1. Properties of rotations number

Proposition 1. *$\rho(f)$ is continuous in C^0 topology.*

Proof. Let $\rho = \rho(f)$, p, q, p', q' be such that $p'/q' < \rho < p/q$. Pick a lift F of f, for which

$$-1 < F^q(x) - x - p \leq 0$$

for some $x \in \mathbb{R}$. Then $F^q(x) < x + p$ for all $x \in \mathbb{R}$ (otherwise $F^q(y) = y + p$ for some $y \in \mathbb{R}$). The function $F^q - id$ is periodic and continuous, attains its maximum: $\exists\, \delta > 0$ such that

$$F^q(x) - x - p - \delta < 0$$

for all $x \in \mathbb{R}$. If $\tilde{F}$ is sufficiently close to F in C^0 topology then the same is true for $\tilde{F}$: $\exists\, \epsilon > 0$ s.t. $|F(x) - \tilde{F}(x)| < \epsilon$ for all $x \in \mathbb{R} \implies$

$$\tilde{F}^q(x) - x - p < 0 \implies \rho(\tilde{F}) < \frac{p}{q}.$$

Similarly, $\exists\, \epsilon > 0$ s.t. $|F(x) - \tilde{F}(x)| < \epsilon$ for all $x \in \mathbb{R} \implies \rho(\tilde{F}) > \frac{p'}{q'}$. Hence, for any $\varepsilon > 0$ $\exists\, \epsilon > 0$ s. t. $\sup_{x \in \mathbb{T}} \tilde{f}(x) - f(x)| < \epsilon \implies |\rho(\tilde{f}) - \rho(f)| < \varepsilon$. $\square$

Define an ordering on a family of OP (orientation preserving) homeos of $\mathbb{T}$. If $f_{t_i} : \mathbb{T} \mapsto \mathbb{T}$, $i = 1, 2$, then set

$$f_t = \frac{(t_2 - t) f_{t_1} + (t - t_1) f_{t_1}}{\|(t_2 - t) f_{t_1} + (t - t_1) f_{t_2}\|},$$

here, $\|\cdot\|$ is a distance function in $\mathbb{R}^2$. This straight line homotopy between f_{t_1} and f_{t_2} is an OP circle homeo (prove!). Lift to $\mathbb{R}$, to get F_t. This specifies a "canonical" choice of two lifts F_{t_i} of f_{t_i}, $i = 1, 2$.

We says that $f_{t_1} < f_{t_2}$ if $F_{t_1}(x) < F_{t_2}(x)$ for all x in $\mathbb{R}$. This ordering is not transitive.

By the definition of the rotation number we have

Proposition 2. *ρ is monotone: $f_1 < f_2 \implies \rho(f_1) \leq \rho(f_2)$.*

Proposition 3. *If $f_1 < f_2$ and $\rho(f_1) \in \mathbb{R} \setminus \mathbb{Q}$ then $\rho(f_1) < \rho(f_2)$.*

Proof. By def. $f_1 < f_2 \implies F_2(x) - F_1(x) > 0$ for the two canonical lifts and all $x \in \mathbb{R}$. $F_i - id$ are periodic with the same period, then so is $F_2 - F_1$, also, continuous, then $\exists$ $\delta > 0$ such that $F_2(x) - F_1(x) > \delta$ for all $x \in \mathbb{R}$. Take p, q s. t.

$$\frac{p - \delta}{q} < \rho(F_1) < \frac{p}{q}.$$

Then, $\exists\ x_0 \in \mathbb{R}$, s.t.

$$F_1^q(x_0) - x_0 > p - \delta$$

(otherwise $\rho(F_1) = \lim_{n\to\infty} \frac{F_1^{nq}(x)-x}{nq} \leq \lim_{n\to\infty} \frac{n(p-\delta)}{nq} = \frac{p-\delta}{q}$).

Next,

$$F_2^q(x_0) = F_2(F_2^{q-1}(x_0)) > F_1(F_2^{q-1}(x_0)) + \delta > F_1(F_1^{q-1}(x_0)) + \delta = F_1^q(x_0) + \delta > x_0 + p.$$

Now we have two cases:

- $F_2^q(x) > x + p$ for all $x \in \mathbb{R}$ - we are done, $\rho(F_2) \geq p/q$;
- if not, by continuity $\exists\ x_1 \in \mathbb{R}$ s.t. $F_2^q(x_1) = x_1 + p$ and $\rho(F_2) = p/q$.

In either case

$$\rho(F_2) \geq \frac{p}{q} > \rho(F_1).$$

□

The Proposition above shows that the property of having an irrational rotation number is not stable under perturbations in an ordered family. However, the rational rotation numbers persist under perturbations, as the next Proposition demonstrates.

Proposition 4. *Let f_t an ordered family of OP homeo of the circle, and suppose that for some t^*, $\rho(f_{t^*}) = \frac{p}{q}$, $p, q \in \mathbb{N}$, and some non-periodic points. Then all sufficiently small perturbations $f_{t^*+\epsilon}$ or all sufficiently small perturbations $f_{t^*-\epsilon}$ have the same rotation number $\frac{p}{q}$.*

Proof. $F_{t^*}^q - id - p$ does not vanish identically for any lift F_{t^*} of f_{t^*}.

1) Suppose $\exists\ x \in \mathbb{R}$ s.t. $F_{t^*}^q(x) - x - p > 0 \implies$ for small $\epsilon > 0$, $f_{t^*-\epsilon}$ the periodic function $F_{t^*-\epsilon}^q - id - p$, where $F_{t^*-\epsilon}$ is compatible with F_{t^*}, is also positive at the same x

$$F_{t^*-\epsilon}^q(x) - x - p > 0 \implies \rho(f_{t^*-\epsilon}) \geq \frac{p}{q}.$$

But by monotonicity of the rotation number: $\rho(f_{t^*-\epsilon}) \leq \rho(f_{t^*}) = \frac{p}{q}$, and therefore

$$\rho(f_{t^*-\epsilon}) = \frac{p}{q}.$$

2) Otherwise, $\exists\ x \in \mathbb{R}$ s.t. $F_{t^*}^q(x) - x - p < 0$. Repeat the argument for $f_{t^*+\epsilon}$, $\epsilon > 0$ and small. □

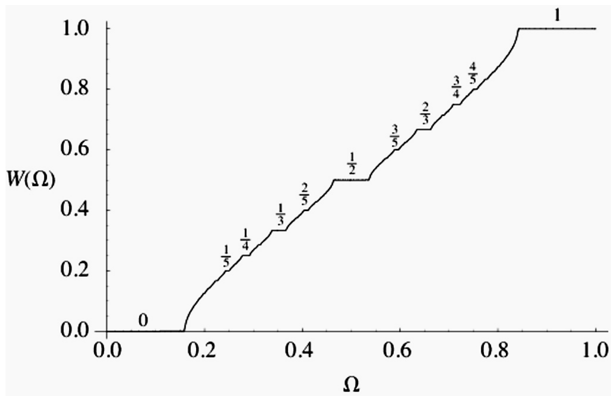

Figure 1. The rotation number (called $W(\Omega)$ here) as a function of parameter Ω in the Arnold family $f(x) = x + \Omega + \frac{1}{2\pi}\sin 2\pi x \mod 1$.

Definition 5. *$\rho : [0,1] \mapsto \mathbb{R}$ is called devil's staircase if there is a family $\{I_\alpha\}$, $\alpha \in A \subset \mathbb{R}$, of disjoint and open subintervals of $[0,1]$ with a dense union in $[0,1]$ s. t. ρ assumes distinct constant values on these subintervals.*

An example of a devil's staircase is Cantor's function.

Let's us denote, for brevity, $\mathbb{Q}' = \mathbb{Q} \cap [0,1]$.

Theorem 6. *Let f_t, $t \in [0,1]$ be a monotone continuous family of OP circle homeos, s.t. $\rho : t \mapsto \rho(f_t)$ is non-constant.*

If $\exists$ a dense subset $S \subset \mathbb{Q}'$ such that no map f_t is topologically conjugate to R_α with $\alpha \in S$, then ρ is a devil's staircase.

Proof. (Proof for $S = \mathbb{Q}'$)

1) By the above propositions $t \mapsto \rho(f_t)$ is monotone and continuous.
2) Since the rational rotation number of a map which is not topologically conjugate to a rigid rotation persists under perturbations, $\rho^{-1}(\mathbb{Q}')$ is a disjoint union of closed intervals of positive length. We need to show that $\rho^{-1}(\mathbb{Q}')$ is dense.
3) ρ is strictly monotone at $T = \rho^{-1}([0,1] \setminus \mathbb{Q}')$. If $t \in T' = \rho^{-1}([0,1) \setminus \mathbb{Q}')$ then $\rho(f_t) \neq \rho(f_{t+\epsilon})$ for any $\epsilon > 0$. by continuity of ρ, density of $\mathbb{Q}'$ in $[0,1]$, and the Intermediate Value Theorem (all values between $\rho(f_t)$ and $\rho(f_{t+\epsilon})$ are attained, in particular, some rational value in $(\rho(f_t), \rho(f_{t+\epsilon}))$), $\exists\, t_1 \in \rho^{-1}(\mathbb{Q}') \cap [t, t+\epsilon]$.

$\square$

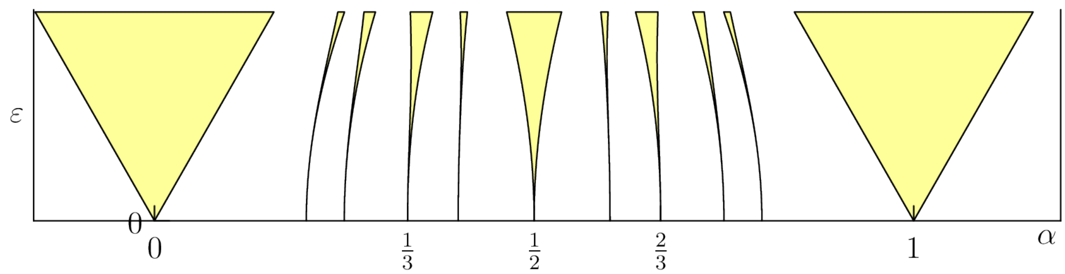

Figure 2. The regions of the constant rotation number in a two-parameter Arnold family $f(x) = x + \alpha + \frac{\epsilon}{2\pi}\sin 2\pi x \mod 1$. The values of the rotation number is marked at the tips of the tongues.

