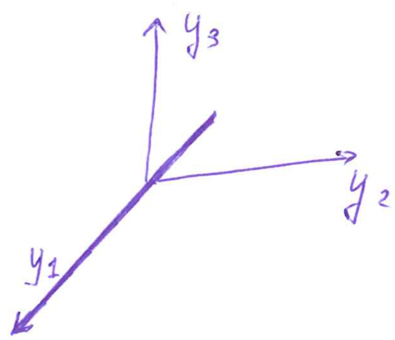
$$F(u+y) - F(u) \geq F'(u)y$$

Problem 3

$$\Delta u = -4\pi \rho$$

$$\rho = \ell(y_1) \cdot \delta(y_2)\, \delta(y_3)$$

$\ell(y_1) \equiv \ell$ - linear charge density



Now, reason in one of the two following ways:

1) A weak solution of the Poisson $\Delta u = f$ eq-n is given by the convolution $u(x) = \int K(x-y) f(y)\,dy$, or

2) The representation theorem for $C^2(\bar{\Omega})$ functions, Ω - bounded, gives.

$$u(x) = \int_{\Omega} K(x-y)\,\Delta u(y)\,dy + \text{boundary terms},$$

and the boundary terms $\to 0$ as $\mathrm{diam}(\Omega) \to \infty$.

Either way:

$$u(x) = -4\pi \int_{\mathbb{R}^3} K(x-y)\,\rho(y)\,dy =$$

$$= -4\pi \int_{\mathbb{R}^3} \frac{1}{(2-n)\,\omega_n} |x-y|^{2-n} \rho\,\delta(y_2)\,\delta(y_3)\,dy \Big|_{n=3}$$

$$= \frac{4\pi}{\omega_3} \int_{\mathbb{R}^3} \frac{\rho\,\delta(y_2)\,\delta(y_3)}{|x-y|}\,dy = \frac{4\pi}{4\pi}\rho \int_{\mathbb{R}^3} \frac{\delta(y_2)\,\delta(y_3)}{\sqrt{(x_1-y_1)^2+(x_2-y_2)^2+(x_3-y_3)^2}}$$

$$= \rho \int_{\mathbb{R}} \frac{dy_1}{\sqrt{(x_1-y_1)^2 + x_2^2 + x_3^2}} =$$

$$= \rho \ln\left((y_1 - x_1) + \sqrt{(x_1-y_1)^2 + x_2^2 + x_3^2} \right) \Big|_{-\infty}^{\infty} \longrightarrow \infty$$

The integral diverges!!!

However, if one considers the difference
$u(x_1) - u(\tilde{x}_2)$, then this is well-defined and is equal to
$$-f(\ln|x| - \ln|\tilde{x}|)$$

Potential is infinity if compared to potential at infinity, but the difference is well-defined