Remark 7. *Consider the two parameter family*

$$f(x) = x + \alpha + \frac{\epsilon}{2\pi}\sin 2\pi x \mod 1,$$

called the Arnold family (Vladimir Arnold, $1937-2010$*, an outstanding Soviet/Russian dynamicist). For a fixed values of the parameter* $\epsilon = const$*, the slice* $\epsilon = const$ *of the diagram* 1 *is a collection of intervals on which the rotation number assume different rational constant values. Notice, that the slice* $\epsilon = 0$ *produces a collection of points, as it should be for the rigid rotations* R_α*, while the section* $\epsilon = 1$ *corresponds to the devil's staircase of picture* 1*. It is only in that case of* $\epsilon = 1$*, that the collection of intervals is dense in* $[0, 1]$*.*

2. Period-doubling in the quadratic family and renormalization

2.1. Period-doubling bifurcations

Proposition 8. *Let* $U \subset \mathbb{R}^m$ *and* $v \subset \mathbb{R}^n$ *, open, and* $f_\mu \mapsto \mathbb{R}^m$*,* $\mu \in V$ *be a family of* C^1 *maps such that*

1) *the maps* $(x, \mu) \mapsto f_\mu(x)$ *is a* C^1*-map;*
2) $f_{\mu_0}(x_0) = x_0$ *for some* $x_0 \in U$ *and* $\mu_0 \in V$*;*
3) 1 *is not an eigenvalue of* $Df_{\mu_0}(x_0)$*,*

then there are open subsets $U' \subset U$ *and* $V' \subset V$*,* $(x_0, \mu_0) \in U' \times V'$*, and a* C^1*-curve* $\gamma : V' \mapsto U'$*, such that* $f_\mu(\gamma(\mu)) = \gamma(mu)$*,* $\gamma(mu)$ *is the unique f.p. in* U'*.*

Proof. Apply the Implicit Function Theorem to $g = f_{\mu_0} - id$. $\square$

Proposition 9. *Let $f_\mu : I \mapsto I$, $I \subset \mathbb{R}$ -open, be a family of C^3 maps, $\mu \in J$, $J \subset \mathbb{R}$ s.t.*

1) *$f(x_0)_{\mu_0} = x_0$ and $f'_{\mu_0}(x_0) = -1$ for some $x_0 \in I$ and $\mu_0 \subset J$ (in particular, the above Prop. applies);*
2) *$\eta = \frac{\partial f'_\mu(\gamma(\mu))}{\partial \mu}_{\mu=\mu_0} < 0$;*
3) *$\zeta = (D^3_x f_\mu(f_\mu(x)))_{(x,\mu)=(x_0,\mu_0)} < 0$*

Then there are $\epsilon > 0$, $\delta > 0$ an C^3 functions $\gamma : (\mu_0 - \delta, \mu_0 + \delta) \mapsto \mathbb{R}$, $\gamma(\mu_0) = x_0$ and $\alpha : (x_0 - \epsilon, x_0 + \epsilon) \mapsto \mathbb{R}$, $\alpha(x_0) = \mu_0$, $\alpha'(x_0) = 0$ and $\alpha''(x_0) = -2\eta/zeta > 0$, such that

1) *Prop. 8 applies with $U = (\mu_0 - \delta, \mu_0 + \delta)$ and $V = (x_0 - \epsilon, x_0 + \epsilon)$;*
2) *$\gamma(\mu)$ is attracting for $\mu \in (\mu_0 - \delta, \mu_0)$ and repelling for $\mu \in (\mu_0, \mu_0 + \delta)$*
3) *for every $\mu \in (\mu_0, \mu_0+\delta)$ f_μ has an attracting period orbit $\{x_1, x_2\}$ in $(x_0-\epsilon, x_0+\epsilon)$, $x_i(\mu) \to x_0$ as $\mu \to \mu_0$, $i = 1, 2$, and $\alpha(x_i) = \mu$.*
4) *for every $\mu \in (\mu_0 - \delta, \mu_0)$, $f^{\circ 2}_\mu$ has one fixed point in $(x_0 - \epsilon, x_0 + \epsilon)$.*

2.2. Period doubling bifurcation cascade in the quadratic family

.

The quadratic family $q_\mu(x) = \mu x(1 - x)$ undergoes a cascade of period doubling bifurcations. Numerically, there exists a sequence of parameter values $\{\mu_k\}, k \in \mathbb{N}$, s. t. q_μ has and attracting periodic orbit of period 2^k for $\mu \in (\mu_k, \mu_{k+1})$, the orbit loses its stability through a period doubling bifurcation at μ_{k+1}, and there are no 2^k periodic orbit for $\mu < \mu_k$ in the real line.

Specifically, one observes two phenomena.

1) **Universality in the parameter plane**. Consider $\{\mu_k\}, k \in \mathbb{N}$. First, $\lim_{k\to\infty} \mu_k$ exists and is equal to an irrational number 3.5699.... Furthermore, the following limit exists

$$\gamma = \lim_{k\to\infty} \frac{\mu_k - \mu_{k-1}}{\mu_{k+1} - \mu_k} = 4.6692\ldots, \tag{2.1}$$

and is, again, an irrational number.

The crucial observation is that the bifurcation cascade exists, and the ration above is define and is equal to the same γ for C^3 perturbations of the quadratic family, and more generally, for a class of maps called "quadratic-like".

2) **Universality in the dynamic plane**. Consider the parameter values μ^*_k for which the superattracting point 0.5 is in the 2^k-th periodic orbit. Call the distances between

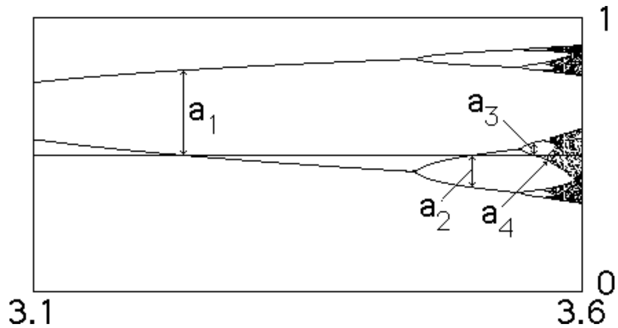

Figure 3. Self-similarity in the dynamic plane for a "quadratic-like" map

0.5 and a neighboring periodic point a_k (see Fig 2.2). Then

$$\alpha = \lim_{k\to\infty} \frac{a_{k+1}}{\tilde{a}_k} = 0.396\ldots, \tag{2.2}$$

again, an irrational number.

The ratio above is define and is equal to the same α for C^3 perturbations of the quadratic family, and more generally, for a class of maps called "quadratic-like".

For references purposes, here the first 14 superattracting periodic points.

2.3. The Cantor attractor for the Feigenbaum function

.

Recall, that the logistic map $\mu x(1-x)$ is affinely conjugate to a quadratic polynomial

$$p_r(x) = 1 - rx^2$$

for $r = r(\mu)$ is uniquely defined by μ.

Pass to these coordinates. p_r is a map of the interval $[-1,1]$ into itself for all $r \in [0,2]$. Assume that such p_r is renormalizable. We will construct an invariant central interval J, such that $p_r \circ p_r : J \mapsto J$. Let

$$a = a(p_r) = -p_r(1),\ b = b(p_r) = p_r(a),\ p_r(b) = c$$

Assume that $0 < a < b$ and that $p_r(b) = p_r(a) < a$. Then

$$p_r : [-a,a] \mapsto [b,1] \mapsto [-a, c \subset [-a,a].$$

Thus the set $\Lambda_1 = [-a,c] \cup [b,1]$ is mapped into itself. If p_r is twice renormalizable, that is $\mathcal{R}p_r$ is renormalizable, then repeat the construction for $\mathcal{R}p_r = \lambda \circ p_r \circ p_r \lambda^{-1}$,

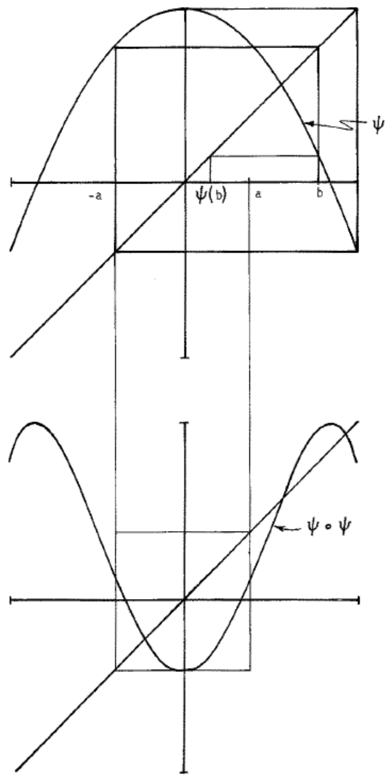

Figure 4. Invariant intervals for $\psi = p_{1.4}$, *from P. Collet, J.-P. Eckmann and O. E. Lanford, 1980*

λ here is the affine rescaling of $[-a, c]$ onto $[1, 1]$. Find a set $[-a', c'] \cup [b', 1]$ invariant under $\mathcal{R}p_r$. Its preimage under λ, $\lambda\left([-a', c'] \cup [b', 1]\right)$ is in $[-a, c]$. Consider the set

$$\Lambda_2 = \lambda\left([-a', c'] \cup [b', 1]\right) \cup p_r\left(\lambda\left([-a', c'] \cup [b', 1]\right)\right)$$

It consists of four intervals which are mapped onto each other by p_r.

Now, suppose p_r is infinitely many time renormalizable. Then we can construct a hierarchy of closed sets Λ_k, each consisting of 2^k closed disjoint intervals. These intervals are labeled in the following way.

- $[-a, c] = \Lambda_1^0$, $[b, 1] = \lambda_1^1$.
- Assume that the intervals upto order k have been labeled. From the middle of each Λ_k^i, $i = 0, 2^k - 1$, remove an open interval, the resulting two intervals are labeled Λ_{k+1}^i and $\Lambda_{k+1}^{i+2^k}$.

With this labeling

$$\Lambda_k^i = p_r^i(\Lambda_k^0).$$

Set $\Lambda = \cap_k \Lambda_k$. We have the following

Theorem 10. *(P. Collet, J.-P. Eckmann, O. E. Lanford, 1980) If f is a C^3 unimodal function in the renormalization stable manifold $\mathcal{W}$ (and, therefore, it is infinitely renormalizable), then f admits an invariant Cantor set Λ such that the action of f on Λ is homeomorphic to that of an "odometer" on Σ_2^+. This homeomorphism $\phi : \Sigma_2^+ \mapsto \Lambda$ is given by*

$$\phi((i_1, i_2, i_3, \ldots)) = \cap_{k=1}^{\infty} \Lambda_k^{j(k)}, \ j(k) = i_1 + i_2 2^1 + i_3 2^2 + \ldots + i_k 2^{k-1}$$

Here, the "odometer" is the following symbolic dynamical system. Consider the formal power series, a representation of $\omega \in \Sigma_2^+$:

$$\omega = (\omega_1, \omega_2, \ldots) \mapsto \sum_{k=1}^{\infty} \omega_k 2^{k-1}.$$

An "odometer" p is the operation of adding 1 in this group. The group Σ_2^+ itself is the inductive limit of the groups W^n of n-tuples $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ with the odometer acting on elements of W^n as

$$p\left(\sum_{k=1}^{n} \omega_k 2^{k-1}\right) = \left(\sum_{k=1}^{n} \omega_k 2^{k-1} + 1\right) \mod 2^n.$$

Proposition 11. *(P. Collet, J.-P. Eckmann, O. E. Lanford, 1980) Suppose that $f \in \mathcal{W}^s$, then*

1) *f has exactly one periodic orbit of each period 2^k and no periodic orbits of other periods. All these periodic orbits are repelling;*
2) *every orbit of f which is not preperiodic converges to the invariant Cantor set.*

3. The Sharkovsky Theorem

Definition 12. *The Sharkovsky ordering of $\mathbb{N}_{Sh} = \mathbb{N} \cup \{2^\infty\}$ is defined by*

$$\begin{aligned}
&1 \prec 2 \prec 2^2 \prec \ldots \prec 2^n \prec 2^\infty \prec \ldots \\
&\quad \ldots \\
&\prec 2^m(2n+1) \prec 2^m(2n-1) \prec \ldots \prec 2^n \cdot 7 \prec 2^m \cdot 5 \prec 2^m \cdot 3 \prec \ldots \\
&\quad \ldots \\
&\prec 2(2n+1) \prec 2(2n-1) \prec \ldots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \ldots \\
&\prec (2n+1) \prec (2n-1) \prec \ldots \prec 7 \prec 5 \prec 3.
\end{aligned}$$

- $\mathbb{N}_{Sh}$ has the least-upper-bound property;
- order on $\mathbb{N}_{Sh}$ is preserved by multiplication by 2.

Theorem 13. *(Sharkovsky) Fro every continous map $f : I \mapsto I$ there is $\alpha \in \mathbb{N}_{Sh}$ such that $MinPer(f) = S(\alpha) := \{k \in \mathbb{N}, \prec \text{ or } = \alpha\}$. Conversely, for every $\alpha \in \mathbb{N}_{Sh}$ there is a continous map $f : I \mapsto I$ with $MinPer(f) = S(\alpha)$.*

Consider a cont. map $f : I \mapsto I$. We say that $J \subset I$ $\underline{f\text{-covers}}$ $K \subset I$ if $K \subset f(J)$, denoted

$$J \mapsto K.$$

• If $J = [a, b]$, then $J \mapsto J$ implies that $\exists\ c, d \in J$, such that $f(c) = a \leq c$ and $f(d) = b \geq d$. By the Intermediate Value Theorem $f(x) - x$ has a zero in J.

• If $J \mapsto K$, and K is closed, then $\exists$ a closed interval $L \subset J$ s. t. $f(L) = K$. Indeed, write $K = [a, b]$. Set

$$c = \max_{over\ preimages} f^{-1}(a),\ d = \min_{over\ preimages} ((c, \infty) \cap f^{-1}(b)),$$

if this is defined, and set $L = [c, d]$. Otherwise, set $L = [c', d']$ with

$$c' = \max_{over\ allpreimages} ((-\infty, c) \cap f^{-1}(b)),\ d = \min_{over\ preimages} ((c', \infty) \cap f^{-1}(a)).$$

• In general, if $J \mapsto K$, there are several $L_i \subset J$ with pairwise disjoint interiors, s.t. $f(L_i) = K$. L_i's are called full components associated to covering $J \mapsto K$. The preimage of K in J may contain infinitely many intervals, but by compactness there only a finite number of full components.

Lemma 14. *If $I_0 \mapsto I_1 \mapsto I_2 \mapsto \ldots \mapsto I_n$ then $\cap_{i=0}^{n} f^{-i}(I_i)$ contains an interval Δ_n such that $f^n(\Delta_n) = I_n$*

Lemma 15. *(Minimal Markov model of an interval map with odd periodic points). Let $I \subset \mathbb{R}$ be a closed interval and $f : I \mapsto I$ a cont. map. Let $x \in I$ be in a periodic orbit of odd period $p > 1$ s.t. there is no periodic orbit of period $1 < q < p$. If x_{min} and x_{max} are the minimum and the maximum of the orbit of x, then the Markov graph of partition C of $[x_{min}, x_{max}]$ induced by the orbit of x contains the following sub-graph.*

One can label the intervals of the partition as $\{I_1, \ldots, I_{p-1}\}$ in such a way that

$$I_1 \mapsto I_1 \mapsto I_2 \mapsto \ldots \mapsto I_{p-1}$$

and

$$I_{p-1} \mapsto I_k$$

for every odd k.

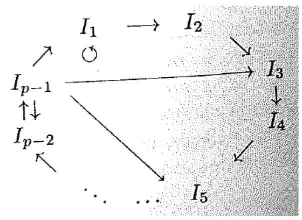

Figure 5. The minimal Markov model of an interval map with periodic points of odd period.

Proof. 1) Set $a = \max\{y \in O(x) : f(y) > y\}$ and $I_1 = [a, b]$ where b is the closest-on-the-right to a point in $O(x)$. Then $f(a) \geq b$ and $f(b) \leq a$, hence $I_1 \subset f(I_1)$, that is $I_1 \mapsto I_1$, and the inclusion is proper since $O(x)$ is of odd period larger than 1 (otherwise, $f(a) = b$ and $f(b) = a$ implies $f^2(a) = a$ - even period). Next, $f(I_1) \subset f(f(I_1))$, by induction,

$$I_1 \subset f(I_1) \subset \ldots \subset f^p(I_1).$$

$f^p(I_1)$ contain <u>all of</u> the orbit of $a \implies$ contains J.

2) We now show that there is an element I' of $C \setminus I_1$ s. t. $I' \mapsto I_1$. Set $l = card\{y \in O(x) : y < a\}$ and $r = card\{y \in O(x) : y > b\}$. Then $l + r = p - 2 \geq 1 \implies l \neq r$. This at least one of the components of $I \setminus I_1$ has a point x' of $O(x)$ s.t. $f(x')$ is in the same component. Also, both components $I \setminus I_1$ have points x'' in $O(x)$ s. t. $f(x'')$ is in the other component (otherwise, all of $O(x)$ would be contained in a single component). Therefore, there are adjacent points c and d such that exactly one of them maps to the other component. Set $I' = [c, d]$, such $I' \mapsto I_1$.

3) Label all other intervals in such a way that

$$]I_1 \mapsto \ldots \mapsto I_k \mapsto I_1 \tag{3.3}$$

is the shortest nontrivial loop that contains I_1. We show that $k = p - 1$.

Since $1 < k \leq p - 1$ by minimality of the loop, it suffices to show that $k \geq p - 1$. Let $q \in \{k, k+1\}$ be odd. By Cor. **??**, existence of the loop $I_1 \mapsto \ldots \mapsto I_k \mapsto I_1$ and (since I_1 covers itself) of the loop $I_1 \mapsto \ldots \mapsto I_k \mapsto I_1 \mapsto I_1$ implies that there is a fixed point y of f^q (with odd q in $\{k, k+1\}$).

We first show that $q \neq 1$. By Lemma 13, there exists $\Delta_n \in I_1 \cap f^{-1}(I_2) \cap \ldots \cap f^{-q}(I_1)$ such that Δ_n f^q-covers I_1, and, thus, $y \in \Delta_n$. But this y is, thus, in $f^{-1}(I_2)$, and $f(y) \in I_2$. Suppose that y is the fixed point of f, then $y \in I_1$ and $y = f(y) \in I_2 \implies y$ is the common boundary point of I_1 and I_2 and is in $O(x)$ - not a fixed point of f.

Therefore, $q > 1$, and is odd by choice, thus by the hypothesis, we have $q \geq p \implies k \geq p - 1$.

4) We show that $I_{p-1} \mapsto I_k$ for odd k. For that we first show that the order of the intervals in $\mathbb{R}$ is as follows:

$$I_{p-1}, I_{p-3}, \ldots, I_2, I_1, I_3, \ldots, I_{p-2}. \tag{3.4}$$

Since the loop (3.3) is the shortest nontrivial loop $I_1 \mapsto \ldots \mapsto I_1$, $I_k \mapsto I_j \implies j \leq k+1$. Now I_1 covers I_1 and I_2, hence, by connectedness, I_2 is adjacent to I_1, so $I_2 = [e, a]$ and since $I_1 \mapsto I_1 \cap I_2$, we must have

$$f(a) = b, f(b) = e.$$

We determine $f(I_2)$. I_2 does not cover I_1 (since (3.3) is the shortest nontrivial) and $f(I_2)$ lies entirely to the right of I_1 (by the def. of a). It covers I_3, so I_3=$[b, d]$. It covers I_3 so $I_3 = [b, d]$. Since I_2 covers no other intervals, $d = f(e) = f^2(b)$. Obtain inductively the ordering (3.4).

5) Writing $a_i = f^i(a)$, we have

$$x_{\min} = a_{p-1} < a_{p-3} < \ldots < a_2 = e < a_0 = a < a_1 = b < a_3 = d < \ldots < a_{p-2} = x_{\max},$$

hence

$$I_{p-1} = [a_{p-1}, a_{p-3}] \mapsto I_k$$

for odd k. □

Lemma 16. *If f has a periodic point of even period then it has a point of period* 2.

Proof. Let p be the smallest even period. Let $O(x)$ be a period p orbit.

1) $p \neq 2$, then, *first suppose that there is a pair c, d of adjacent points in $O(x)$, s.t. $I_k = [c, d] \neq I_1$ covers I_1* (I_1 as in the previous Lemma). Label the intervals, such that

$$I_1 \mapsto \ldots \mapsto I_k \mapsto I_1 \tag{3.5}$$

is the shortest nontrivial loop $I_1 \mapsto \ldots \mapsto I_1$, then $k \leq p - 1$.

Let $q \in \{k, k+1\}$ be even. Then $q \leq p$. Since the existence of the loop (3.5) or

$$I_1 \mapsto \ldots \mapsto I_k \mapsto I_1 \mapsto I_1$$

implies existence of a fixed point y of f^q.

This y can not be a fixed point of f (same argument as in the proceeding Lemma). Hence, $q \geq p$ and $k \geq p-1$.

As in the previous Lemma, argue that the intervals are ordered like in (3.4), hence

$$I_{p-1} \mapsto I_k$$

for even k. Thus the loop

$$I_{p-1} \mapsto I_{p-2} \mapsto I_{p-1}$$

gives a period 2 orbit.

2) $p \neq 2$, and, *there is no* $I_k = [c, d]$ *with* c *and* d *in* $O(x)$ *covering* I_1. We show that $[x_{min}, a] \mapsto [b, x_{max}] \mapsto [x_{min}, a]$.

$f(x) \geq b$ so $f([x_{min}, a])$ contains points to the right of I_1. Our assumption in 2) implies that $[x_{min}, a]$ does not cover I_1, so all of $f([x_{min}, a])$ is to the right of I_1. Similarly $f([b, x_{max}]$ lies to the left of b. But since f always send a point from the left of I_1 to a point on the right and vice versa, and permutes points of $O(x)$, we have that

$$[x_{min}, a] \mapsto [b, x_{max}] \mapsto [x_{min}, a],$$

which implies existence of a period 2 point. $\square$

Proof. of the Sharkovsky Theorem.
(0) $p = 2^k$. We show that there are periods 2^l, $l < k$.

If x is a periodic period with a minimal period p, then by 14 period $1 = 2^0$ exists by the self-loop in Fig. 5.

Consider $g = f^{2^{l-1}}$ it has a periodic orbit of period 2^{k-l+1}, then by Lemma 15, g has a periodic orbit of period 2, while f that of period 2^l.

(1) $p = r2^k$, r odd. Consider the map $g = f^{2^k}$. It has an odd period.

a) $q = s2^k$, s even. Assume that r is minimal, that is f has no periodic points of period $t2^k$ for $t < r$, odd. r is the minimal odd period for f^{2^k}. By Lemma 14 there is a non-trivial loop of length s:

$$I_{r-1} \mapsto I_{r-2} \mapsto \ldots \mapsto I_{r-2} \mapsto I_{r-1}$$

if $s < r$. Otherwise

$$I_1 \mapsto I_2 \mapsto \ldots \mapsto I_{r-1} \mapsto I_1 \mapsto I_1 \ldots \mapsto I_1.$$

Thus $g = f^{2^k}$ has a periodic point of minimal period s, $s2^k$ is period for f, and it is minimal, since, otherwise, $s/2$ would be a period for f^{2^k} - impossible by minimality of s.

b) $q = 2^l$, $l \le k$. Apply *a)* with $s = 2$, get a periodic orbit of period 2^{k+1}. By (0), there are orbits of all periods 2^l, $l \le k$.

c) $q = s2^k$, $s > r$ odd. The loop

$$I_1 \mapsto I_2 \mapsto \ldots \mapsto I_{r-1} \mapsto I_1 \mapsto I_1 \ldots \mapsto I_1$$

gives a point of the minimal period s for f^{2^k}. If the minimal period for f in this case is $s2^k$, then we are done. Otherwise it is $s2^t$ for some $t < k$. But then, take $p' = s2^t$ and $s' = s2^{k-t}$ and, by case *a)* get a periodic orbit of minimal period $s'2^t = s2^k$.

□

4. The Hartman-Grobman Theorem

Let F be a map of a subset of $\mathcal{A} = \mathbb{R}^n$ or $\mathcal{A} = \mathbb{C}$ onto itself, and let $\|\cdot\|$ be a norm in $\mathcal{A}$.

Let $\mathcal{C}^0_b(\mathcal{U}, \mathcal{A})$ be the Banach space of continuous functions defined on a subset $\mathcal{U}$ of $\mathcal{A}$.

Set

$$Lip(F) = \sup_{x \neq y \in \mathcal{A}} \frac{\|F(x) - F(y)\|}{\|x - y\|} \tag{4.6}$$

This, for example, is defined on the following Banach space

$$\mathcal{C}^1_b = \{F \in \mathcal{C}^1(\mathcal{A}, \mathcal{A}) \cap \mathcal{C}^0_b(\mathcal{A}, \mathcal{A}) : \sup_{x \in \mathcal{A}} \|DF(x)\| < \infty\}.$$

The set of all functions in $F \in C^0(\mathcal{U}, \mathcal{A})$, $\mathcal{U} \subset \mathcal{A}$ such that $Lip(F) < \infty$ is denoted $\mathcal{L}(\mathcal{U}, \mathcal{A})$ - *Lipschitz functions.*

Theorem 17. *Global Hartman-Grobman Theorem Suppose that a map $F \in \mathcal{L}(\mathcal{U}, \mathcal{A})$ is hyperbolic and invertible. Then $\exists$ an $\epsilon > 0$ such that for every $g \in \mathcal{C}^1_b(\mathcal{U}, \mathcal{A})$ satisfying $Lip(G) < \epsilon$, there is a unique function $v \in \mathcal{C}^0_b(\mathcal{U}, \mathcal{A})$ such that*

$$G(h(x)) = h(F(x))$$

where

$$h = id + v, \quad G = F + g,$$

and h is a homeomorphisms onto the image of $\mathcal{U}$.

In particular, this theorem tells us that if G is a perturbation of a linear map F, then there is a linearizing coordinate h which conjugates it to its "normal form" F. We will not proof this theorem (we will rather concentrate on a much stronger Hadamard-Perron Theorem): the proof can be found in KH, or, in these lecture notes.

The Hartman-Grobman theorem gives us a sufficient (but not necessary) condition for a conjugacy to exist. But it does not give a a simple way to construct the conjugacy, nor does it tell us how smooth the conjugacy is.

5. The Hadamard-Perron Theorem

Definition 18.

- *Let $\mu < \lambda$. A sequence of invertible linear maps $L_m : \mathbb{R}^n \mapsto \mathbb{R}^n$, $m \in \mathbb{Z}$ admits a* ***(μ, λ)-splitting*** *if there exists decompositions $\mathbb{R}^n = E_m^+ \oplus E_m^-$ such that $L_m E_m^\pm = E_{m+1}^\pm$ and*

$$\|L_m|_{E_m^-}\| \leq \mu, \|L_m^{-1}|_{E_{m+1}^+}\| \leq \lambda^{-1}.$$

- *We will say that $\{L_m\}$ admits an* ***exponential splitting*** *if $\mu < 1$, $\dim E_m^- \geq 1$* ***or*** *$\lambda > 1$, $\dim E_m^+ \geq 1$*
- *$\{L_m\}$ is* ***uniformly hyperbolic*** *if it admits a (μ, λ)-splitting with $\mu < 1 < \lambda$.*

Theorem 19. *Let $\mu < \lambda$, $r \geq 1$, and for each $m \in \mathbb{Z}$ let $F_m : \mathbb{R}^n \mapsto \mathbb{R}^n$ be a surjective C^r-diffeomorphism such that for $(x, y) \in \mathbb{R}^k \oplus \mathbb{R}^{n-k}$,*

$$F_m(x, y) = (A_m x + \alpha_m(x, y), B_m y + \beta_m(x, y))$$

for some invertible linear maps $A_m : \mathbb{R}^k \mapsto \mathbb{R}^k$ and $B_m : \mathbb{R}^{n-k} \mapsto \mathbb{R}^{n-k}$ with $\|A_m^{-1}\| \leq \lambda^{-1}$, $\|B_m\| \leq \mu$ and $\alpha_m(0) = 0$, $\beta_m(0) = 0$.

Then $\exists\ \gamma_0 = \gamma_0(\mu, \lambda)$, s.t. for every $0 < \gamma < \gamma_0$ $\exists\ \delta_0 = \delta_0(\mu, \lambda, \gamma_0)$, s.t. for every $0 < \delta < \delta_0$ the following holds.

If $\|\alpha_m\|_{C^1} < \delta$, $\|\beta_m\|_{C^1} < \delta$ for all $m \in \mathbb{Z}$, then there is a unique family $\{\mathcal{W}_m^+\}_{m\in\mathbb{Z}}$ of k-dimensional C^1-manifolds

$$\mathcal{W}^+ = \{(x, \phi_m^+(x)) : x \in \mathbb{R}^k\} = \mathrm{graph}\phi_m^+$$

and a unique family $\{\mathcal{W}_m^-\}_{m\in\mathbb{Z}}$ of $n-k$-dimensional C^1-manifolds

$$\mathcal{W}^- = \{(x, \phi_m^-(x)) : x \in \mathbb{R}^k\} = \mathrm{graph}\phi_m^-$$

where $\phi_m^+ ; \mathbb{R}^k \mapsto \mathbb{R}^k$, $\phi_m^- ; \mathbb{R}^{n-k} \mapsto \mathbb{R}^k$, $\sup_{m\in\mathbb{Z}} \|D\phi_m^\pm\| < \gamma$

- $F_m(\mathcal{W}_m^{\pm}) = \mathcal{W}_{m+1}^{\pm}$;
- $\|F_m(z)\| < \mu'\|z\|$ *for* $z \in \mathcal{W}_m^-$ *and* $\|F_m^{-1}(z)\| < (\lambda')^{-1}\|z\|$ *for* $z \in \mathcal{W}_m^+$ *where*

$$\mu < \mu' = \mu'(\gamma, \delta, \mu) < \lambda' = \lambda'(\gamma, \delta, \lambda) < \lambda.$$

- *If* $\mu < 1 < \lambda$ *then* $\{\mathcal{W}_m^{\pm}\}m \in \mathbb{Z}$ *are* C^r*-manifolds.*

Proof. *case* $F_m \equiv T$ *and* $n = 2$

i) Existence of the local unstable manifold:

C. Liverani's notes.

ii) C^1*-regularity*:

We would like to show that the tangent vector to the curve γ_* exists. The proof is a slight modification and clarification of that in C. Liverani's notes (in particular, the notation has been smoothed out).

Below, the notation $o(\delta)$ means little "o", and stands for any/some function of δ such that $\lim_{\delta \to 0} \frac{o(\delta)}{\delta} = 0$.

1) Define the cone field

$$C_{\theta,h}(x; v) := \left\{\xi \in B_h(x) \subset \mathbb{R}^2 : \xi - x = (a_0, b_0), a_0 \neq 0, |b_0/a_0 - v| \leq \theta\right\}$$

this is a cone at point x, symmetric around the direction $(1, v)$, of an opening that depends only on θ, intersected with an h-nbhd of x in $\mathbb{R}^2$. We also impose the conditions that $|v| < c$, $\theta \leq c\delta$ and $h \leq \delta$.

2) Given $x \in \gamma_*$, the piece of the curve γ_*,

$$\gamma_*^{h,x} := B_h(x) \cap \gamma_*$$

lies in $C_{c,h}(x; 0)$. Indeed, if $x \in \gamma_*$, then $(a_0, b_0) = \xi - x = \gamma_*(t_1) - \gamma_*(t_2) = (t_1 - t_2, u_*(t_1) - u_*(t_2)$ satisfies

$$\left|\frac{b_0}{a_0} - 0\right| = \left|\frac{u_*(t_1) - u_*(t_2)}{t_1 - t_2}\right| \leq c$$

by the Lipschitz property of u_* (i.e. we have that $v = 0$ and $\theta = c$ in $C_{\theta,h}(x; v)$).

3) Apply maps $\hat{T}$ and its affine approximation $L(\xi) := T(x) + DT[x](\xi - x)$ to $C_{\theta_0,h}(x; v)$:

$$T\left(C_{\theta_0,h}(x; v)\right) = \{T(\xi),\ \text{where } \xi \in C_{\theta_0,h}(x; v)\}, \quad L\left(C_{\theta_0,h}(x; v)\right) = \{L(\xi),\ \text{where } \xi \in C_{\theta_0,h}(x; v)\}$$

We notice that for $\xi \in C_{\theta_0,h}(x;v)$,

$$(a_1,b_1) := T(\xi) - T(x) = DT[x](a_0,b_0) + o(\|a_0,b_0\|) = L(\xi) - L(x) + o(\|a_0,b_0\|,$$

since T is a C^1 map. Therefore, sets $T\left(C_{\theta_0,h}(x;v)\right)$ and $L\left(C_{\theta_0,h}(x;v)\right)$ are not too far from each other: there is a constant $C_1 = C_1(\delta) = o(\delta)$, such that

$$(B_h(T(x)) \cap T\left(C_{\theta_0,h}(x;v)\right)) \subset (B_h(T(x)) \cap L\left(C_{\theta_0+C_1,h}(x;v)\right)). \tag{5.7}$$

The second set here can be "computed": since point x is δ-close to 0, there exist a constant $C_2 = C_2(\delta) = o(\delta)$, such that

$$\|(a_1', b_1') - (\lambda a_0, \mu b_0)\|_\infty \le C_2,$$

where $(a_1', b_1') := (L(\xi) - L(x))$ and $\|\cdot\|_\infty$ is the vector l_∞-norm. Therefore, denoting the normalized action of the derivative map on the second component of a vector as $S[x]v$:

$$S_x(v) := \frac{\pi_2 DT[x](1,v)}{\pi_1 DT[x](1,v)}, \quad \pi_i - \text{projection on the i} - \text{th component},\ i = 1,2,$$

we get,

$$\begin{aligned} \left|\frac{b_1'}{a_1'} - S_x(v)\right| = \quad & \left|\frac{\mu}{\lambda}\frac{b_0}{a_0} - \frac{\mu}{\lambda}v + o(\delta)\right| = \frac{\mu}{\lambda}\left|b_0/a_0 - v + o(\delta)\right| \implies \\ \implies & \left|\frac{b_1'}{a_1'} - S_x(v)\right| < \left(\frac{\mu}{\lambda} + C_3\right)(\theta_0 + C_1), \end{aligned}$$

for some $C_3 = C_3(\delta) = o(\delta)$. Set, inductively,

$$\theta_n = \left(\frac{\mu}{\lambda} + o(\delta)\right)(\theta_{n-1} + C_1), \quad n = 1,2,3,\ldots, \tag{5.8}$$

then

$$\begin{aligned} (B_h(T(x)) \cap L\left(C_{\theta_0+C_1,h}(x;v)\right)) \; = \; & \left\{\xi \in B_h(T(x)) \subset \mathbb{R}^2 : \xi - x = (a_1', b_1'), a_1' \ne 0, \left|\frac{b_1'}{a_1'} - S_x(v)\right| \le \theta_1\right\} \\ & \subset C_{\theta_1,h}(T(x), S_x(v)). \end{aligned}$$

Finally, we have by (5.7), that

$$(B_h(T(x)) \cap T\left(C_{\theta_0,h}(x;v)\right)) \subset C_{\theta_1,h}(T(x), S_x(v)). \tag{5.9}$$

4) Consider

$$\gamma_*^{\theta_n h, T^{-n}x} := B_{\theta_n h}(T^{-n}(x)) \cap \gamma_*.$$

By part 2) this lies in $C_{c,\theta_n h}(T^{-n}(x);0)$. Where does the piece of the stable manifold $B_{\theta_n h} \cap T^n(\gamma_*^{\theta_n h, T^{-n}x}) \ni x$ lie? In

$$\left(B_{\theta_n h} \cap T^n(\gamma_*^{\theta_n h, T^{-n}x})\right) \subset T^n\left(C_{c,\theta_n h}(T^{-n}(x);0)\right),$$

and, by (5.9), this is contained in

$$C_{\theta_n c,\theta_n h}(x, S_{T^{-1}(x)} \circ \ldots \circ S_{T^{-n+1}(x)} \circ S_{T^{-n}(x)}(0))$$

5) Use the above machinery to show that for any point $x \in \gamma_*$ and a sequence $\xi_n \in \gamma_*$ such that $\lim_{n\to\infty} \xi_n = x$, the limit:

$$\lim_{n\to\infty} \frac{\pi_2(\xi_n - x)}{\pi_1(\xi_n - x)}$$

exist, and is a continuous function of x. The C^1 property of γ_* follows. $\square$

6. Hyperbolic sets and shadowing

6.1. Hyperbolic sets

Let M be a C^1 Riemannian manifold, $U \subset M$ a non-empty open subset, $f : U \mapsto f(U)$ - a C^1 diffeomorphism.

A compact f-invariant subset Λ is *hyperbolic* if $\exists\ \lambda \in (0,1)$ and families of subspaces $E^{\pm}(x) \subset T_xM$, $x \in \Lambda$, s.t. for every $x \in \Lambda$:

- $T_xM = E^+(x) \oplus E^-(x)$;
- $\|D(f^n)(x)|_{E^+(x)}\| \le C\lambda^n$ and $n \ge 0$;
- $\|D(f^{-n})(x)|_{E^-(x)}\| \le C\lambda^n$ and $n \ge 0$;
- $Df(x)E^{\pm}(x) = E^{\pm}(f(x))$.

Definition 20. *If $\Lambda = M$ then f is called an Anosov diffeomorphism.*

Theorem 21. *Let Λ be a hyperbolic set for f. Then the subspaces $E^{\pm}(x)$ depend continuously on $x \in \Lambda$.*

Theorem 22. *If Λ is a hyperbolic set of f with constant C and λ, then for every $\epsilon > 0$ $\exists$ a C^1 Riemannian adapted metric $\|\cdot\|'$ in a neighborhood of Λ with respect to which the family $Df(f^m(x))$ is uniformly hyperbolic (i.e. $C = 1$) with $\lambda' = \lambda + \epsilon$ and $\mu' = \lambda^{-1}$ and the subspaces $E^{\pm}(x)$ are ϵ-orthogonal, i.e. $\langle v^+, v^- \rangle' < \epsilon$ for all unit vectors $v^{\pm} \in E^{\pm}(x)$, $x \in \Lambda$.*

Proof. For any $v, v' \in E^+(x)$, $x \in \Lambda$, define the following inner product on $E^+(x)$:

$$\langle v, v' \rangle' = \sum_{n \geq 0} \lambda'^{-2n} \langle Df^n(x)v, Df^n(x)v' \rangle. \tag{6.10}$$

For any $v, v' \in E^-(x)$, $x \in \Lambda$, define the following inner product on $E^-(x)$:

$$\langle v, v' \rangle' = \sum_{n \geq 0} \lambda'^{-2n} \langle Df^{-n}(x)v, Df^{-n}(x)v' \rangle. \tag{6.11}$$

Both converge uniformly for $\|v^{\pm}\| \leq 1$ and $x \in \Lambda$. Set $\|v^{\pm}\|' = \sqrt{\langle v, v \rangle'}$, $v \in E^{\pm}(x)$, and for a vector $v = v^+ + v^-$, $\|v\|' = \sqrt{\|v^+\|^2 + \|v^-\|^2}$. Finally, for any two vectors v and w in T_xM set

$$\langle v, w \rangle' = \frac{1}{2}\left(\|v + w\|' - \|v\|' - \|w\|'\right)$$

□

6.2. Horseshoe: an example of a hyperbolic set

A *rectangle* in $\mathbb{R}^{k+l}$ will mean a set of the form $D_1 \times D_2 \subset$ where D_i are disks, $\pi_1 : \mathbb{R}^{k+l} \mapsto \mathbb{R}^k$ and $\pi_2 : \mathbb{R}^{k+l} \mapsto \mathbb{R}^l$ will be two orthogonal projections. $\mathbb{R}^k$ will be called the "horizontal" direction, $\mathbb{R}^l$ - the vertical.

Definition 23. *Full component Suppose $\Delta \subset U \subset \mathbb{R}^{k+l}$ is a rectangle and $f : U \mapsto \mathbb{R}^{k+l}$ is a diffeo. A connected component $\Delta_0 = f(\Delta_0')$ of $\Delta \cap f(\Delta)$ is called full, if*
1) $\pi_2(\Delta_0') = D_2$;
2) *for any $z \in \Delta_0'$, $\pi_1|_{f(\Delta_0' \cap (D_1 \times \pi_2(z)))}$ is a bijection onto D_1.*

Definition 24. *(Horseshoe) If $U \subset \mathbb{R}^{k+l}$ is open then a rectangle $\Delta = D_1 \times D_2 \subset U \subset \mathbb{R}^k \oplus \mathbb{R}^l$ is called a horseshoe for a diffeo $f : U \mapsto \mathbb{R}^{k+l}$ if $\Delta \cap f(\Delta)$ contains at least two full components Δ_0 and Δ_1 such that for $\Delta' = \Delta_0 \cap \Delta_1$..*
1) $\pi_2(\Delta') \subset \mathrm{int} D_2$, $\quad \pi_1(f^{-1}(\Delta')) \subset \mathrm{int} D_1$;
2) $D\left(f|_{f^{-1}(\Delta')}\right)$ *preserves and expands a horizontal cone family on $f^{-1}(\Delta')$;*
3) $D\left(f^{-1}|_{\Delta'}\right)$ *preserves and expands a vertical cone family on $f^{-1}(\Delta')$.*

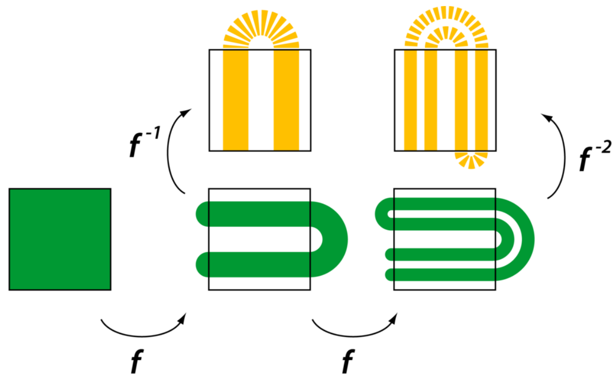

Figure 6. Generating a horseshoe.

• Let us study the maximal invariant subset of Δ. Denote Δ_{ω_0}, $\omega_0 = 0, 1$, the two full components of $\Delta \cap f^1(\Delta)$, and $\Delta^{\omega_0} = f^{-1}(\Delta_{\omega_0})$, $\omega_0 = 0, 1$.

• The intersection $\Delta \cap f(\Delta) \cap f^2(\Delta)$ consists of four horizontal rectangles:

$$\Delta_{\omega_1 \omega 2} = \Delta_{\omega_1} \bigcap f(\Delta_{\omega_2}) = f(\Delta^{\omega_1}) \bigcap f^2(\Delta^{\omega_2}),$$

$\omega_i \in \{0, 1\}$.

• Inductively, the set $\cap_{i=1}^n f^i(\Delta)$ consists of 2^n disjoint horizontal rectangles of exponentially decreasing heights.

$$\Delta_{\omega_1 \dots \omega_n} := \bigcap_{i=1}^{n} f^i(\Delta^{\omega_i}),\ \omega_i \in \{0, 1\}.$$

Each infinite intersection

$$\Delta_\omega := \bigcap_{i=0}^{n} f^i(\Delta^{\omega_i}), \omega = (\omega_1 \dots, \omega_n, \dots) \in \Sigma_2^+,$$

is a horizontal fiber (a curve connecting the left and the right sides of Δ, such that the projection π_1 on the disk D_1 is a bijection).

• Similarly, the sets

$$\Delta^{\omega_{-n} \dots \omega_0} := \bigcap_{i=0}^{n} f^{-i}(\Delta^{\omega_{-i}}),\ \omega_{-i} \in \{0, 1\},$$

are vertical rectangles, the sets

$$\Delta^\omega := \bigcap_{i=0}^{n} f^{-i}(\Delta^{\omega_{-i}}),\ \omega = (\dots, \omega_{-n}, \dots, \omega_{-1}, \omega_0) \in \Sigma_2^+,$$

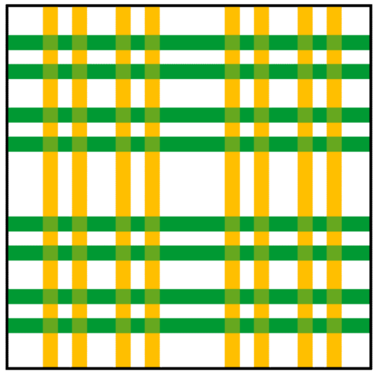

Figure 7. An approximation of the invariant hyperbolic set.

are vertical fibers.

• The intersection of any vertical fiber with the set of horizontal fibers projects to a Cantor set Λ_2 in D_2, while the intersection of any horizontal fiber with the vertical ones projects to a Cantor set Λ_1 in D_1:

$$\Lambda_2 := \Delta^{\dots\omega_{-n}\dots\omega_{-1},\omega_0} \bigcap \left(\bigcap_{i=1}^{\infty} f^i(\Delta) \right),$$

$$\Lambda_1 := \Delta_{\omega_1\dots\omega_n\dots} \bigcap \left(\bigcap_{i=0}^{\infty} f^{-i}(\Delta) \right).$$

• Finally, the set

$$\Lambda := \bigcap_{i=-\infty}^{\infty} f^{-i}(\Delta)$$

is an invariant set, equal to the product of two Cantor sets Λ_1 and Λ_2, hence a Cantor set itself. The map $h : \Sigma_2 \mapsto \Lambda$, given by

$$h(\omega) = \bigcap_{i=-\infty}^{\infty} f^{-i}(\Delta^{\omega_i})$$

is the homeomorphism conjugating the shift $\sigma|_{\Sigma_2}$ to $f|_\Lambda$.

Corollary 25. *The horseshoe is a hyperbolic set.* $f|_\Lambda$ *is topologically conjugate to* $\sigma|_{\Sigma_2}$.

Proof. Hyperbolicity follows from the invariance of the cone families and stretching of the vectors inside the cones. □

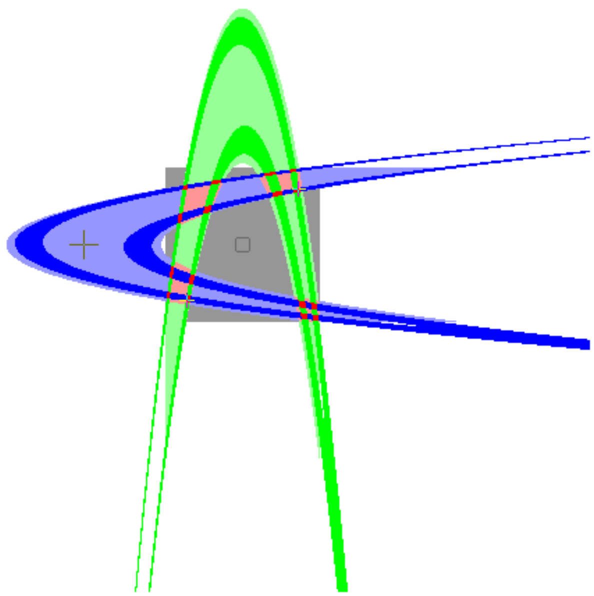

Figure 8. Horseshoe for a Hénon map, taken from this applet.

Corollary 26. *$f|_\Lambda$ is topologically mixing. Periodic points of f are dense in Λ, and the number of periodic points of period p is 2^p.*

For stable/unstable manifolds, horseshoe, the attractor, etc for the Hénon family check this applet.

6.3. Homoclinic and heteroclinic intersections

Definition 27. *(Homoclinic points) Let p be a hyperbolic periodic point of a diffeo $f : U \mapsto M$. A point q is homoclinic to p if $q \neq p$ and $q \in W^s(p) \cap W^u(p)$. It is transverse homoclinic if, additionally, $W^s(p)$ and $W^u(p)$ intersect transversely at q.*

Definition 28. *(Heteroclinic points) Suppose $p_1, \ldots, p_k$ be periodic points (of possibly different periods) of $f : U \mapsto M$. Suppose $W^u(p_i)$ intersects $W^s(p_{i+1}$ at q_i, $i = 1, \ldots, k$ ($p_{k+1} = p_1$. q_i are called heteroclinic points.*

Theorem 29. *Let p be a hyperbolic periodic point of a diffeo $f : U \mapsto M$ and let q be a transverse homoclinic point to p. Then for every $\epsilon > 0$ the union of ϵ-neighborhoods of the orbits of p and q contains a horseshoe of f.*

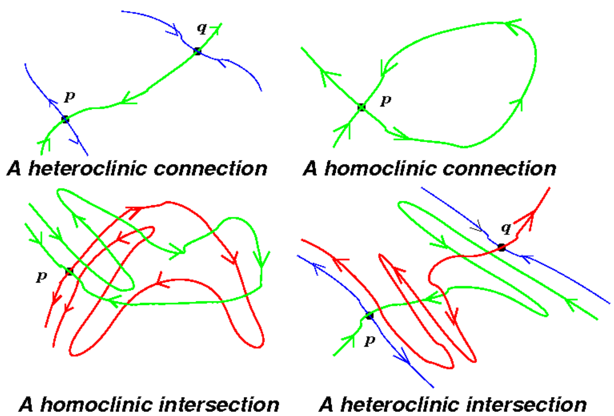

Figure 9. Some possible configurations of homoclinic/heteroclinic intersections

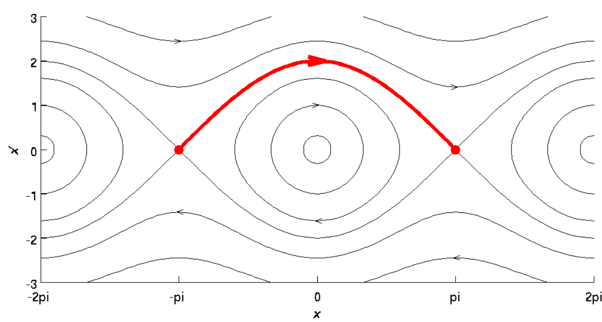

Figure 10. A heteroclinic connection in a pendulum

6.4. Shadowing

An ϵ-orbit (a pseudo-orbit) if $f : U \mapsto M$ is a finite or infinite set $\{x_n\}$ s.t. s to $\mathrm{dist}(f(x_n), x_{n+1}) < \epsilon$ for all n.

Question: *When are orbits of a perturbed dynamical system are ϵ-orbits of the original one? This might give a us a way to conjugate the perturbed and the original systems.*

The following theorem answers this question.

Theorem 30. *(Shadowing Theorem) Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1-diffeo $f : M \mapsto M$ of a smooth manifold M. Then there exists a nbhd. U of Λ and a neighborhood W of f in $C^1(M, M)$ such that for all $\delta > 0$ there exists $\epsilon > 0$ s. t. for all topological spaces X, homeos $g : X \mapsto X$ and continuous maps $h_0 : X \mapsto U$ the following holds.*

If $\tilde{f} \in W$ is such that $d_{C^0}(h_0 \circ g, \tilde{f} \circ h_0) < \epsilon$ then

1) *(existence of a conjugacy) there is a continuous $h_1 : X \mapsto U$ s.t.*

$$h_1 \circ g = \tilde{f} \circ h_1, \text{ and } d_{C^0}(h_0, h_1) < \delta;$$

2) *(uniqueness of the conjugacy) $\exists\ \delta_0 = \delta_0(\Lambda, f) > 0$, s.t. if $h_1' : X \mapsto U$ is a cont. map satisfying $h_1' \circ g = \tilde{f} \circ h_1'$ and $d_{C^0}(h_1', h_1) < \delta_0$ then $h_1' = h_1$;*
3) *(continuity of the conjugacy) h_1 depends continuously on $\tilde{f}$*

Proof. The proof will be based on the Contraction Mapping Principle: we look for the desired h_1 as the fixed point of the operator

$$F : C^0(X, U) \mapsto C^0(X, M),\ F(h) := \tilde{f} \circ h \circ g^{-1}.$$

1) Set

$$C^0_{h_0}(X, h_0^* TM) = \left\{ \xi \in C^0(X, TM) : \xi(x) \in T_{h_0(x)} M, x \in X \right\},$$

the space of continous vector fields field "along" h_0, endowed with the sup. norm. Now, let U_1 be any relatively compact nbhd. of Λ.

There is $\theta_0 = \theta_0(U_1, M) > 0$ such that for all $0 < \theta < \theta_0$, the following map $\mathcal{A} : B_\theta(h_0) \subset C^0(X, U_1) \mapsto C^0_{h_0}(X, h_0^* TM)$ is defined

$$\mathcal{A}(h)(x) := \exp^{-1}_{h_0(x)}(h(x))$$

Sidenote: for $v \in T_x M$ we now denote by c_v the geodesic with $c(0) = x$, $\dot{c}(0) := v$, the exponential map

$$\exp_x(v) := c_v(\epsilon)$$

is an embedding of $\{v \in T_x M : \|v\| \leq R\}$ into M, where ϵ is possibly a very small constant, depending on R.

Lemma 31. *Let $c : [0, T] \mapsto M$ be a geodesic and $\tau : [0, T/a] \mapsto [0, T]$, $t \mapsto \tau(t) = at$. Then $\tilde{c} = c \circ \tau$ is a geodesic.*

Therefore, for $\delta = \epsilon R$ we obtain a smooth embedding of

$$\exp_x : \overline{B(0, \delta)}) := \{v \in T_x M : \|v\| \leq \delta\} \mapsto M,\ \exp_x(v) := c_v(1).$$

Define $r_x > 0$ to be the supremum of those δ for which $\exp_x$ is injective on the δ-ball $B(0, \delta)$, the injectivity radius.

---

2) Suppose that v is a f. p. of

$$F^{h_0} := \mathcal{A} \circ F \circ \mathcal{A}^{-1} : B_\theta^{h_0}(0) \mapsto C^0_{h_0}(X, h_0^* TM),$$

$$F^{h_0}(v)(x) := \exp^{-1}_{h_0(x)}(\tilde{f}(\exp_{h_0(g^{-1}(x))}(v(g^{-1}(x))))).$$

Then $\mathcal{A}^{-1}v$ is a fixed point of F.

F^{h_0} is differentiable in v:

$$\begin{aligned}(DF^{h_0}|_v \xi)(x) : &= (D \exp^{-1}_{h_0(x)})|_{\tilde{f}(\exp_{h_0(g^{-1}(x))} v(g^{-1}(x)))} \cdot D\tilde{f}|_{\exp_{h_0(g^{-1}(x))} v(g^{-1}(x))} \\ &\cdot \ (D \exp_{h_0(g^{-1}(x))})|_{v(g^{-1}(x))} \xi(g^{-1}(x)).\end{aligned}$$

Notice, for $v_1, v_2 \in B_\theta^{h_0}(0)$

$$\|(DF^{h_0}|_{v_1}\xi)(x) - (DF^{h_0}|_{v_2}\xi)(x)\| \le C\|v_1 - v_2\|,$$

for some C.

Lemma 32. *There exists a neighborhood $U \supset \Lambda$, $\epsilon_0, \epsilon > 0$, and $R > 0$ independent of X, g and h_0, such that*

$$\| \left(DF^{h_0}|_0 - Id\right)^{-1} \| < R,$$

whenever $d_{C^1}(f, \tilde{f}) < \epsilon_0$, $d_{C^0}(h_0 \circ g, \tilde{f} \circ h_0) < \epsilon$.

$$F^{h_0}(v) = DF^{h_0}|_0 v + H(v).$$

A f. p. v of F^{h_0} satisfies

$$((DF^{h_0})_0 - Id)v = -H(v),$$

or

$$v = -(DF^{h_0}|_0 - Id)^{-1} H(v) =: T(v).$$

DH is Lipschitz, since DF^{h_0} is. This Lipschitz constant K is independent of X, g, h_0. Therefore,

$$\|T(v_1) - T(v_2)\| < RK \max(\|v_1\|, \|v_2\|)\|v_1 - v_2\|,$$

and, therefore, T is a contraction near 0.

Next,

$$H(0)(x) = F^{h_0}(0)(x) = \exp^{-1}_{h_0(x)}(\tilde{f}(h_0(g^{-1}(x)))),$$

we have

$$\|H(0)\| = d_{C^0}(h_0, \tilde{f} \circ h_0 \circ g^{-1}) = d_{C^0}(h_0 \circ g, \tilde{f} \circ h_0),$$

and

$$\|T(0)\| < R\|H(0)\| = Rd_{C^0}(h_0 \circ g, \tilde{f} \circ h_0).$$

Now, take $\delta_0 = 1/2RK$, $\theta = \min(\delta, \delta_0)$, and $\epsilon < \theta/2R$ as in the Lemma. Then

$$\|T(v_1) - T(v_2)\| < \frac{1}{2}\|v_1 - v_2\|$$

for $v_i \in B^{h_0}_{\delta_0}(0) \subset C^0_{h_0}(X, h_0^*TM)$, and $\|T(0)\| < \theta/2$, whenever h_0 is such that $d_{c^0}(h_0 \circ g, \tilde{f} \circ h_0) < \epsilon$.

In conclusion, $T(B^{h_0}_{\delta_0}(0) \subset B^{h_0}_{\delta_0}(0)$. By CMP, T has a unique f. p. v in $B^{h_0}_{\delta_0}(0)$ and F has a unique f. p. $\beta = \mathcal{A}^{-1}v \in B_{\delta_0}(h_0)$, which is, in fact, in $B_\delta(h_0)$ since $T(B^{h_0}_{\delta_0}(0) \subset B^{h_0}_{\delta_0}(0) \subset B^{h_0}_{\delta}(0)$. □

Definition 33. *Let (X, f) be a dyn. sys. on a metric space X. An ϵ-psedu-orbit $\{x_k\}_{k\in\mathbb{Z}}$ is δ-shadowed by an orbit of of $x \in X$ under f if $d_X(x_k, f^k(x)) < \delta$ for all $k \in \mathbb{Z}$.*

Orbits of a hyperbolic dynamical system shadow pseudo-orbits:

Corollary 34. *(Shadowing Lemma) Let Λ be a hyperbolic set for $f : U \mapsto M$. Then $\exists$ an open nbhd $V \supset \Lambda$ s.t. for every $\delta > 0$ there is $\epsilon > 0$ so that every ϵ-pseudo-orbit in V is δ-shadowed by an orbit of f.*

Furthermore, there is δ_0 s. t. if $\delta < \delta_0$ then the orbit of f shadowing the given pseudo-orbit is unique.

Proof. Take $X = \mathbb{Z}$ (with discrete topology); $g : X \mapsto X$ given by $g(k) = k + 1$; $h_0 : X \mapsto V$ given by $h_0(k) = x_k$; and $\tilde{f} = f$. By the Shadowing Theorem $\exists$ $h_1 : X \mapsto V$ such that $h_1 \circ g = f \circ h_1$ and $d_{C^0}(h_0, h_1) < \delta$, i.e.

$$h_1(k+1) = f(h_1(k)), \text{ for all } k \in \mathbb{Z} \text{ or } h_1(k) = f^k(x),$$

where $x = h_1(0)$, and $d(x_k f^k(x)) < \delta$ for all $k \in \mathbb{Z}$ as requested. □

Periodic orbits of a hyperbolic dynamical system shadow pseudo-orbits "uniformly":

Corollary 35. *(Anosov Closing Lemma) Let Λ be a hyperbolic set for $f : U \mapsto M$. Then $\exists$ an open nbhd $V \supset \Lambda$ and $C, \epsilon_0 > 0$, s.t. for every $\epsilon < \epsilon_0$ and any periodic ϵ-orbit $(x_0, x_1, \dots, x_m) \subset V$, there is a point $y \in U$ s. t. $f^m(y) = y$ and $\operatorname{dist}(f^k(y), x_k) < C\epsilon$ for $k = 0, 1, \dots, m-1$.*

Proof. Choose $X = \mathbb{Z}_m$, $g(k) = k + 1 \bmod m$, $h_0(k) = x_k$ and $\tilde{f} = f$ in the Shadowing Theorem. $\square$

Remark 36. *In particular, consider an almost periodic orbit, i.e. an orbit segment s. t. $\operatorname{dist}(f^m(x_0), x_0) < \epsilon$ (this is a pseudo-orbit). Thus Anosov Closing Lemma implies that close to any orbit in a hyperbolic set Λ that "almost" returns to itself, there is a true periodic orbit (but not necessarily in Λ).*

Finally, the Shadowing Theorem leads to the *structural stability* of hyperbolic sets:

Theorem 37. *(Persistence of hyperbolic sets) Let $\Lambda \subseteq M$ be a hyperbolic set for a C^1-diffeo $f : M \mapsto M$ Then there exists an open nbhd. $V \supset \Lambda$ s.t. for any C^1 diffeo $g : M \mapsto M$ sufficiently C^1-close to f, the completely invariant set*

$$\Lambda_V^g = \bigcap_{m \in \mathbb{Z}} g^m(\bar{V})$$

is hyperbolic for g, if not empty. In particular, $\Lambda_V^f \supseteq \Lambda$ is hyperbolic.

Proof. 1) Extend the invariant splitting $T_x M = E_x^+ \oplus E_x^-$ defined for $x \in \Lambda$ to a continuous (but not nec. invariant splitting) on an open $V_1 \supset \Lambda$. Given $\gamma > 0$, let

$$H_x^\gamma := \left\{ u + v \in T_x M : u \in E_x^+, v \in E_x^-, \|v\| \le \gamma \| \|u\| \right\}$$

be the corresponding horizontal cone in $T_x M$, and let V_x^g be the complimentary vertical cone.

2) $\exists$ (λ, μ)-splitting on $\Lambda \implies$

$$\begin{aligned} Df[x]\left(H_x^\gamma\right) \quad &\subseteq H_{f(x)}^{\gamma\lambda/\mu} \subset \operatorname{int} H_{f(x)}^\gamma \cap \{0\}, \\ (Df[x])^{-1}\left(V_{f(x)}^\gamma\right) &\subseteq V_x^{\gamma\lambda/\mu} \subset \operatorname{int} V_x^\gamma \cap \{0\}, \end{aligned}$$

and

$$u + v \in H_x^\gamma \implies \|Df[x](u+v)\| \ge \frac{\mu - \lambda\gamma}{1+\gamma} \|u + v\|, \tag{6.12}$$

$$u + v \in (Df[x])^{-1}\left(V_{f(x)}^\gamma\right) \implies \|Df[x](u+v)\| \le (1+\gamma)\lambda \|u + v\|. \tag{6.13}$$

Now, by continuity, for any $\delta > 0$ we can find a rel. compact nbhd $V \subseteq V_1$ of Λ and a nbhd f in C^1-topology s.t. (6.12) and (6.13) remain valid with μ substituted by $\mu - \delta$ and λ by $\lambda + \delta$ for all $x \in V$ and $g \in W$.

3) Consider the set Λ_V^g.

The sequence of differentials $Dg(g^m(x))$ admits a (λ', μ') splitting with

$$\lambda' = (1+\gamma)(\lambda+\delta),$$
$$\mu' = \frac{\mu - \lambda\gamma - (1+\gamma)\delta}{1+\gamma},$$

and if δ and γ are small, we still have $\lambda' < 1 < \mu'$, the set Λ_V^g is hyperbolic for g. In particular, the subspaces

$$E^+(x) := \cap_{n\geq 0} Dg^{-n}(g^n)V^\gamma(g^n(x)), \tag{6.14}$$
$$E^-(x) := \cap_{n\geq 0} Dg^{n}(g^{-n})H^\gamma(g^{-n}(x)), \tag{6.15}$$

and the definition of a hyperbolic set checks with $\lambda := \max(\lambda', (\mu')^{-1}$ □

Theorem 38. *(Structural stability of hyperbolic sets) Let $\Lambda \subseteq M$ be a hyperbolic set for C^1 diffeomorphism $f : M \mapsto M$ of a smooth manifold M. Then for every open nbhd. V of Λ and every $\eta > 0$ there exists a nbhd. W of f in $C^1(M,M)$ such that for all diffeomorphisms $\tilde{f} \in W$ there is a hyperbolic set $\tilde{\Lambda} \subset V$, and a homeomorphism $H : \Lambda \mapsto \tilde{\Lambda}$ with*

$$h \circ f = \tilde{f} \circ h$$

on Λ and $d_{C^0}(\mathrm{id}, h) + d_{C^0}(\mathrm{id}, h^{-1}) < \eta$. Furthermore, h is unique if δ is small enough.

Proof.
i) Apply the Shadowing Theorem taking $\delta < \min\{\delta_0, \eta/2\}$, $X = \Lambda$, $h_0 = \mathrm{id}_\Lambda$ and $g = f$. Get a nbhd $V_1 \subset V$ of Λ, and a nbhd W_1 of f, such that $d_{C^0}(\tilde{f}, f) < \epsilon$ for all $\tilde{f} \in W_1$, and a unique $h_1 : \Lambda \mapsto V_1$ such that $h_1 \circ f = \tilde{f} \circ h_1$ and $d_{C^0}(\mathrm{id}_\Lambda, h_1) < \delta$.

In particular, $\tilde{\Lambda} = h_1(\Lambda)$ is completely $\tilde{f}$-invariant and hyperbolic by Theorem 36 (after, possibly, a shrinking of W_1).

ii) To prove that h_1 is injective, we apply the Shadowing Theorem again taking δ as before, $X = \tilde{\Lambda}$ and $h_0 := \mathrm{id}_{\tilde{\Lambda}}$ and $g = \tilde{f}$, we get the same nbhd W_1 as soon as ϵ is small. Then we have a unique $h_2 : \tilde{\Lambda} \mapsto V$ s.t. $h_2 \circ \tilde{f} = f \circ h_2$ and $d_{C^0}(\mathrm{id}_{\tilde{\Lambda}}, h_2) < \delta$.

iii) To end the proof, it is sufficient to show that $h_2 \circ h_1 = \mathrm{id}_\Lambda$. We apply again the Shadowing Theorem with $X = \Lambda$, $h_0 = \mathrm{id}_{\tilde{\Lambda}}$ and $g = \tilde{f} = f$. Since

$$d_{C^0}(\mathrm{id}_\Lambda, h_2 \circ h_1) \leq d_{C^0}(\mathrm{id}_\Lambda, h_1) + d_{C^0}(h_1, h_2 \circ h_1) = d_{C^0}(\mathrm{id}_\Lambda, h_1) + d_{C^0}(\mathrm{id}_{\tilde{\Lambda}}, h_2) < 2\delta < \delta_0,$$

we can apply the uniqueness statement in the Shadowing Theorem to get

$$h_2 \circ h_1 = \mathrm{id}_\Lambda,$$

because they both commute with f and are close to h_1. $\square$